

# How long is long enough? Finite-horizon approximation of energy storage scheduling problems

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## ABSTRACT

Energy storage scheduling problems, where a storage is operated to maximize its profit in response to a price signal, are essentially infinite-horizon optimization problems as storage systems operate continuously, without a foreseen end to their operation. Such problems can be solved to optimality with a rolling-horizon approach, provided that the planning horizon over which the problem is solved is long enough. Such a horizon is termed a *forecast horizon*. However, the length of the planning horizon is usually chosen arbitrarily for such applications. We introduce an easy-to-check condition that confirms whether a planning horizon is a forecast horizon, and which can be used to derive a bound on suboptimality when it is not the case. By way of an example, we demonstrate that the existence of forecast horizons is not guaranteed for this problem. We also derive a lower bound on the length of the minimum forecast horizon. We show how the condition introduced can be used as part of an algorithm to determine the minimum forecast horizon of the problem, which ensures the determination of optimal solutions at the lowest computational and forecasting costs. Finally, we provide insights into the implications of different planning horizons for a range of storage system characteristics.

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## 1. Introduction

Energy storage systems are key enablers in the transition to decarbonized energy systems. They guarantee the reliable operation of networks in which a large share of the production, coming from renewable sources, is variable and uncertain. For these reasons, their presence in power systems is expected to increase significantly in the coming years. As an example, worldwide investment in battery storage has doubled year-on-year since 2020 (IEA, 2024). Storage systems with different durations of charge and discharge – which corresponds to the time they need to fully charge or discharge – are needed to cover flexibility needs at different time scales, including seasonal storage (Schmidt and Staffell, 2023; Yang et al., 2024; Zhang et al., 2021). Therefore, the question of how to optimally operate these storage systems has never been more relevant.

Scheduling an energy storage system involves determining when to charge or discharge to maximize a desired objective. As storage systems introduce links between periods, and as there is no specific end date to their operation, this problem should be modeled as having an infinite horizon. This infinite-horizon aspect is usually disregarded in the literature; however, there are some exceptions. Nascimento and Powell (2013) consider an infinite-horizon problem, but their approach is inexact as it is based on approximating

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the objective function of the problem. Van de Ven et al. (2013) do treat the problem as having an infinite horizon and show that the optimal policy has a structure with two energy threshold levels. When the state of energy is below the lower threshold, the system charges, and it discharges when the state of energy is above the upper threshold. However, their results are based on the assumption that costs are discounted, and they only manage to get an analytical expression for these thresholds in cases with perfect efficiencies. A similar result is obtained in Harsha and Dahleh (2015), under the assumption of positive prices. Other papers, such as Finnah et al. (2022) mention the extension to an infinite horizon as a future research direction. Currently, an exact method that can solve the infinite-horizon problem while considering the possibility of inefficiencies, negative prices, and undiscounted future costs does not exist.

To ensure that decisions made for scheduling an energy storage system are future-aware, a common approach is to solve a rolling-horizon version of the problem. The problem is solved over a longer horizon, but only the decisions in the first periods are implemented. The rest of the horizon is only advisory. The window is then shifted, to make the next decisions. The horizon corresponding to the decisions that must be implemented is called the *decision horizon*, and the longer horizon over which the problem is solved is the *planning horizon*. The advantages of a rolling horizon are demonstrated in, e.g., Secomandi (2015). If the planning horizon is long enough, this approach may actually obtain an optimal solution of the infinite-horizon problem. In this case, the planning horizon is said to be a *forecast horizon*. However, if the planning horizon is not a forecast horizon, then the best solution obtainable might be much worse than that of the infinite-horizon problem. For most energy storage systems technologies, the investment costs are very high (Harsha and Dahleh, 2015). Therefore, operating them closer to the optimal infinite-horizon schedule is crucial to ensure profitability (Weitzel and Glock, 2018). On the other hand, if the planning horizon is too long, it can result in unnecessarily high forecasting costs, which may diminish their operational profits (Bardhan et al., 2013). It is therefore essential to study the question of forecast horizons for problems with energy storage systems and to be able to determine the *minimum forecast horizon*.

Though the length of the planning horizon in the rolling-horizon approach is crucial, it is rarely discussed, and often simply given as a number, with no justification (Diller et al., 2024; Mercier et al., 2023; Weitzel and Glock, 2018). Identifying an appropriate planning horizon is described as a challenge and a research gap for problems with storage systems by Sioshansi et al. (2021). Cuisinier et al. (2022) propose to aggregate future time intervals to increase the length of the planning horizon at a minimum computational cost. This aggregation is again arbitrary. Some authors do discuss the length of the planning horizon but use an empirical approach, by looking at the variation of their solution when the horizon is varied, see, e.g., del Real et al. (2014); Houwing et al. (2007); Kannan and Zavala (2011); Mayhorn et al. (2017). The conclusions are only applicable to the respective case study considered and do not give guidance as to how to proceed in general. A systematic approach to evaluating planning horizons is thus missing.

An attempt at proposing a theoretical analysis of planning horizons was made by Cruise et al. (2019), who introduced an algorithm to determine planning and decision horizons for a finite-horizon energy storage scheduling problem. The algorithm finds the respective lengths of the decision and planning horizon for which the solution to their full-horizon problem is obtained. This algorithm is applied in Cruise et al. (2018) and further extended in Anjos et al. (2020). However, this approach does not provide the minimum planning

horizon that gives the solution of the full-horizon problem. Moreover, as they assume that the problem is finite, there is no indication regarding how the solution compares to that of the infinite-horizon problem when the end of the finite horizon is reached. In other works, a sufficient condition to identify a forecast horizon has been obtained. If in the resulting schedule, the storage system hits both its minimum and maximum capacity limits (in any order), the horizon is a forecast horizon; see Flatley et al. (2016); Jesudasan and Andrew (2014); Zhao et al. (2019). A similar result has long been known for the wheat trading problem, see for example Hartl (1986). However, this condition is not necessary and therefore, can also not be used to determine the minimum forecast horizon.

The finite-horizon approximation of infinite-horizon dynamic problems is not a new area. The problem has been extensively studied in the literature, particularly within the fields of inventory management and lot sizing, see e.g., Chand et al. (2002). However, in the case of energy storage scheduling, some specific assumptions such as capacities, rate of charge/discharge, losses, and time-varying prices are needed. All of these have not been simultaneously considered previously, and the results of the papers surveyed by Chand et al. (2002), cannot be immediately extended. However, some insights from these applications are useful. Bhaskaran and Sethi (1987) show that, in the case of production planning and warehousing problems, if the solutions over the decision horizons are the same when solving two versions of the problem, for both extreme values of the terminal conditions, a forecast horizon has been found. This condition is extended to a stochastic set-up and included in an algorithm for detecting forecast horizons by Cheevaprawatdomrong and Smith (2004). A similar condition is obtained in Zhao et al. (2012) for hydro-reservoir operation. However, their theoretical result is based on the assumption of diminishing marginal utility, which does *not* apply in the case of energy storage system scheduling. There is also no mention of the minimum forecast horizon. Here, we use a similar idea and apply it to recover the minimum forecast horizon for energy storage scheduling problems.

This paper introduces a simple *necessary and sufficient condition* for a chosen planning horizon to be a forecast horizon for the scheduling problem of an energy storage system, considering losses due to leakage and inefficiencies when charging and discharging, as well as the possibility of negative prices. We show that this condition can also be used to obtain a bound on suboptimality when the planning horizon is not a forecast horizon. We illustrate that the existence of a forecast horizon is not guaranteed, which can limit the performance of methods based on a finite horizon. Though the length of the minimum forecast horizon cannot be evaluated without solving the problem, we obtain an analytical formula that gives a lower bound based on the characteristics of the storage system. Finally, we give an algorithm that uses the identified condition to determine the minimum forecast horizon for the scheduling problem. We apply it for varying the characteristics of the storage system. We also compare the results obtained to those of a rolling horizon with a planning horizon of arbitrary length.

The rest of this paper is organized as follows. In Section 2, we introduce the storage scheduling problem and formally define the different horizons. In Section 3, properties of forecast horizons are presented, including the simple condition to identify them, and a lower bound on the forecast horizon length. An algorithm to retrieve the minimum forecast horizon is also described. In Section 4, the previous results are

applied in various case studies. Section 5 concludes the paper. The complete proofs of the main results, including intermediary results and their proofs, are given in the appendices.

## 2. Problem set-up and definitions

### 2.1. Model and assumptions

We consider the problem of scheduling a price-taker energy storage system, using energy price forecasts, and taking into account the operating constraints. We use a minimum representation of the storage system, including leakage and charging and discharging efficiencies, as well as limits for the state of energy and for charging and discharging. We consider that the storage system can be continuously dispatched between the minimum and maximum capacities.

The model when scheduling the operation of the storage over the time periods  $\mathcal{T} = \{1, 2, \dots, T\}$  is

$$\max_{\mathbf{p}^D, \mathbf{p}^C, \mathbf{s}} \sum_{t \in \mathcal{T}} \Delta t C_t (p_t^D - p_t^C) \quad (1a)$$

$$\text{s.t. } s_1 = \rho S^{\text{init}} + \Delta t \left( \eta^C p_1^C - \frac{1}{\eta^D} p_1^D \right), \quad (1b)$$

$$s_t = \rho s_{t-1} + \Delta t \left( \eta^C p_t^C - \frac{1}{\eta^D} p_t^D \right), \quad \forall t \in \mathcal{T} \setminus \{1\}, \quad (1c)$$

$$\underline{S} \leq s_t \leq \bar{S}, \quad \forall t \in \mathcal{T}, \quad (1d)$$

$$0 \leq p_t^C \leq \bar{P}^C, \quad \forall t \in \mathcal{T}, \quad (1e)$$

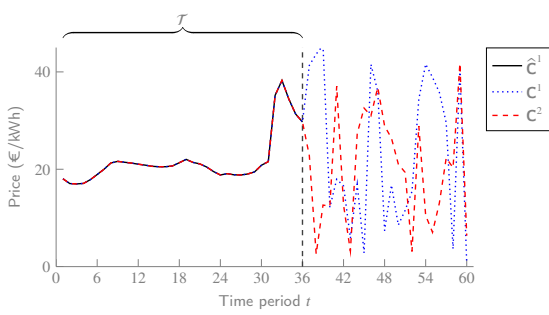
$$0 \leq p_t^D \leq \bar{P}^D, \quad \forall t \in \mathcal{T} \quad (1f)$$

$$p_t^C p_t^D = 0, \quad \forall t \in \mathcal{T}. \quad (1g)$$

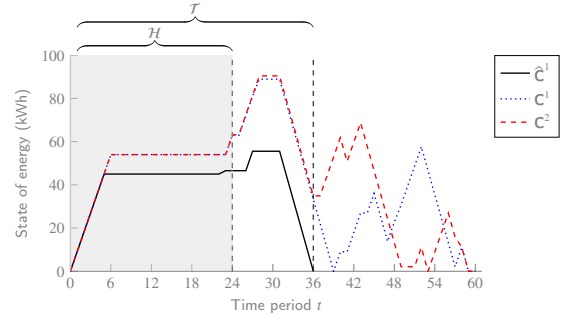
The decision variables are the state of energy of the storage system,  $s_t$ , and the power charged and discharged during each time period of duration  $\Delta t$ ,  $p_t^C$ , and  $p_t^D$ . These variables are bounded in (1d)-(1f). Parameters  $\underline{S}$  and  $\bar{S}$  are the minimum and maximum state of energy,  $\bar{P}^C$  is the maximum rate of charge, and  $\bar{P}^D$  is the maximum rate of discharge. The objective, given in (1a), is to maximize the profit from arbitrage, using the storage system. The energy price paid if charging, or received if discharging, is  $C_t$ . We gather these prices in the vector  $\mathbf{C} \in C_{\mathcal{T}}$ , where  $C_{\mathcal{T}}$  is the set of all possible price vectors over  $\mathcal{T}$ . Constraints (1b) and (1c) update the state of energy for the first time period and for the rest of the time periods, respectively. The level of energy initially available in the storage is given by  $S^{\text{init}}$ . We consider that the storage system has leakage, such that the energy left at the end of each time period is multiplied by a factor  $\rho \in ]0, 1]$  at the beginning of the next time period (Cruise et al., 2019; Jesudasan and Andrew, 2014). We also consider losses when charging and discharging using efficiencies  $\eta^C \in ]0, 1]$  and  $\eta^D \in ]0, 1]$ , and denote the round-trip efficiency  $\eta = \eta^C \eta^D$ . Finally, constraint (1g) prevents the simultaneous charge and discharge of the storage. This can be linearized using binary variables, as explained by Pozo (2022). We name this model  $\mathbf{S}(\mathcal{T}, \mathbf{C})$  and refer to its infinite-horizon variant as  $\mathbf{S}(\mathbb{N}^+, \mathbf{C})$ .

## 2.2. Definition of the different horizons

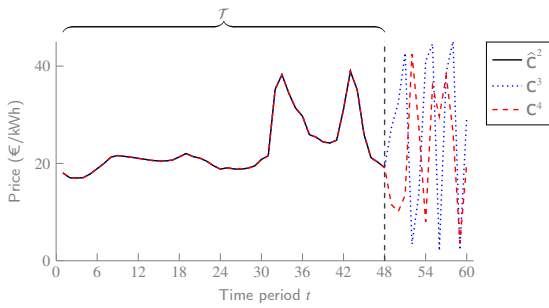
We formally define the different types of horizons and illustrate them with the help of Figure 1. We assume that the decision horizon, which corresponds to the horizon for which we commit the decisions, is fixed by the decision-maker. In the following,  $\mathcal{H}$  represents the set of time periods in the decision horizon, with  $|\mathcal{H}| = H$  being its length. In Figure 1b and Figure 1d,  $H = 24$ , so only the 24 first decisions are committed. The set of time periods in the planning horizon, which is the horizon used to solve the problem, is  $\mathcal{T}$ , and is assumed to have a length  $|\mathcal{T}| = T \geq H$ . In the illustrative examples, we have two different planning horizons, in particular,  $T = 36$  in Figure 1a and Figure 1b, and  $T = 48$  in Figure 1c and Figure 1d.



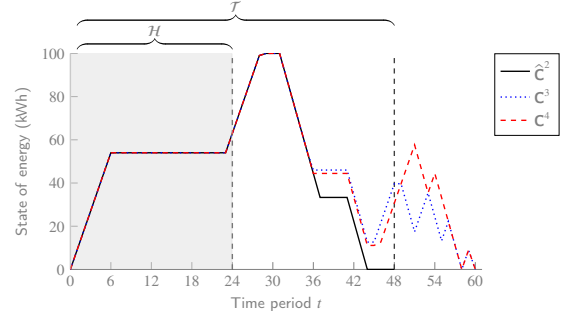
(a) Price forecast and two scenarios for  $T = 36$



(b) Resulting state of energy for  $T = 36$



(c) Price forecast and two scenarios for  $T = 48$



(d) Resulting state of energy for  $T = 48$

**Figure 1:** Illustrative example of the different horizons, for a decision horizon  $H = 24$  and for two different planning horizons

The planning horizon is a forecast horizon if it results in an optimal solution of the infinite-horizon problem (Chand et al., 2002; Ghatge, 2011). We introduce the notation  $\mathcal{X}_{\mathcal{H}}(\mathbf{P})$  to represent the set of optimal solutions of problem  $\mathbf{P}$  over the decision horizon  $\mathcal{H}$ . Finally, a subscript on  $\mathbf{C}$  indicates the restriction of the vector to the given set of time periods, for example,  $\mathbf{C}_{\mathcal{T}}$ .

**Definition 1** (Forecast horizon). *Consider a decision horizon  $\mathcal{H}$  and a planning horizon  $\mathcal{T}$ . The planning horizon  $\mathcal{T}$  is a **forecast horizon** for the price forecast  $\hat{\mathbf{C}}$  if and only if an optimal solution over  $\mathcal{H}$  is also optimal over  $\mathcal{H}$  in the infinite-horizon problem, regardless of the value of the prices after the end of the planning horizon:*

$$\mathcal{X}_{\mathcal{H}}(\mathcal{S}(\mathcal{T}, \hat{\mathbf{C}})) \subseteq \mathcal{X}_{\mathcal{H}}(\mathcal{S}(\mathbb{N}^+, \mathbf{C})), \forall \mathbf{C} \in \mathbb{C}_{\mathbb{N}^+}, \mathbf{C}_{\mathcal{T}} = \hat{\mathbf{C}}.$$

We also use the expression *long enough* to qualify a planning horizon that is a forecast horizon.

In Figure 1, we can see that  $\mathcal{T} = 36$  is not a forecast horizon for the forecast  $\hat{\mathbf{C}}^1$  since when we solve the problem for two different price vectors  $\mathbf{C}^1$  and  $\mathbf{C}^2$ , which are such that  $\mathbf{C}_{\mathcal{T}}^1 = \mathbf{C}_{\mathcal{T}}^2 = \hat{\mathbf{C}}^1$ , as seen in Figure 1a, the resulting state of energy, shown in Figure 1b is not the same over the decision horizon. However,  $\mathcal{T} = 48$  might be a forecast horizon for the price vector  $\hat{\mathbf{C}}^2$ , since for two different price vectors  $\mathbf{C}^3$  and  $\mathbf{C}^4$ , which are such that  $\mathbf{C}_{\mathcal{T}}^3 = \mathbf{C}_{\mathcal{T}}^4 = \hat{\mathbf{C}}^2$ , as seen in Figure 1c, the solutions over  $\mathcal{H}$  coincide, as shown in Figure 1d. However, to claim that  $\mathcal{T} = 48$  is a forecast horizon for  $\hat{\mathbf{C}}^2$ , this property must hold for all  $\mathbf{C} \in \mathcal{C}_{\mathbb{N}^+}$ ,  $\mathbf{C}_{\mathcal{T}} = \hat{\mathbf{C}}^2$ .

Note that if  $\mathcal{T}$  is a forecast horizon, any longer planning horizon is also a forecast horizon. We are interested in finding the shortest of these, which we call the *minimum forecast horizon*.

**Definition 2** (Minimum forecast horizon). *The planning horizon  $\mathcal{T}$  is the **minimum forecast horizon** for the price forecast  $\hat{\mathbf{C}}$  if it is a forecast horizon and if all shorter planning horizons are not forecast horizons.*

Next, we show some properties of (minimum) forecast horizons for the storage scheduling problem.

### 3. Characterization of forecast horizons

#### 3.1. Condition for the identification of forecast horizons

It is possible to evaluate the minimum reachable level at the end of a given planning horizon  $T$ ,  $\underline{S}_T$ , by considering that, from the initial level, the maximum quantity is discharged over the full planning horizon, or until the minimum state of energy is reached, also accounting for losses when discharging and for leakage:

$$\underline{S}_T = \max \left\{ \underline{S}, \rho^T S^{\text{init}} - \Delta t \sum_{t=0}^{T-1} \rho^t \frac{1}{\eta^{\text{D}}} \overline{P}^{\text{D}} \right\}. \quad (2)$$

Similarly, the maximum reachable level at the end of the planning horizon,  $\overline{S}_T$ , can be obtained by considering that, from the initial level, the maximum quantity is charged over the full planning horizon, or until the maximum state of energy is reached, accounting for charging losses and for leakage:

$$\overline{S}_T = \min \left\{ \overline{S}, \rho^T S^{\text{init}} + \Delta t \sum_{t=0}^{T-1} \rho^t \eta^{\text{C}} \overline{P}^{\text{C}} \right\}. \quad (3)$$

We introduce  $\mathbf{F}(\mathcal{T}, \mathbf{C}, S^{\text{end}})$ , which is a finite version of (1), for which we have an additional constraint fixing the final state of energy in the storage:

$$s_T = S^{\text{end}}. \quad (4)$$

We identify  $\underline{\mathcal{X}}$  as the set of optimal solutions for problem  $\mathbf{F}(\mathcal{T}, \mathbf{C}, \underline{S}_T)$ . We use  $\underline{\mathbf{x}}$  to identify an element of  $\underline{\mathcal{X}}$ , and  $\underline{x}_t$  to identify the value of variable  $x$  at  $t$  for solution  $\underline{\mathbf{x}}$ . Similarly,  $\overline{\mathcal{X}}$  is the set of optimal solutions for problem  $\mathbf{F}(\mathcal{T}, \mathbf{C}, \overline{S}_T)$ , with elements  $\overline{\mathbf{x}}$ , and  $\overline{x}_t$  is the value of variable  $x$  at  $t$  for solution  $\overline{\mathbf{x}}$ . More generally,  $\mathcal{X}^*$  is the set of optimal solutions for problem  $\mathbf{F}(\mathcal{T}, \mathbf{C}, S^{\text{end}})$ , with  $\underline{S}_T \leq S^{\text{end}} \leq \overline{S}_T$ , with elements  $\mathbf{x}^*$ , and  $x_t^*$  is the value of variable  $x$  at  $t$  for solution  $\mathbf{x}^*$ .

Our result, formalized in Theorem 1, states that if, when solving  $\mathbf{F}(\mathcal{T}, \mathbf{C}, \underline{s}_T)$  and  $\mathbf{F}(\mathcal{T}, \mathbf{C}, \overline{s}_T)$ , the state of energy at the end of the decision horizon is the same for both problems, then the planning horizon  $\mathcal{T}$  is a forecast horizon. The proof is based on different intermediary results and is available in Appendix A.

**Theorem 1.** *The planning horizon  $\mathcal{T}$  is a forecast horizon if and only if  $\exists \underline{\mathbf{x}} \in \underline{\mathcal{X}}$  and  $\exists \overline{\mathbf{x}} \in \overline{\mathcal{X}}$  such that  $\underline{s}_H = \overline{s}_H$ .*

This result covers the case of multiple optimal solutions: it is enough to find a solution to each problem such that  $\underline{s}_H = \overline{s}_H$ . In Section 3.5, we describe an algorithm for finding the minimum forecast horizon that evaluates the condition in the case of multiple optimal solutions.

If there are no solutions such that  $\underline{s}_H = \overline{s}_H$ , the planning horizon is not a forecast horizon. However, a small gap between these two values indicates that the error made in this case will be limited. We next show how the gap between  $\underline{s}_H$  and  $\overline{s}_H$  is related to suboptimality.

### 3.2. An upper bound on suboptimality

Evaluating suboptimality with respect to the solution to the infinite-horizon problem is far from being a simple task. Indeed, making poor decisions in the current decision horizon could still have an impact long after those decisions were made, and errors in future decision horizons might further amplify the overall suboptimality. Therefore, we need to finitely constrain what we understand by suboptimality to be able to evaluate it. Accordingly, here we assess suboptimality by assuming that an error is only made in the current decision horizon and that all future decisions are optimal (i.e., all future planning horizons are forecast horizons).

We further assume that future prices, outside of the decision horizon, are bounded from below by  $\underline{C} \leq 0$  and from above by  $\overline{C} \geq 0$ . With these assumptions and conditions in place, the following result applies. The proof is given in Appendix B.

**Proposition 1.** *For a given level  $s_H$  of the storage at the end of the decision horizon such that  $\underline{s}_H \leq s_H \leq \overline{s}_H$ , the deviation from the infinite-horizon objective  $\Delta Z$  is upper-bounded as follows:*

$$\Delta Z \leq Z^{\text{opt,DH}} - Z^{\text{DH}} + \max \left\{ -\underline{C} \frac{1}{\eta^{\text{C}}} (s_H - \underline{s}_H), \overline{C} \eta^{\text{D}} (\overline{s}_H - s_H) \right\}, \quad (5)$$

where  $Z^{\text{DH}}$  is the value of the objective function over the decision horizon corresponding to  $s_H$ , and  $Z^{\text{opt,DH}}$  is the objective value achieved when solving the scheduling problem over the decision horizon only and considering that the final storage at the end of the decision horizon is a variable bounded between  $\underline{s}_H$  and  $\overline{s}_H$ .

Therefore, the lower the gap between  $\overline{s}_H$  and  $\underline{s}_H$ , the tighter the upper bound on the suboptimality gap with respect to the infinite-horizon solution.

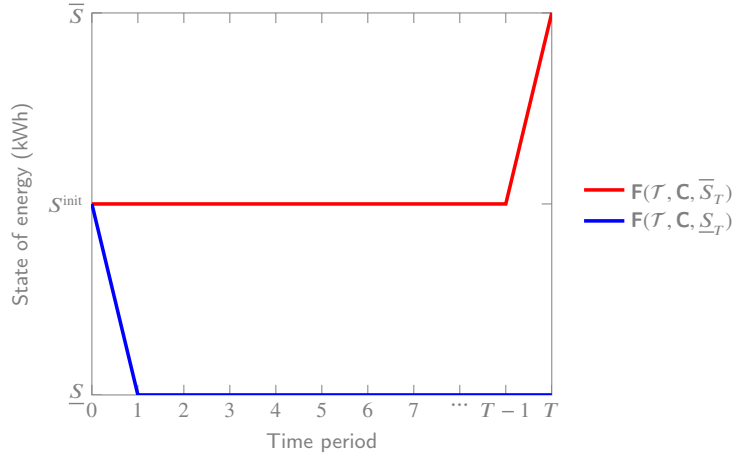
We can also choose  $s_H$  so that this upper bound on suboptimality is minimized.

This bound can most likely be improved by including more precise information on future prices, such as probability distributions, and leakage. However, this is a complete topic in itself, and we only aim to provide a starting point to undertake such a complex study.

### 3.3. Existence of forecast horizons

Prior to determining forecast horizons, it is useful to establish whether they exist or not. In fact, the non-existence of forecast horizons under certain conditions is well-known from the literature on infinite-horizon optimization problems (Chand et al., 2002; Cheevaprawatdomrong et al., 2007; Lortz et al., 2015). By way of an example, we demonstrate that the existence of a forecast horizon is not guaranteed in energy storage scheduling problems.

Consider a storage system with charging and discharging rates such that the storage system can be charged or discharged completely in one time period, with  $\eta < 1$ , and with an initial level such that  $\underline{S} < S^{\text{init}} < \bar{S}$ . The decision horizon is one time period. The price sequence is such that  $C_1 > 0$  and  $\eta C_1 < C_t < C_1, \forall t > 1$ . The resulting states of energy from solving  $\mathbf{F}(\mathcal{T}, \mathbf{C}, \underline{S}_T)$  and  $\mathbf{F}(\mathcal{T}, \mathbf{C}, \bar{S}_T)$  are shown in Figure 2.



**Figure 2:** Example of non-existence of a forecast horizon

To have a forecast horizon, the respective states of energy must be equal in the first time period. For the red curve to match the blue curve in the first time period, it needs to be profitable to discharge in the first period, at price  $C_1$ , and later charge at a lower price  $C_t$ . However, we also have to account for inefficiencies:  $\Delta s = s_t - s_{t-1} = \Delta t \left( \eta^C p_t^C - \frac{1}{\eta^D} p_t^D \right)$  (without leakage). So if  $\Delta s \leq 0$ , the corresponding discharge is  $p_t^D = \frac{-\Delta s \eta^D}{\Delta t}$ . And if  $\Delta s \geq 0$ , the corresponding charge is  $p_t^C = \frac{\Delta s}{\eta^C \Delta t}$ . The change in the objective function  $\Delta Z$  associated with the discharge of  $\Delta s$  in the first period and the charge of  $\Delta s$  at a later  $t$  is  $\Delta Z = C_1 \Delta s \eta^D - C_t \frac{\Delta s}{\eta^C}$ , so it is only profitable to discharge in the first period if there is  $t > 1$  such that  $C_t \leq \eta C_1$ .

On the other hand, for the blue curve to match the red curve in the first time period, it needs to be profitable to discharge less in the first time period and discharge more later, which is the case if there is  $t > 1$  such that  $C_t \geq C_1$ .

Therefore, if  $\eta C_1 < C_t < C_1$  for all  $t > 1$ , the two decisions in the first period will never match, which means that there exists no forecast horizon.



The analysis above suggests that the existence of a forecast horizon depends on the evolution of the price, also accounting for efficiency. The lower the efficiency, the larger the variation of prices needed to make a charging-discharging action profitable.

The fact that the existence of forecast horizons is not guaranteed can limit the applicability of algorithms for discovering forecast horizons. Algorithms based on iteratively increasing the length of the planning horizon may fail to converge if there is no forecast horizon at all. This can be avoided by setting a maximum to the length of the planning horizon, as in Cruise et al. (2019); however, there is no guarantee that a forecast horizon is found by the time the maximum is reached.

### 3.4. Necessary condition for forecast horizons

For the problem considered, it is not possible to calculate the minimum forecast horizon without solving the problem, as it highly depends on the value of the prices. However, we can obtain a necessary condition for forecast horizons prior to solving the problem, as given by Proposition 2. The proof is available in Appendix C.

**Proposition 2.** *For a planning horizon  $\mathcal{T}$ , of length  $T$ , to be a forecast horizon, it must satisfy*

$$\min \left\{ \begin{aligned} & \bar{S} - \underline{S} - \left( \sum_{t=0}^{T-H-1} \rho^t \right) \left( \Delta t \eta^C \bar{P}^C + \frac{\Delta t}{\eta^D} \bar{P}^D \right), \\ & \rho^T S^{\text{init}} - \underline{S} + \Delta t \eta^C \bar{P}^C \left( \sum_{t=T-H}^{T-1} \rho^t \right) - \frac{\Delta t}{\eta^D} \bar{P}^D \left( \sum_{t=0}^{T-H-1} \rho^t \right), \\ & \bar{S} - \rho^T S^{\text{init}} - \Delta t \eta^C \bar{P}^C \left( \sum_{t=0}^{T-H-1} \rho^t \right) + \frac{\Delta t}{\eta^D} \bar{P}^D \left( \sum_{t=T-H}^{T-1} \rho^t \right) \end{aligned} \right\} \leq 0. \quad (6)$$

If a planning horizon  $\mathcal{T}$  does not satisfy (6), we can conclude that it is not a forecast horizon, without having to resort to Theorem 1. Expression (6) is similar to the one proposed in Ela and O'Malley (2015) that is used for determining the length of their planning horizon. In the case of storage system scheduling, (6) is not sufficient to guarantee that the planning horizon is long enough but can provide a starting point for an algorithm built to determine the minimum forecast horizon as presented in Section 4. This starting point, referred to as  $T^{\min}$ , is the first  $T \geq H$  such that (6) is satisfied.

### 3.5. Algorithm to determine the minimum forecast horizon

The results established above can be used to obtain the minimum forecast horizon of (1). We describe an algorithm to do so.

The main idea of Algorithm 1 is to iteratively increase the length of the planning horizon until the condition given in Theorem 1 is satisfied, which is the case when  $gap = 0$ , where  $gap$  is initialized to  $M > 0$ . This approach is similar to those of Garcia and Smith (2000) and Cheevaprawatdomrong and Smith (2004), but adapted to the energy storage scheduling problem. The final  $gap$  is used to provide an upper bound on suboptimality, denoted by *subopt*, following, for example, the approach in Section 3.2. The starting point for the planning horizon, indicated by  $T^{\min}$ , corresponds to the lower bound obtained with Proposition 2.

**Algorithm 1:** Minimum forecast horizon

---

**Data:**  $T^{\max} \geq H$

Calculate  $T^{\min}$  (Proposition 2);

$T \leftarrow T^{\min}$ ;

$gap \leftarrow M$ ;

$subopt \leftarrow M$ ;

**while**  $T \leq T^{\max}$  **and**  $gap > 0$  **do**

$\underline{s}_H \leftarrow \text{Solve } \mathbf{F}(\mathcal{T}, \mathbf{C}, \underline{S}_T)$ ;

$\bar{s}_H \leftarrow \text{Solve } \mathbf{F}(\mathcal{T}, \mathbf{C}, \bar{S}_T)$ ;

$gap \leftarrow \bar{s}_H - \underline{s}_H$ ;

**if**  $gap > 0$  **then**

$gap \leftarrow \text{Solve } min\_gap$ ;

**end**

$T \leftarrow T + 1$ ;

**end**

**if**  $gap > 0$  **then**

    Calculate  $subopt$ ;

**else**

$subopt \leftarrow 0$ ;

**end**

**return**  $T, subopt$

---

The limit  $T^{\max}$  should be chosen such that there is enough data available to cover it and a solution can be obtained in finite time. If  $T^{\max}$  is reached, a strictly positive value of  $subopt$  indicates that, if it exists, the minimum forecast horizon is greater than  $T^{\max}$ . If this happens, and if it is possible to obtain data further into the future, one can repeat the process using a larger value of  $T^{\max}$  and starting from the previous  $T^{\max}$ .

Note that the algorithm also includes a test for solution multiplicity, denoted as  $min\_gap$ . Even if  $\underline{s}_H \neq \bar{s}_H$ , there might be other optimal solutions such that this condition is satisfied. One way to check this is to solve a third problem that includes both scheduling problems, for  $S^{\text{end}} = \underline{S}_T$  and for  $S^{\text{end}} = \bar{S}_T$ . The objective function minimizes the difference between  $\underline{s}_H$  and  $\bar{s}_H$ , and constraints enforce that the profit for each scheduling problem is equal to the optimal value previously obtained when the problems were solved separately. Thus, this problem will identify if there exist optimal solutions such that  $\underline{s}_H = \bar{s}_H$ .

To the best of our knowledge, this is the first algorithm that can determine the minimum forecast horizon in energy storage scheduling problems. Once the minimum forecast horizon is known, it is possible to quantify the implications of using either a too-short or too-long horizon. In the future, more efficient

**Table 1**

Data for the storage systems of the test cases

	$\overline{P}^C$ and $\overline{P}^D$ (kW)	$\underline{S}$ (kWh)	$\overline{S}$ (kWh)	$\eta^C$ and $\eta^D$	Duration of charge (h)	Duration of discharge (h)	$\rho$
Fast storage	1	0	10	0.9	12	9	1.0
Fast storage (low eff.)	1.5 and 0.7	0	10	0.6	12	9	1.0
Slow storage	1	0	50	0.9	56	45	1.0
Slow storage (leakage)	1	0	50	0.9	56	45	0.99

algorithms could be obtained, for example, applying a bisection method or using the solution of the previous iteration as a warm start.

## 4. Application

We apply Algorithm 1 to different storage systems to evaluate the minimum forecast horizon for various storage characteristics and compare the results from solving the problem with a forecast horizon to those obtained with common approaches. The code for the case studies is available at <https://github.com/eLeaprat/stg-horizons>.

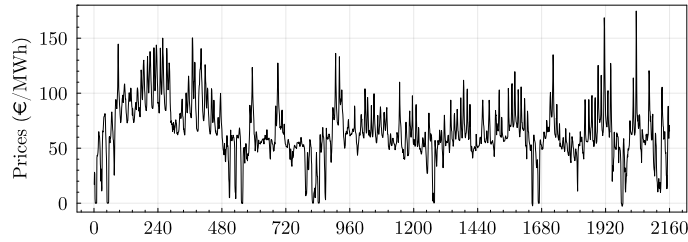
### 4.1. Test case

The data for four storage systems is given in Table 1. The first two have short durations of charge and discharge, which correspond to the minimum time needed to completely fill or empty the storage system, calculated as  $\frac{\overline{S}-S}{\eta^C \overline{P}^C}$  for the duration of charge and  $\eta^D \frac{\overline{S}-S}{\overline{P}^D}$  for the duration of discharge. The durations of charge and discharge are five times as long in the last two cases. The difference between the first two is that the second has lower efficiencies (eff.), while the durations of charge and discharge are kept. This results in distinct values for the maximum rates of charge and discharge. The last storage system allows us to evaluate the impact of leakage.

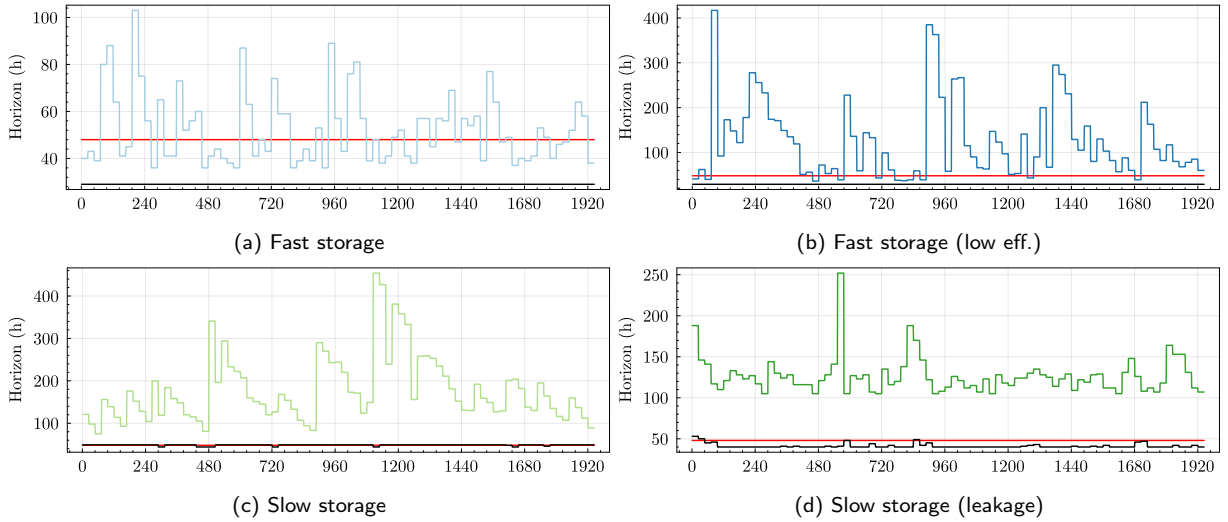
We compare the schedules over three months. The initial level is half capacity, as well as the level at the end of the three months, also used as a final target for the rolling horizon with forecast horizon. This is only done for ease of comparison between the different approaches. In reality, we would keep applying the same procedure for an indefinite amount of time. In all the studies, we set  $T^{\max}$  to be equal to the maximum horizon available in the data, i.e., until the end of the three months. The prices are from the day-ahead market for Denmark (DK1) for the first three months of 2024. They are plotted in Figure 3.

### 4.2. Study of the minimum forecast horizon

We study the minimum forecast horizon for the four case studies, following Algorithm 1. We obtain the plots in Figure 4, which give the length of the minimum horizon in hours for each day of the three months considered and for each test system. These plots also show the lower bound on the minimum forecast horizon obtained with Proposition 2 in black, and for comparison, the red line corresponds to 48 hours.

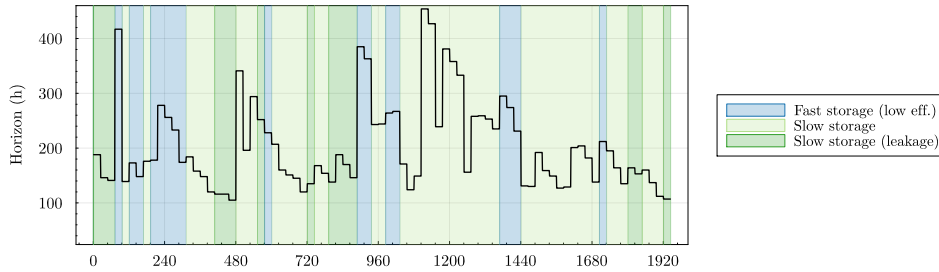


**Figure 3:** Prices used in the case study (day-ahead prices of DK1 between January and March 2024)



**Figure 4:** Minimum forecast horizon for the four storage systems, plotted against time (hours). The red line indicates 48 hours and the black line indicates the lower bound on the minimum forecast horizon  $T^{\min}$ .

The minimum forecast horizon is almost always above 48 hours, except for the fast storage, for which it is often below. Overall, this casts doubt on why 48 hours is usually chosen as the length of the planning horizon. Also note that it varies significantly over time in all cases, which is an argument for not using fixed planning horizons. For all cases, there is a large difference between the lower bound on the minimum forecast horizon  $T^{\min}$  and the actual minimum forecast horizon. As  $T^{\min}$  is calculated based on the technical characteristics of the storage system only, thus disregarding the prices and their evolution, this large difference emphasizes the importance prices have on the length of the minimum forecast horizon. The results of Figure 4a and Figure 4b suggest that, for the same duration of charge and discharge, the efficiency of the storage has a significant impact on the length of the minimum forecast horizon, which is in line with the comments of Section 3.3. Comparing Figure 4a and Figure 4c, we see that the minimum forecast horizon increases for a storage system that takes longer to charge and discharge. For the same charging and discharging duration, the minimum forecast horizon decreases when there is leakage, as we can observe by comparing Figure 4c and Figure 4d.



**Figure 5:** Minimum forecast horizon for the four storage systems, plotted against time (hours).

Note that at the end of the horizon, the length of the planning horizon decreases as we do not look further than the three months here and rather impose the final level at the end of these three months, as discussed above. Therefore, these plots stop before that.

In the case where these four storage systems were to be scheduled in the same problem under the same price, the plot in Figure 5 indicates which of the storage systems would determine the minimum forecast horizon for the complete problem. The plotted curve corresponds to the maximum of the minimum forecast horizons of the four storage systems and the background color shows which specific system sets this maximum. Indeed, a problem with multiple storage systems can be decomposed into one sub-problem per storage system, and the results in Section 3 apply to these subproblems. We observe that the slow storage is often the one determining the minimum forecast horizon, but sometimes it is also the fast storage with low efficiency, or the slow storage with leakage. On the contrary, the fast storage never sets the length of the minimum forecast horizon in this case study and thus, could have been simply disregarded from the analysis.

### 4.3. Comparison with myopic approaches

We now discuss the results from the optimization of the schedules of the storage systems, comparing the use of a rolling horizon with a forecast horizon (also referred to as “Fcst. hor.”) with two other common approaches Weitzel and Glock (2018). One of these approaches, which we call “Fixed level,” is to solve for the next 24 hours only and set the final level to be equal to the initial one. The other is similar but uses an arbitrary rolling horizon of two days, i.e., a planning horizon of 48 hours with a decision horizon of 24 hours (“Hor. 2 days”).

We show the evolution of the state of energy over time in Figure 6. For the storage system with a fast charge and high efficiencies, there is not too much difference between the schedules obtained by the rolling-horizon method with a 2-day planning horizon and with a forecast horizon. The state of energy with a fixed level at the end of the day presents fewer variations and rarely discharges completely, as it always has to return to the same level. For the storage system with a fast charge and low efficiencies, this behavior is more pronounced. In the results for a rolling horizon of two days, the storage system seldom discharges, more than for a fixed final level, but still not as much as with a forecast horizon. For the slow storage systems, the approach with a forecast horizon is the only one for which the storage system reaches its bounds. In the

other two cases, it does not happen as the horizon considered is not long enough for the state of energy to reach the bounds and still end up in the initial level.

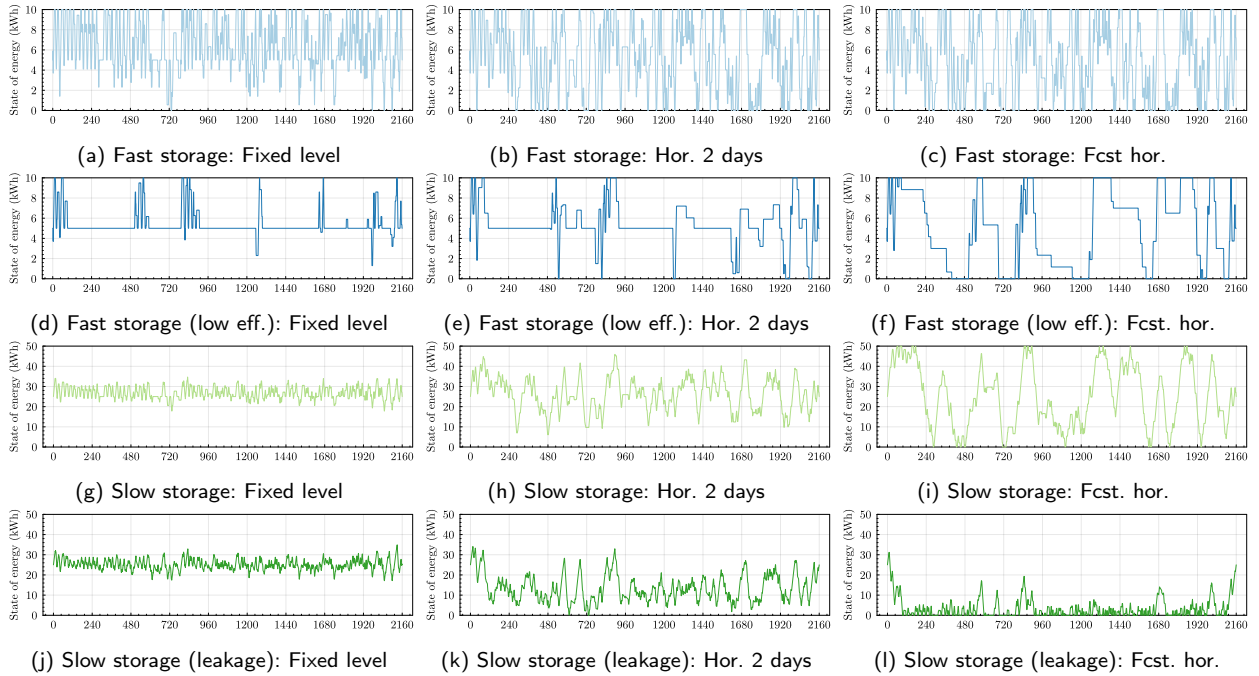


Figure 6: State of energy for the different test cases

Other results are gathered in Table 2. For the profit, the numbers in parentheses represent the profit loss compared to the case with a forecast horizon. The advantage of using a forecast horizon is clear. It is less critical for faster storage systems and in this case, it is the most critical for the slow storage system with leakage. This is due to the fact that the optimal strategy is to operate around the minimum level, while the other approaches push the operation closer to half of the capacity instead. These results illustrate that using a planning horizon that is too short can even lead to a negative profit. Compared to the profit obtained when using a forecast horizon, the loss in profit can be as much as 362% with a fixed level and 136% with a horizon of two days. Finally, it appears essential to properly account for charging and discharging efficiencies, as the very different results between the two first examples suggest.

The storage system use corresponds to the total amount of energy charged and discharged to and from the storage system over the three months. Often, the storage system use is increased for the methods with a too-short horizon, which shows another advantage of a long-enough horizon. The profit is higher while the storage is less utilized, which also limits its degradation.

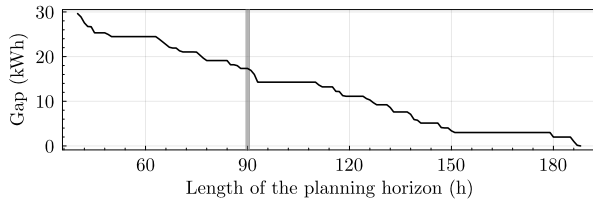
#### 4.4. Evaluation of suboptimality

We use (5) to compute an upper bound on the suboptimality gap for the slow storage with leakage. As  $\underline{C}$  and  $\overline{C}$ , we use the minimum and maximum prices imposed on the Nordpool day-ahead market, which are  $-500\text{€}/\text{MWh}$  and  $4000\text{€}/\text{MWh}$  respectively. We evaluate suboptimality for the first day and analyze how

**Table 2**

Profit and storage system use for the different test cases

	Profit (€)			Storage system use (kWh)		
	Fixed level	Hor. 2 days	Fcst. hor.	Fixed level	Hor. 2 days	Fcst. hor.
Fast storage	12.32 (17%)	14.73 (0.3%)	<b>14.78</b>	1061.46	1041.20	<b>1035.95</b>
Fast storage (low eff.)	2.49 (49%)	3.86 (22%)	<b>4.93</b>	<b>213.75</b>	241.93	241.55
Slow storage	13.26 (37%)	18.24 (14%)	<b>21.11</b>	<b>1185.62</b>	1291.98	1273.01
Slow storage (leakage)	-25.17 (362%)	-3.49 (136%)	<b>9.61</b>	1229.07	1267.86	<b>943.99</b>


 (a) Gap between  $\bar{s}_H$  and  $\underline{s}_H$ 


(b) Upper bound on the suboptimality gap

**Figure 7:** Suboptimality depending on the length of the planning horizon. The gray line highlights the results for a planning horizon of 90 hours.

it evolves as we increase the planning horizon. The storage level at the end of the decision horizon  $s_H$  is chosen such that  $\bar{s}_H \leq s_H \leq \underline{s}_H$ , and such that the profit over the decision horizon is maximized, i.e.,  $Z^{\text{DH}} = Z^{\text{opt,DH}}$ . The plots in Figure 7a and Figure 7b respectively illustrate the evolution of the gap between  $\bar{s}_H$  and  $\underline{s}_H$ , and the evolution of the upper bound on suboptimality, for an increasing planning horizon. Both follow the same trend.

To provide more insight, we further analyze the results obtained for a planning horizon of 96 hours, which are highlighted in Figure 7. In this case, the upper bound on suboptimality is 51.40 €, or 2253%. The actual suboptimality is obtained by continuing to solve the scheduling problem over the following days. For each of these next days, a forecast horizon is used. We stop when the resulting state of energy at the end of a day for the problem with a suboptimal decision on the first day is equal to that of the infinite-horizon problem, which occurs after 3 more days. Comparing the sum of the objective values over each decision horizon, we obtain a suboptimality of 0.003 €, or 0.1%, compared to the infinite horizon. If instead of the Nordpool minimum and maximum prices we use the minimum and maximum prices observed between 2019 and 2023 for that zone, which are -440.1 €/MWh and 871 €/MWh, the upper bound on the suboptimality gap is 11.19 €, or 490%. In both cases, we note that the bound is quite pessimistic. Indeed, not only does it assume that extreme prices are going to arise, but also that they will arise at a moment when the storage system is unable to react to them. This would be the case for example if the storage system were empty at the end of the day and the maximum possible price occurs in the first hour of the next day. If it were not the first hour, then the storage system could first charge and then discharge at the maximum price later. The upper bound should be refined to include these considerations.

## 5. Conclusion

In this paper, we consider the problem of determining the minimum forecast horizon for storage scheduling problems to be used in a rolling-horizon approach. We demonstrate that the existence of a forecast horizon is not guaranteed. We formally introduce an easily verifiable condition to check if a chosen planning horizon is long enough. It involves solving two problems, where the final state of energy is set to its minimum and maximum possible value, respectively. If the state of energy at the end of the decision horizon is the same for these two problems, then the planning horizon is a forecast horizon. We propose an iterative algorithm to find the minimum forecast horizon. This procedure is initialized using a lower bound on the minimum forecast horizon that is theoretically computed using the features of the storage device and is valid for any price profile.

Numerical experiments show that several factors related to the storage system characteristics influence the length of the minimum forecast horizon. These include its charging and discharging efficiencies, the duration of charge and discharge, and the rate of self-discharge or leakage. Another key factor is how prices vary, which is not straightforward to characterize. Comparing the use of a forecast horizon to common arbitrary approaches, we demonstrate the advantage of forecast horizons. For some storage characteristics, using a planning horizon much shorter than the minimum forecast horizon can even lead to a negative profit.

An extension of this work would be to obtain tighter bounds on suboptimality when the horizon is too short. Furthermore, in this paper, we place ourselves in a context of perfect (and free) information. In reality, the quality of forecasts decreases as we look further into the future. Obtaining longer forecasts also comes with higher costs. It would be essential to integrate these parameters into the choice of an appropriate horizon.

### A. Proof of Theorem 1

In all our proofs, we use the following expression of the state of energy at any time period  $t_2$  with the state of energy at any previous time period  $t_1$ , obtained by merging constraints (1c) between  $t_1$  and  $t_2$ :

$$s_{t_2} = \rho^{t_2-t_1} s_{t_1} + \Delta t \sum_{t=t_1+1}^{t_2} \rho^{t_2-t} \left( \eta^C p_t^C - \frac{1}{\eta^D} p_t^D \right), \quad \forall (t_1, t_2) \in \mathcal{T}, t_1 \leq t_2, s_0 = S^{\text{init}}. \quad (7)$$

A first useful result is that the state of energy obtained when setting the final level to the maximum reachable level is an upper bound for an optimal solution to the problem with any final level. Similarly, the state of energy obtained when setting the final level to the minimum reachable level is a lower bound for an optimal solution to the problem with any final level.

**Lemma 1.** *The following is true*

1. Given  $\bar{\mathbf{x}} \in \bar{\mathcal{X}}$  and  $S^{\text{end}}$ , with  $\underline{S}_T \leq S^{\text{end}} \leq \bar{S}_T$ , there exists  $\mathbf{x}^* \in \mathcal{X}^*$  such that  $s_t^* \leq \bar{s}_t, \forall t \in \mathcal{T}$ .
2. Given  $\underline{\mathbf{x}} \in \underline{\mathcal{X}}$  and  $S^{\text{end}}$ , with  $\underline{S}_T \leq S^{\text{end}} \leq \bar{S}_T$ , there exists  $\mathbf{x}^* \in \mathcal{X}^*$  such that  $s_t^* \geq \underline{s}_t, \forall t \in \mathcal{T}$ .

*Proof.* We consider any solution  $\bar{\mathbf{x}} \in \bar{\mathcal{X}}$ , and any final level  $S^{\text{end}}$ , with  $\underline{S}_T \leq S^{\text{end}} \leq \bar{S}_T$ . We suppose that for all solutions of  $\mathbf{F}(\mathcal{T}, \mathbf{C}, S^{\text{end}})$ , there exists  $t \in \mathcal{T}$  such that  $s_t^* > \bar{s}_t$  and show a contradiction.



Consider any of these solutions,  $\mathbf{x}^* \in \mathcal{X}^*$ . We identify  $t_1 \in \mathcal{T}$ , such that  $s_t^* \leq \bar{s}_t, \forall t < t_1$  and  $s_{t_1}^* > \bar{s}_{t_1}$ . Since  $s_T^* = S^{\text{end}}, \bar{s}_T = \bar{S}_T$  and  $S^{\text{end}} \leq \bar{S}_T, \exists t_2 \in \mathcal{T}, t_2 > t_1$  such that  $s_t^* > \bar{s}_t, \forall t \in [t_1, t_2 - 1]$  and  $s_{t_2}^* \leq \bar{s}_{t_2}$ .

We build solutions  $\mathbf{x}'$  and  $\bar{\mathbf{x}}'$ , based on  $\mathbf{x}^*$  and  $\bar{\mathbf{x}}$  respectively, and derive some properties of the problem.

We first build  $\mathbf{x}'$  based on  $\mathbf{x}^*$ . We modify the solution at  $t_1$ , decreasing charge if  $p_{t_1}^{C*} > 0$  or increasing discharge if  $p_{t_1}^{C*} = 0$ , which implies  $p_{t_1}^{D*} \geq 0$ . Therefore, the complementarity constraint at  $t_1$  still holds.

If  $p_{t_1}^{C*} > 0$ : We decrease the charge at  $t_1$  by a small quantity  $\epsilon > 0$ , with  $\epsilon \leq p_{t_1}^{C*}$ , such that  $p_{t_1}^{C'} = p_{t_1}^{C*} - \epsilon$  and the bounds on charge are respected. We apply (7) between  $t_1$  and  $t \in [t_1, t_2 - 1]$  to obtain the modified state of energy at  $t$ :

$$s'_t = \rho^{t-t_1} s'_{t_1} + \Delta t \sum_{k=t_1+1}^t \rho^{t-k} \left( \eta^C p_k^{C'} - \frac{1}{\eta^D} p_k^{D'} \right) = s_t^* - \rho^{t-t_1} \Delta t \eta^C \epsilon. \quad (8)$$

We choose  $\epsilon$  such that the lower bound on the state of energy is respected, i.e.  $\epsilon \leq \frac{s_t^* - \underline{S}}{\rho^{t-t_1} \Delta t \eta^C}, \forall t \in [t_1, t_2 - 1]$ , which is possible since  $\underline{S} \leq \bar{s}_t < s_t^*, \forall t \in [t_1, t_2 - 1]$ . Since  $s_t^* \leq \bar{S}$ , the upper bound is still respected.

We now modify the solution at  $t_2$  to return on the same trajectory, decreasing discharge if  $p_{t_2}^{D*} > 0$  or increasing charge if  $p_{t_2}^{D*} = 0$  and  $p_{t_2}^{C*} \geq 0$ . Therefore, the complementarity constraint at  $t_2$  still holds.

- If  $p_{t_2}^{D*} > 0$ : We decrease discharge at  $t_2$  by  $\rho^{t_2-t_1} \eta^C \eta^D \epsilon$ , with  $\epsilon \leq \frac{p_{t_2}^{D*}}{\rho^{t_2-t_1} \eta^C \eta^D}$ , such that  $p_{t_2}^{D'} = p_{t_2}^{D*} - \rho^{t_2-t_1} \eta^C \eta^D \epsilon$  and the bounds on discharge are respected. At  $t_2$ , the modified state of energy is

$$s'_{t_2} = \rho s_{t_2-1}^* - \rho^{t_2-t_1} \Delta t \eta^C \epsilon + \Delta t \eta^C p_{t_2}^{C*} - \Delta t \frac{1}{\eta^D} p_{t_2}^{D*} + \Delta t \frac{1}{\eta^D} \rho^{t_2-t_1} \eta^C \eta^D \epsilon = s_{t_2}^*. \quad (9)$$

We are back on the same trajectory, which completes the demonstration that the new solution is feasible. The change in the objective function compared to  $\mathbf{x}^*$  is  $\Delta t C_{t_1} \epsilon - \Delta t \rho^{t_2-t_1} \eta^C \eta^D C_{t_2} \epsilon$ . Since  $\mathbf{x}'$  cannot be better than  $\mathbf{x}^*$ , we derive

$$C_{t_1} \leq \rho^{t_2-t_1} \eta^C \eta^D C_{t_2}. \quad (10)$$

- If  $p_{t_2}^{D*} = 0$  and  $p_{t_2}^{C*} \geq 0$ : We increase charge at  $t_2$  by  $\rho^{t_2-t_1} \epsilon$ , with  $\epsilon \leq \frac{\bar{P}^C - p_{t_2}^{C*}}{\rho^{t_2-t_1}}$ , such that  $p_{t_2}^{C'} = p_{t_2}^{C*} + \rho^{t_2-t_1} \epsilon$  and the bounds on charge are respected. Note that  $\epsilon > 0$  is guaranteed by  $p_{t_2}^{C*} < \bar{P}^C$ . Indeed, since  $\bar{s}_{t_2} < s_{t_2}^*, \bar{s}_{t_2} \geq s_{t_2}^*$  would not be possible otherwise. At  $t_2$ , the modified state of energy is

$$s'_{t_2} = \rho s_{t_2-1}^* - \rho^{t_2-t_1} \Delta t \eta^C \epsilon + \Delta t \eta^C p_{t_2}^{C*} + \Delta t \eta^C \rho^{t_2-t_1} \epsilon - \Delta t \frac{1}{\eta^D} p_{t_2}^{D*} = s_{t_2}^*. \quad (11)$$

We are back on the same trajectory, which completes the demonstration that the new solution is feasible. The change in the objective function compared to  $\mathbf{x}^*$  is  $\Delta t C_{t_1} \epsilon - \Delta t \rho^{t_2-t_1} C_{t_2} \epsilon$ . Since  $\mathbf{x}'$  cannot be better than  $\mathbf{x}^*$ , we derive

$$C_{t_1} \leq \rho^{t_2-t_1} C_{t_2}. \quad (12)$$

If  $p_{t_1}^{C*} = 0$  and  $p_{t_1}^{D*} \geq 0$ : We increase the discharge at  $t_1$  by a small quantity  $\epsilon > 0$ , with  $\epsilon \leq \bar{P}^D - p_{t_1}^{D*}$ , such that  $p_{t_1}^{D'} = p_{t_1}^{D*} + \epsilon$  and the bounds on discharge are respected. Note that  $\epsilon > 0$  is guaranteed by  $p_{t_1}^{D*} < \bar{P}^D$ . Indeed, since  $s_{t_1}^* < \bar{s}_{t_1}$ ,  $s_{t_1}^* \geq \bar{s}_{t_1}^*$  would not be possible otherwise. We apply (7) between  $t_1$  and  $t \in [t_1, t_2 - 1]$  to obtain the modified state of energy at  $t$ :

$$s'_t = \rho^{t-t_1} s'_{t_1} + \Delta t \sum_{k=t_1+1}^t \rho^{t-k} \left( \eta^C p_k^{C'} - \frac{1}{\eta^D} p_k^{D'} \right) = s_t^* - \rho^{t-t_1} \Delta t \frac{1}{\eta^D} \epsilon. \quad (13)$$

We choose  $\epsilon$  such that the lower bound on the state of energy is respected, i.e.  $\epsilon \leq \frac{\eta^D (s_t^* - \underline{S})}{\rho^{t-t_1} \Delta t}$ ,  $\forall t \in [t_1, t_2 - 1]$ , which is possible since  $\underline{S} \leq \bar{s}_t < s_t^*$ ,  $\forall t \in [t_1, t_2 - 1]$ . Since  $s_t^* \leq \bar{S}$ , the upper bound is still respected.

We then modify the solution at  $t_2$  in a similar way as before.

- If  $p_{t_2}^{D*} > 0$ : We decrease discharge at  $t_2$  by  $\rho^{t_2-t_1} \epsilon$ , with  $\epsilon \leq \frac{p_{t_2}^{D*}}{\rho^{t_2-t_1}}$ , such that  $p_{t_2}^{D'} = p_{t_2}^{D*} - \rho^{t_2-t_1} \epsilon$  and the bounds on discharge are respected. At  $t_2$ , the modified state of energy is

$$s'_{t_2} = \rho s_{t_2-1}^* - \rho^{t_2-t_1} \Delta t \frac{1}{\eta^D} \epsilon + \Delta t \eta^C p_{t_2}^{C*} - \Delta t \frac{1}{\eta^D} p_{t_2}^{D*} + \Delta t \frac{1}{\eta^D} \rho^{t_2-t_1} \epsilon = s_{t_2}^*. \quad (14)$$

We are back on the same trajectory, which completes the demonstration that the new solution is feasible. The change in the objective function compared to  $\mathbf{x}^*$  is  $\Delta t C_{t_1} \epsilon - \Delta t \rho^{t_2-t_1} C_{t_2} \epsilon$ . Since  $\mathbf{x}'$  cannot be better than  $\mathbf{x}^*$ , we derive

$$C_{t_1} \leq \rho^{t_2-t_1} C_{t_2}. \quad (15)$$

- If  $p_{t_2}^{D*} = 0$  and  $p_{t_2}^{C*} \geq 0$ : We increase charge at  $t_2$  by  $\frac{\rho^{t_2-t_1} \epsilon}{\eta^C \eta^D}$ , with  $\epsilon \leq \frac{\eta^C \eta^D (\bar{P}^C - p_{t_2}^{C*})}{\rho^{t_2-t_1}}$ , such that  $p_{t_2}^{C'} = p_{t_2}^{C*} + \frac{\rho^{t_2-t_1} \epsilon}{\eta^C \eta^D}$ . The bounds on discharge are respected. At  $t_2$ , the modified state of energy is

$$s'_{t_2} = \rho s_{t_2-1}^* - \rho^{t_2-t_1} \Delta t \frac{1}{\eta^D} \epsilon + \Delta t \eta^C p_{t_2}^{C*} + \Delta t \eta^C \frac{\rho^{t_2-t_1} \epsilon}{\eta^C \eta^D} - \Delta t \frac{1}{\eta^D} p_{t_2}^{D*} = s_{t_2}^*. \quad (16)$$

We are back on the same trajectory, which completes the demonstration that the new solution is feasible. The change in the objective function compared to  $\mathbf{x}^*$  is  $\Delta t C_{t_1} \epsilon - \Delta t \frac{\rho^{t_2-t_1}}{\eta^C \eta^D} C_{t_2} \epsilon$ . Since  $\mathbf{x}'$  cannot be better than  $\mathbf{x}^*$ , we derive

$$\eta^C \eta^D C_{t_1} \leq \rho^{t_2-t_1} C_{t_2}. \quad (17)$$

Next, we build  $\bar{\mathbf{x}}'$  based on  $\bar{\mathbf{x}}$ . We modify the solution at  $t_1$ , decreasing discharge if  $\bar{p}_{t_1}^D > 0$  or increasing charge if  $\bar{p}_{t_1}^D = 0$ , which implies  $\bar{p}_{t_1}^C \geq 0$ . Therefore, the complementarity constraint at  $t_1$  still holds.

If  $\bar{p}_{t_1}^D > 0$ : We decrease the discharge at  $t_1$  by a small quantity  $\bar{\epsilon} > 0$ , with  $\bar{\epsilon} \leq \bar{p}_{t_1}^D$ , such that  $\bar{p}_{t_1}^{D'} = \bar{p}_{t_1}^D - \bar{\epsilon}$  and the bounds on discharge are respected. We apply (7) between  $t_1$  and  $t \in [t_1, t_2 - 1]$  to obtain the modified state of energy at  $t$ :

$$\bar{s}'_t = \rho^{t-t_1} \bar{s}'_{t_1} + \Delta t \sum_{k=t_1+1}^t \rho^{t-k} \left( \eta^C \bar{p}_k^{C'} - \frac{1}{\eta^D} \bar{p}_k^{D'} \right) = \bar{s}_t + \rho^{t-t_1} \Delta t \frac{1}{\eta^D} \bar{\epsilon}. \quad (18)$$

We choose  $\bar{\epsilon}$  such that the upper bound on the state of energy is respected, i.e.  $\bar{\epsilon} \leq \frac{\eta^D(\bar{S}-\bar{s}_t)}{\rho^{t-1}\Delta t}$ ,  $\forall t \in [t_1, t_2 - 1]$ , which we know is possible since  $\bar{s}_t < s_t^* \leq \bar{S}$ ,  $\forall t \in [t_1, t_2 - 1]$ . Since  $\bar{s}_t \geq \underline{S}$ , the lower bound is still respected.

We then modify the solution at  $t_2$  to return on the same trajectory, decreasing charge if  $\bar{p}_{t_2}^C > 0$  or increasing discharge if  $\bar{p}_{t_2}^C = 0$  and  $\bar{p}_{t_2}^D \geq 0$ . Therefore, the complementarity constraint at  $t_2$  still holds.

- If  $\bar{p}_{t_2}^C > 0$ : We decrease charge at  $t_2$  by  $\frac{\rho^{t_2-t_1}\bar{\epsilon}}{\eta^C\eta^D}$ , with  $\bar{\epsilon} \leq \frac{\eta^C\eta^D\bar{p}_{t_2}^C}{\rho^{t_2-t_1}}$ , such that  $\bar{p}_{t_2}^{C'} = \bar{p}_{t_2}^C - \frac{\rho^{t_2-t_1}\bar{\epsilon}}{\eta^C\eta^D}$  and the bounds on charge are respected. At  $t_2$ , the modified state of energy is

$$\bar{s}'_{t_2} = \rho\bar{s}_{t_2-1} + \rho^{t_2-t_1}\Delta t\frac{1}{\eta^D}\bar{\epsilon} + \Delta t\eta^C\bar{p}_{t_2}^{C'} - \Delta t\eta^C\frac{\rho^{t_2-t_1}\bar{\epsilon}}{\eta^C\eta^D} - \Delta t\frac{1}{\eta^D}\bar{p}_{t_2}^D = \bar{s}_{t_2}. \quad (19)$$

We are back on the same trajectory, which completes the demonstration that the new solution is feasible. The change in the objective function compared to  $\mathbf{x}^*$  is  $-\Delta t C_{t_1}\bar{\epsilon} + \Delta t\frac{\rho^{t_2-t_1}}{\eta^C\eta^D}C_{t_2}\bar{\epsilon}$ . Since  $\mathbf{x}'$  cannot be better than  $\mathbf{x}^*$ , we derive

$$\eta^C\eta^D C_{t_1} \geq \rho^{t_2-t_1} C_{t_2}. \quad (20)$$

- If  $\bar{p}_{t_2}^C = 0$  and  $\bar{p}_{t_2}^D \geq 0$ : We increase discharge at  $t_2$  by  $\rho^{t_2-t_1}\bar{\epsilon}$ , with  $\bar{\epsilon} \leq \frac{\bar{P}^D - \bar{p}_{t_2}^D}{\rho^{t_2-t_1}}$ , such that  $\bar{p}_{t_2}^{D'} = \bar{p}_{t_2}^D + \rho^{t_2-t_1}\bar{\epsilon}$ . The bounds on discharge are respected. Note that  $\bar{\epsilon} > 0$  is guaranteed by  $\bar{p}_{t_2}^D < \bar{P}^D$ . Indeed, since  $\bar{s}_{t_2} < s_{t_2}^*$ ,  $\bar{s}_{t_2} \geq s_{t_2}^*$  would not be possible otherwise. At  $t_2$ , the state of energy is

$$\bar{s}'_{t_2} = \rho\bar{s}_{t_2-1} + \rho^{t_2-t_1}\Delta t\frac{1}{\eta^D}\bar{\epsilon} + \Delta t\eta^C\bar{p}_{t_2}^{C'} - \Delta t\frac{1}{\eta^D}\bar{p}_{t_2}^{D'} - \Delta t\frac{1}{\eta^D}\rho^{t_2-t_1}\bar{\epsilon} = \bar{s}_{t_2}. \quad (21)$$

We are back on the same trajectory, which completes the demonstration that the new solution is feasible. The change in the objective function compared to  $\mathbf{x}^*$  is  $-\Delta t C_{t_1}\bar{\epsilon} + \Delta t\rho^{t_2-t_1}C_{t_2}\bar{\epsilon}$ . Since  $\mathbf{x}'$  cannot be better than  $\mathbf{x}^*$ , we derive

$$C_{t_1} \geq \rho^{t_2-t_1} C_{t_2}. \quad (22)$$

If  $\bar{p}_{t_1}^D = 0$  and  $\bar{p}_{t_1}^C \geq 0$ : We increase the charge at  $t_1$  by a small quantity  $\bar{\epsilon} > 0$ , with  $\bar{\epsilon} \leq \bar{P}^C - \bar{p}_{t_1}^C$ , such that  $\bar{p}_{t_1}^{C'} = \bar{p}_{t_1}^C + \bar{\epsilon}$  and the bounds on charge are respected. Note that  $\bar{\epsilon} > 0$  is guaranteed by  $\bar{p}_{t_1}^C < \bar{P}^C$ . Indeed, since  $s_{t_1}^* < \bar{s}_{t_1}$ ,  $s_{t_1}^* \geq \bar{s}_{t_1}^*$  would not be possible otherwise.

We apply (7) between  $t_1$  and  $t \in [t_1, t_2 - 1]$  to obtain the modified state of energy at  $t$ :

$$\bar{s}'_t = \rho^{t-t_1}\bar{s}'_{t_1} + \Delta t \sum_{k=t_1+1}^t \rho^{t-k} \left( \eta^C\bar{p}_k^{C'} - \frac{1}{\eta^D}\bar{p}_k^{D'} \right) = \bar{s}_t + \rho^{t-t_1}\Delta t\eta^C\bar{\epsilon}. \quad (23)$$

We choose  $\bar{\epsilon}$  such that the upper bound on the state of energy is respected, i.e.  $\bar{\epsilon} \leq \frac{\bar{S}-\bar{s}_t}{\rho^{t-1}\Delta t\eta^C}$ ,  $\forall t \in [t_1, t_2 - 1]$ , which is possible since  $\bar{s}_t < s_t^* \leq \bar{S}$ ,  $\forall t \in [t_1, t_2 - 1]$ . Since  $\bar{s}_t \geq \underline{S}$ , the lower bound is still respected.

We then modify the solution at  $t_2$  in a similar way as before.

- If  $\bar{p}_{t_2}^C > 0$ : We decrease charge at  $t_2$  by  $\rho^{t_2-t_1}\bar{e}$ , with  $\bar{e} \leq \frac{\bar{p}_{t_2}^C}{\rho^{t_2-t_1}}$ , such that  $\bar{p}_{t_2}^{C'} = \bar{p}_{t_2}^C - \rho^{t_2-t_1}\bar{e}$  and the bounds on charge are respected. At  $t_2$ , the state of energy is

$$\bar{s}'_{t_2} = \rho\bar{s}_{t_2-1} + \rho^{t_2-t_1}\Delta t \eta^C \bar{e} + \Delta t \eta^C \bar{p}_{t_2}^{C'} - \Delta t \eta^C \rho^{t_2-t_1}\bar{e} - \Delta t \frac{1}{\eta^D} \bar{p}_{t_2}^D = \bar{s}_{t_2}. \quad (24)$$

We are back on the same trajectory, which completes the demonstration that the new solution is feasible. The change in the objective function compared to  $\mathbf{x}^*$  is  $-\Delta t C_{t_1} \bar{e} + \Delta t \rho^{t_2-t_1} C_{t_2} \bar{e}$ . Since  $\mathbf{x}'$  cannot be better than  $\mathbf{x}^*$ , we derive

$$C_{t_1} \geq \rho^{t_2-t_1} C_{t_2}. \quad (25)$$

- If  $\bar{p}_{t_2}^C = 0$  and  $\bar{p}_{t_2}^D \geq 0$ : We increase discharge at  $t_2$  by  $\rho^{t_2-t_1}\eta^C \eta^D \bar{e}$ , with  $\bar{e} \leq \frac{\bar{p}_{t_2}^D}{\rho^{t_2-t_1}\eta^C \eta^D}$ , such that  $\bar{p}_{t_2}^{D'} = \bar{p}_{t_2}^D + \rho^{t_2-t_1}\eta^C \eta^D \bar{e}$ . The bounds on discharge are respected. At  $t_2$ , the modified state of energy is

$$\bar{s}'_{t_2} = \rho\bar{s}_{t_2-1} + \rho^{t_2-t_1}\Delta t \eta^C \bar{e} + \Delta t \eta^C \bar{p}_{t_2}^{D'} - \Delta t \frac{1}{\eta^D} \bar{p}_{t_2}^D - \Delta t \frac{1}{\eta^D} \rho^{t_2-t_1}\eta^C \eta^D \bar{e} = \bar{s}_{t_2}. \quad (26)$$

We are back on the same trajectory, which completes the demonstration that the new solution is feasible. The change in the objective function compared to  $\mathbf{x}^*$  is  $-\Delta t C_{t_1} \bar{e} + \Delta t \rho^{t_2-t_1} C_{t_2} \eta^C \eta^D \bar{e}$ . Since  $\mathbf{x}'$  cannot be better than  $\mathbf{x}^*$ , we derive

$$C_{t_1} \geq \rho^{t_2-t_1} \eta^C \eta^D C_{t_2}. \quad (27)$$

To ensure that  $s_{t_1-1}^* \leq \bar{s}_{t_1-1}$ ,  $s_{t_1}^* > \bar{s}_{t_1}$ ,  $s_{t_2-1}^* > \bar{s}_{t_2-1}$ , and  $s_{t_2}^* \leq \bar{s}_{t_2}$ , the possible combinations of charge and discharge at  $t_1$  are limited to the following:

- At  $t_1$ ,  $p_{t_1}^{C*} > 0$  and  $\bar{p}_{t_1}^D = 0$ . At  $t_2$ ,  $p_{t_2}^{D*} = 0$  and  $\bar{p}_{t_2}^C > 0$ . Then, with (12) and (25), we get

$$C_{t_1} = \rho^{t_2-t_1} C_{t_2}. \quad (28)$$

- At  $t_1$ ,  $p_{t_1}^{C*} > 0$  and  $\bar{p}_{t_1}^D = 0$ . At  $t_2$ ,  $p_{t_2}^{D*} > 0$  and  $\bar{p}_{t_2}^C > 0$ . Then, with (10) and (25), we get

$$C_{t_1} = \rho^{t_2-t_1} C_{t_2}, \quad (29)$$

and  $\eta^C \eta^D = 1$  and/or  $C_{t_1} = C_{t_2} = 0$ .

- At  $t_1$ ,  $p_{t_1}^{C*} > 0$  and  $\bar{p}_{t_1}^D = 0$ . At  $t_2$ ,  $p_{t_2}^{D*} > 0$  and  $\bar{p}_{t_2}^C = 0$ . Then, with (10) and (27), we get

$$C_{t_1} = \rho^{t_2-t_1} \eta^C \eta^D C_{t_2}. \quad (30)$$

- At  $t_1$ ,  $p_{t_1}^{C*} > 0$  and  $\bar{p}_{t_1}^D > 0$ . At  $t_2$ ,  $p_{t_2}^{D*} = 0$  and  $\bar{p}_{t_2}^C > 0$ . Then, with (12) and (20), we get

$$C_{t_1} = \rho^{t_2-t_1} C_{t_2}, \quad (31)$$

and  $\eta^C \eta^D = 1$  and/or  $C_{t_1} = C_{t_2} = 0$ .

- At  $t_1$ ,  $p_{t_1}^{C*} > 0$  and  $\bar{p}_{t_1}^D > 0$ . At  $t_2$ ,  $p_{t_2}^{D*} > 0$  and  $\bar{p}_{t_2}^C > 0$ . Then, with (10) and (20), we get

$$C_{t_1} = \rho^{t_2-t_1} C_{t_2}, \quad (32)$$

and  $\eta^C \eta^D = 1$  and/or  $C_{t_1} = C_{t_2} = 0$ .

- At  $t_1$ ,  $p_{t_1}^{C*} > 0$  and  $\bar{p}_{t_1}^D > 0$ . At  $t_2$ ,  $p_{t_2}^{D*} > 0$  and  $\bar{p}_{t_2}^C = 0$ . Then, with (10) and (22), we get

$$C_{t_1} = \rho^{t_2-t_1} C_{t_2}, \quad (33)$$

and  $\eta^C \eta^D = 1$  and/or  $C_{t_1} = C_{t_2} = 0$ .

- At  $t_1$ ,  $p_{t_1}^{C*} = 0$  and  $\bar{p}_{t_1}^D > 0$ . At  $t_2$ ,  $p_{t_2}^{D*} = 0$  and  $\bar{p}_{t_2}^C > 0$ . Then, with (17) and (20), we get

$$\eta^C \eta^D C_{t_1} = \rho^{t_2-t_1} C_{t_2}. \quad (34)$$

- At  $t_1$ ,  $p_{t_1}^{C*} = 0$  and  $\bar{p}_{t_1}^D > 0$ . At  $t_2$ ,  $p_{t_2}^{D*} > 0$  and  $\bar{p}_{t_2}^C > 0$ . Then, with (15) and (20), we get

$$C_{t_1} = C_{t_2} = \rho^{t_2-t_1} C_{t_2}, \quad (35)$$

and  $\eta^C \eta^D = 1$  and/or  $C_{t_1} = C_{t_2} = 0$ .

- At  $t_1$ ,  $p_{t_1}^{C*} = 0$  and  $\bar{p}_{t_1}^D > 0$ . At  $t_2$ ,  $p_{t_2}^{D*} > 0$  and  $\bar{p}_{t_2}^C = 0$ . Then, with (15) and (22), we get

$$C_{t_1} = \rho^{t_2-t_1} C_{t_2}. \quad (36)$$

Focusing on  $\mathbf{x}^*$ , we have the following results:

1. If  $p_{t_1}^{C*} > 0$  and  $p_{t_2}^{D*} = 0$ ,

$$C_{t_1} = \rho^{t_2-t_1} C_{t_2}. \quad (37)$$

2. If  $p_{t_1}^{C*} > 0$  and  $p_{t_2}^{D*} > 0$ ,

$$C_{t_1} = \rho^{t_2-t_1} \eta^C \eta^D C_{t_2}, \quad (38)$$

since in (29), (32) and (33), either  $\eta^C \eta^D = 1$  and/or  $C_{t_1} = C_{t_2} = 0$  so this result stands in any case.

3. If  $p_{t_1}^{C*} = 0$  and  $p_{t_2}^{D*} = 0$ ,

$$\eta^C \eta^D C_{t_1} = \rho^{t_2-t_1} C_{t_2}. \quad (39)$$

4. If  $p_{t_1}^{C*} = 0$  and  $p_{t_2}^{D*} > 0$ ,

$$C_{t_1} = \rho^{t_2-t_1} C_{t_2}. \quad (40)$$

Next, we iteratively build a new solution  $\mathbf{x}'$  based on  $\mathbf{x}^*$ , and satisfying point 1 of the lemma, as follows:

**1<sup>st</sup> step:** Depending on the case, we modify the solution  $\mathbf{x}^*$  at  $t_1$  and  $t_2$  to obtain  $\mathbf{x}'$ . We explain below how.

**2<sup>nd</sup> step:** For  $\mathbf{x}'$ , we identify  $t_1$  and  $t_2$ , corresponding to the first interval for which the state of energy is strictly greater. If there is no  $t_1$ , then  $s'_t \leq \bar{s}_t, \forall t \in \mathcal{T}$  and the procedure terminates.

**3<sup>rd</sup> step:** We update  $\mathbf{x}^*$  to  $\mathbf{x}'$  and go back to the 1<sup>st</sup> step.

We show that this procedure always converges and that the solution built is indeed optimal, which completes our proof. We consider separately the 4 cases identified previously.

1.  $p_{t_1}^{C*} > 0$  and  $p_{t_2}^{D*} = 0$ . We modify the solution at  $t_1$  to  $p_{t_1}^{C'} = p_{t_1}^{C*} - \epsilon$  and at  $t_2$  to  $p_{t_2}^{C'} = p_{t_2}^{C*} + \rho^{t_2-t_1}\epsilon$ , with

$$\epsilon = \min \left\{ p_{t_1}^{C*}, \left\{ \frac{s_t^* - \underline{S}}{\rho^{t-t_1} \Delta t \eta^C}, \forall t \in [t_1, t_2 - 1] \right\}, \frac{\overline{P}^C - p_{t_2}^{C*}}{\rho^{t_2-t_1}} \right\}, \quad (41)$$

which we saw previously is a feasible solution, with a change in the objective of

$$\Delta t C_{t_1} \epsilon - \Delta t \rho^{t_2-t_1} C_{t_2} \epsilon = 0, \quad (42)$$

with (37). Therefore this new solution is also optimal.

We show that for this solution,  $s'_t > \bar{s}_t$  for less time periods than for  $\mathbf{x}^*$ . We evaluate separately the 3 possibilities:

- $\epsilon = p_{t_1}^{C*}$ . Then  $p_{t_1}^{C'} = 0$  and we move to case 3, for which we show below that we can further modify the solution in a way that  $s'_t > \bar{s}_t$  for less time periods than for  $\mathbf{x}^*$ .
- $\epsilon = \min \left\{ \frac{s_t^* - \underline{S}}{\rho^{t-t_1} \Delta t \eta^C}, t \in [t_1, t_2 - 1] \right\}$ . Then for at least one  $t \in [t_1, t_2 - 1]$ ,  $s'_t = \underline{S} \leq \bar{s}_t$ , and the result stands.
- $\epsilon = \frac{\overline{P}^C - p_{t_2}^{C*}}{\rho^{t_2-t_1}}$ . Then  $p_{t_2}^{C'} = \overline{P}^C$ . We have  $s'_{t_2} - \rho s'_{t_2-1} = s_{t_2}^* - \rho s'_{t_2-1} = \Delta t \eta^C \overline{P}^C \geq \bar{s}_{t_2} - \rho \bar{s}_{t_2-1}$ . Since  $s_{t_2}^* \leq \bar{s}_{t_2}$ , it means that  $s'_{t_2-1} \leq \bar{s}_{t_2-1}$  and the result stands.

2.  $p_{t_1}^{C*} > 0$  and  $p_{t_2}^{D*} > 0$ . We modify the solution at  $t_1$  to  $p_{t_1}^{C'} = p_{t_1}^{C*} - \epsilon$  and at  $t_2$  to  $p_{t_2}^{D'} = p_{t_2}^{D*} - \rho^{t_2-t_1} \eta^C \eta^D \epsilon$ , with

$$\epsilon = \min \left\{ p_{t_1}^{C*}, \left\{ \frac{s_t^* - \underline{S}}{\rho^{t-t_1} \Delta t \eta^C}, \forall t \in [t_1, t_2 - 1] \right\}, \frac{p_{t_2}^{D*}}{\rho^{t_2-t_1} \eta^C \eta^D} \right\}, \quad (43)$$

which we saw previously is a feasible solution, with a change in the objective of

$$\Delta t C_{t_1} \epsilon - \Delta t \rho^{t_2-t_1} C_{t_2} \eta^C \eta^D \epsilon = 0, \quad (44)$$

with (38). Therefore this new solution is also optimal.

We show that for this solution,  $s'_t > \bar{s}_t$  for less time periods than for  $\mathbf{x}^*$ . We evaluate separately the 3 possibilities:

- $\epsilon = p_{t_1}^{C*}$ . Then  $p_{t_1}^{C'} = 0$  and we move to case 3, for which we show below that we can further modify the solution in a way that  $s'_t > \bar{s}_t$  for less time periods than for  $\mathbf{x}^*$ .
- $\epsilon = \min \left\{ \frac{s_t^* - \underline{S}}{\rho^{t-t_1} \Delta t \eta^C}, t \in [t_1, t_2 - 1] \right\}$ . Then for at least one  $t \in [t_1, t_2 - 1]$ ,  $s'_t = \underline{S} \leq \bar{s}_t$ , and the result stands.
- $\epsilon = \frac{p_{t_2}^{D*}}{\rho^{t_2-t_1} \eta^C \eta^D}$ . Then  $p_{t_2}^{D'} = 0$  and we move to case 1, for which we have shown that we can further modify the solution in a way that  $s'_t > \bar{s}_t$  for less time periods than for  $\mathbf{x}^*$ .

3.  $p_{t_1}^{C*} = 0$  and  $p_{t_2}^{D*} = 0$ . We modify the solution at  $t_1$  to  $p_{t_1}^{D'} = p_{t_1}^{D*} + \epsilon$  and at  $t_2$  to  $p_{t_2}^{C'} = p_{t_2}^{C*} + \frac{\rho^{t_2-t_1}}{\eta^C \eta^D} \epsilon$ , with

$$\epsilon = \min \left\{ \overline{P}^D - p_{t_1}^{D*}, \left\{ \frac{\eta^D (s_t^* - \underline{S})}{\rho^{t-t_1} \Delta t}, \forall t \in [t_1, t_2 - 1] \right\}, \frac{\eta^C \eta^D (\overline{P}^C - p_{t_2}^{C*})}{\rho^{t_2-t_1}} \right\}, \quad (45)$$

which we saw previously is a feasible solution, with a change in the objective of

$$\Delta t C_{t_1} \epsilon - \Delta t C_{t_2} \frac{\rho^{t_2-t_1}}{\eta^C \eta^D} \epsilon = 0, \quad (46)$$

with (39). Therefore this new solution is also optimal.

We show that for this solution,  $s'_t > \bar{s}_t$  for less time periods than for  $\mathbf{x}^*$ . We evaluate separately the 3 possibilities:

- $\epsilon = \overline{P}^D - p_{t_1}^{D*}$ . Then  $p_{t_1}^{D'} = \overline{P}^D$ . We have  $s'_{t_1} - \rho s'_{t_1-1} = s'_{t_1} - \rho s_{t_1-1}^* = -\Delta t \frac{1}{\eta^D} \overline{P}^D \leq \bar{s}_{t_1} - \rho \bar{s}_{t_1-1}$ . Since  $s_{t_1-1}^* \leq \bar{s}_{t_1-1}$ , it means that  $s'_{t_1} \leq \bar{s}_{t_1}$  and the result stands.
- $\epsilon = \min \left\{ \eta^D \frac{s_t^* - \underline{S}}{\rho^{t-t_1} \Delta t}, t \in [t_1, t_2 - 1] \right\}$ . Then for at least one  $t \in [t_1, t_2 - 1]$ ,  $s'_t = \underline{S} \leq \bar{s}_t$ , and the result stands.
- $\epsilon = \frac{\eta^C \eta^D (\overline{P}^C - p_{t_2}^{C*})}{\rho^{t_2-t_1}}$ . Then  $p_{t_2}^{C'} = \overline{P}^C$ . We have  $s'_{t_2} - \rho s'_{t_2-1} = s_{t_2}^* - \rho s'_{t_2-1} = \Delta t \eta^C \overline{P}^C \geq \bar{s}_{t_2} - \rho \bar{s}_{t_2-1}$ . Since  $s_{t_2}^* \leq \bar{s}_{t_2}$ , it means that  $s'_{t_2-1} \leq \bar{s}_{t_2-1}$  and the result stands.

4.  $p_{t_1}^{C*} = 0$  and  $p_{t_2}^{D*} > 0$ . We modify the solution at  $t_1$  to  $p_{t_1}^{D'} = p_{t_1}^{D*} + \epsilon$  and at  $t_2$  to  $p_{t_2}^{D'} = p_{t_2}^{D*} - \rho^{t_2-t_1} \epsilon$ , with

$$\epsilon = \min \left\{ \overline{P}^D - p_{t_1}^{D*}, \left\{ \eta^D \frac{s_t^* - \underline{S}}{\rho^{t-t_1} \Delta t}, \forall t \in [t_1, t_2 - 1] \right\}, \frac{p_{t_2}^{D*}}{\rho^{t_2-t_1}} \right\}, \quad (47)$$

which we saw previously is a feasible solution, with a change in the objective of

$$\Delta t C_{t_1} \epsilon - \Delta t \rho^{t_2-t_1} C_{t_2} \epsilon = 0, \quad (48)$$

with (40). Therefore this new solution is also optimal.

We show that for this solution,  $s'_t > \bar{s}_t$  for less time periods than for  $\mathbf{x}^*$ . We evaluate separately the 3 possibilities:

- $\epsilon = \overline{P}^D - p_{t_1}^{D*}$ . Then  $p_{t_1}^{D'} = \overline{P}^D$ . We have  $s'_{t_1} - \rho s'_{t_1-1} = s'_{t_1} - \rho s_{t_1-1}^* = -\Delta t \frac{1}{\eta^D} \overline{P}^D \leq \bar{s}_{t_1} - \rho \bar{s}_{t_1-1}$ . Since  $s_{t_1-1}^* \leq \bar{s}_{t_1-1}$ , it means that  $s'_{t_1} \leq \bar{s}_{t_1}$  and the result stands.
- $\epsilon = \min \left\{ \eta^D \frac{s_t^* - \underline{S}}{\Delta t}, t \in [t_1, t_2 - 1] \right\}$ . Then for at least one  $t \in [t_1, t_2 - 1]$ ,  $s'_t = \underline{S} \leq \bar{s}_t$ , and the result stands.
- $\epsilon = \frac{p_{t_2}^{D*}}{\rho^{t_2-t_1}}$ . Then  $p_{t_2}^{D'} = 0$  and we move to case 3, for which we have shown that we can further modify the solution in a way that  $s'_t > \bar{s}_t$  for less time periods than for  $\mathbf{x}^*$ .

By induction, starting again the procedure on the new solution, with updated  $t_1$  and  $t_2$ , we can build a solution  $\mathbf{x}' \in \mathcal{X}^*$  for which  $s'_t \leq \bar{s}_t, \forall t \in \mathcal{T}$ , which proves point 1 of the lemma.

The proof for point 2 of the lemma is similar.  $\square$

Based on Lemma 1, the following can also be derived:

**Corollary 1.** For any  $\bar{\mathbf{x}} \in \bar{\mathcal{X}}$  and  $\underline{\mathbf{x}} \in \underline{\mathcal{X}}$ , such that  $\underline{s}_t \leq \bar{s}_t, \forall t \in \mathcal{T}$ , and for any  $S^{\text{end}}, \underline{S}_T \leq S^{\text{end}} \leq \bar{S}_T, \exists \mathbf{x}^* \in \mathcal{X}^*$  such that  $\underline{s}_t \leq s_t^* \leq \bar{s}_t, \forall t \in \mathcal{T}$ .

*Proof.* First, given  $\bar{\mathbf{x}} \in \bar{\mathcal{X}}$ , we know from Lemma 1 that there exists  $\underline{\mathbf{x}} \in \underline{\mathcal{X}}$ , such that  $\underline{s}_t \leq \bar{x}_t, \forall t \in \mathcal{T}$ . For the second part, using Lemma 1, we know that there exists  $\mathbf{x}^* \in \mathcal{X}^*$  such that  $s_t^* \geq \underline{s}_t, \forall t \in \mathcal{T}$ . Supposing that there exists  $t \in \mathcal{T}$  such that  $s_t^* > \bar{s}_t$ , we can build a new solution  $\mathbf{x}' \in \mathcal{X}^*$  such that  $\underline{s}_t \leq s'_t \leq \bar{x}_t, \forall t \in \mathcal{T}$ , similarly to what was done in the proof of Lemma 1. The only difference is that the  $\epsilon$  used to modify the solution should be such that it stays above  $\underline{s}_t$  instead of  $\underline{S}$ . For example, (41) becomes

$$\epsilon = \min \left\{ p_{t_1}^{C*}, \left\{ \frac{s_t^* - \underline{s}_t}{\rho^{t-t_1} \Delta t \eta^C}, \forall t \in [t_1, t_2 - 1] \right\}, \frac{\bar{P}^C - p_{t_2}^{C*}}{\rho^{t_2-t_1}} \right\} \quad (49)$$

where  $\underline{S}$  has been replaced by  $\underline{s}_t$ . The same goes for (43), (45) and (47).  $\square$

We also need the following result:

**Lemma 2.** If there exist  $\bar{\mathbf{x}} \in \bar{\mathcal{X}}, \underline{\mathbf{x}} \in \underline{\mathcal{X}}$  and  $\tau \in \mathcal{T}$  such that  $\underline{s}_\tau = \bar{s}_\tau = S$ , then there exists  $\bar{\mathbf{x}} \in \bar{\mathcal{X}}$  and  $\underline{\mathbf{x}} \in \underline{\mathcal{X}}$  such that  $\underline{s}_t = \bar{s}_t, \forall t \in [1, \tau]$  and  $\underline{s}_t \leq \bar{s}_t, \forall t \in [\tau + 1, T]$ .

*Proof.* Let's identify as  $S$  the state of energy at  $\tau$  for which  $\underline{s}_\tau = \bar{s}_\tau = S$ . Augmenting problem  $\mathbf{F}(\mathcal{T}, \mathbf{C}, \underline{S}_T)$  with the constraint  $s_\tau = S$  will return the same optimal objective value. For this augmented problem, we can solve independently for  $t \in [1, \tau]$  and for  $t \in [\tau + 1, T]$ , since in the only constraint linking those two periods,  $s_{\tau+1} = \rho s_\tau + \Delta t \left( \eta^C p_\tau^C - \frac{1}{\eta^D} p_\tau^D \right)$ ,  $s_\tau$  can be replaced by  $S$ . Similarly, augmenting  $\mathbf{F}(\mathcal{T}, \mathbf{C}, \bar{S}_T)$  with the constraint  $s_\tau = S$  will return the same optimal objective value, and for this augmented problem, we can solve independently for  $t \in [1, \tau]$  and for  $t \in [\tau + 1, T]$ . The two problems are the same over the first period, i.e. for  $t \in [1, \tau]$ , so their optimal solutions are the same. Over the second period, as the starting point is the same, the results from Lemma 1 can be applied for  $t \in [\tau + 1, T]$  to show that there exists  $\underline{\mathbf{x}} \in \underline{\mathcal{X}}$ , such that  $\underline{s}_t \leq \bar{x}_t, \forall t \in [\tau + 1, T]$ . Combining the solutions over the two periods, we get the desired result.  $\square$

With Lemma 2, if  $\exists \underline{\mathbf{x}} \in \underline{\mathcal{X}}$  and  $\exists \bar{\mathbf{x}} \in \bar{\mathcal{X}}$  such that  $\underline{s}_H = \bar{s}_H = S$ , then  $\exists \bar{\mathbf{x}}' \in \bar{\mathcal{X}}$  and  $\exists \underline{\mathbf{x}}' \in \underline{\mathcal{X}}$  such that  $\underline{s}'_t = \bar{s}'_t, \forall t \in [1, \tau]$  and  $\underline{s}'_t \leq \bar{s}'_t, \forall t \in [1, \tau]$ . With Corollary 1, for any  $S^{\text{end}}, \underline{S}_T \leq S^{\text{end}} \leq \bar{S}_T, \exists \mathbf{x}^* \in \mathcal{X}^*$  such that  $\underline{s}'_t \leq s_t^* \leq \bar{s}'_t, \forall t \in \mathcal{T}$ , and in particular, such that  $\underline{s}'_t = s_t^* = \bar{s}'_t, \forall t \in [1, H]$ .

Solving the problems over a horizon longer by one time period,  $T + 1$ , the optimal level at  $T$ , called  $S$ , obtained from solving  $\mathbf{F}([1, T + 1], \mathbf{C}, \underline{S}_{T+1})$  is such that  $\underline{S}_T \leq S \leq \bar{S}_T$ , since  $\underline{S}_T$  and  $\bar{S}_T$  are the lowest and highest reachable levels at  $T$ . We can add the constraint  $s_T = S$  and split the problem over the two intervals,  $[1, T]$  and  $[T + 1]$  without changing the optimal value of the objective function. The problem on



$[1, T]$  is then  $\mathbf{F}(\mathcal{T}, \mathbf{C}, S)$ , for which we have already shown that the solution over the decision horizon will be the same. The same applies for  $\mathbf{F}([1, T + 1], \mathbf{C}, \bar{S}_{T+1})$ , and therefore for  $\mathbf{F}([1, T + 1], \mathbf{C}, S^{\text{end}})$ , with any  $S^{\text{end}}$ , such that  $\underline{S}_{T+1} \leq S^{\text{end}} \leq \bar{S}_{T+1}$ . By induction, we obtain that the solution over the decision horizon is optimal for the infinite horizon problem.

On the other hand, if  $T$  is a forecast horizon, by definition,  $\exists \underline{\mathbf{x}} \in \underline{\mathcal{X}}$  and  $\exists \bar{\mathbf{x}} \in \bar{\mathcal{X}}$  such that  $\underline{s}_H = \bar{s}_H$ .

## B. Proof of Proposition 1

For a given  $s_H \in [\underline{s}_H, \bar{s}_H]$ , the difference in objective value with respect to the solution of the infinite-horizon problem is the sum of the difference due to the divergent schedule over the decision horizon and the difference due to starting the remainder of the horizon with a different storage level:

$$\Delta Z = Z^{*,\text{DH}} - Z^{\text{DH}} + Z^{*,\text{R}} - Z^{\text{R}} \quad (50)$$

where  $Z^{*,\text{DH}}$  is the value of the objective function over the decision horizon for the infinite-horizon problem, and  $Z^{*,\text{R}}$  and  $Z^{\text{R}}$  are the values of the objective function over the rest of the horizon for the infinite-horizon problem and for the problem actually solved, respectively.

It is possible to bound  $Z^{*,\text{DH}}$  from above with  $Z^{\text{opt,DH}}$ , the maximum profit obtainable if considering the decision horizon in isolation and allowing the final storage at the end of the decision horizon to be a variable bounded between  $\underline{s}_H$  and  $\bar{s}_H$ , the interval within which the optimal storage level in the infinite horizon problem lies.

Over the rest of the horizon, we can define two extreme cases. In the first case,  $s_H \geq s_H^*$ , where  $s_H^*$  is the optimal level at the end of the decision horizon in the infinite-horizon problem, and we miss an opportunity to charge the difference later at a lower price, and in the worst case at  $\underline{C}$ . Therefore we have

$$Z^{*,\text{R}} - Z^{\text{R}} \leq -\underline{C} \frac{1}{\eta^{\text{C}}} (s_H - s_H^*) \leq -\underline{C} \frac{1}{\eta^{\text{C}}} (s_H - \underline{s}_H). \quad (51)$$

In the second case,  $s_H \leq s_H^*$ , and we miss an opportunity to discharge the difference  $s_H^* - s_H$  later at a higher price, and in the worst case at  $\bar{C}$ . Therefore we have

$$Z^{*,\text{R}} - Z^{\text{R}} \leq \bar{C} \eta^{\text{D}} (s_H^* - s_H) \leq \bar{C} \eta^{\text{D}} (\bar{s}_H - s_H). \quad (52)$$

Combining (51) and (52) gives

$$\Delta Z \leq Z^{\text{opt,DH}} - Z^{\text{DH}} + \max \left\{ -\underline{C} \frac{1}{\eta^{\text{C}}} (s_H - \underline{s}_H), \bar{C} \eta^{\text{D}} (\bar{s}_H - s_H) \right\}. \quad (53)$$

## C. Proof of Proposition 2

We prove the result by contradiction. Consider  $T \in \mathbb{N}^+$ , such that the expression in (6) is strictly positive and suppose that  $T$  is a forecast horizon. We then have the following:

$$\bar{S} - \underline{S} - \left( \sum_{t=0}^{T-H-1} \rho^t \right) \left( \Delta t \eta^{\text{C}} \bar{P}^{\text{C}} + \frac{\Delta t}{\eta^{\text{D}}} \bar{P}^{\text{D}} \right) > 0, \quad (54)$$

$$\rho^T S^{\text{init}} - \underline{S} + \Delta t \eta^C \overline{P}^C \left( \sum_{t=T-H}^{T-1} \rho^t \right) - \frac{\Delta t}{\eta^D} \overline{P}^D \left( \sum_{t=0}^{T-H-1} \rho^t \right) > 0, \quad (55)$$

and

$$\overline{S} - \rho^T S^{\text{init}} - \Delta t \eta^C \overline{P}^C \left( \sum_{t=0}^{T-H-1} \rho^t \right) + \frac{\Delta t}{\eta^D} \overline{P}^D \left( \sum_{t=T-H}^{T-1} \rho^t \right) > 0, \quad (56)$$

Since  $T$  is a forecast horizon, with Theorem 1,  $\exists \underline{x} \in \underline{\mathcal{X}}$  and  $\exists \overline{x} \in \overline{\mathcal{X}}$  such that  $\underline{s}_H = \overline{s}_H$ . We consider such  $\underline{x}$  and  $\overline{x}$ , and we introduce  $S$  to represent this value, i.e.  $S = \underline{s}_H = \overline{s}_H$ . Tight lower and upper bounds on  $S$  can be obtained by considering that the maximum quantity is discharged during the whole decision horizon or until the minimum level is reached:

$$S \geq \max \left\{ \underline{S}, \rho^H S^{\text{init}} - \frac{\Delta t}{\eta^D} \overline{P}^D \sum_{t=0}^{H-1} \rho^t \right\}, \quad (57)$$

and by considering that the maximum quantity is discharged during the whole decision horizon or until the minimum level is reached:

$$S \leq \min \left\{ \overline{S}, \rho^H S^{\text{init}} + \Delta t \eta^C \overline{P}^C \sum_{t=0}^{H-1} \rho^t \right\}. \quad (58)$$

Moreover, we know that the minimum and the maximum reachable levels at the end of the planning horizon,  $\underline{S}_T$  and  $\overline{S}_T$ , can be reached from  $S$ . In other words, if discharging the maximum quantity after the end of the decision horizon and until the end of the planning horizon  $\underline{S}_T$  should at least be reached:

$$\rho^{T-H} S - \frac{\Delta t}{\eta^D} \overline{P}^D \left( \sum_{t=0}^{T-H-1} \rho^t \right) \leq \underline{S}_T. \quad (59)$$

Similarly, if charging the maximum quantity after the end of the decision horizon and until the end of the planning horizon,  $\overline{S}_T$  should at least be reached:

$$\rho^{T-H} S + \Delta t \eta^C \overline{P}^C \left( \sum_{t=0}^{T-H-1} \rho^t \right) \geq \overline{S}_T. \quad (60)$$

Next we use the fact that  $\underline{S}_T = \max \left\{ \underline{S}, \rho^T S^{\text{init}} - \frac{\Delta t}{\eta^D} \overline{P}^D \sum_{t=0}^{T-1} \rho^t \right\}$  and  $\overline{S}_T = \min \left\{ \overline{S}, \rho^T S^{\text{init}} + \Delta t \eta^C \overline{P}^C \sum_{t=0}^{T-1} \rho^t \right\}$  to define the four following cases:

1.  $\underline{S}_T = \underline{S}$  and  $\overline{S}_T = \overline{S}$
2.  $\underline{S}_T = \underline{S}$  and  $\overline{S}_T = \rho^T S^{\text{init}} + \Delta t \eta^C \overline{P}^C \sum_{t=0}^{T-1} \rho^t$
3.  $\underline{S}_T = \rho^T S^{\text{init}} - \frac{\Delta t}{\eta^D} \overline{P}^D \sum_{t=0}^{T-1} \rho^t$  and  $\overline{S}_T = \overline{S}$
4.  $\underline{S}_T = \rho^T S^{\text{init}} - \frac{\Delta t}{\eta^D} \overline{P}^D \sum_{t=0}^{T-1} \rho^t$  and  $\overline{S}_T = \rho^T S^{\text{init}} + \Delta t \eta^C \overline{P}^C \sum_{t=0}^{T-1} \rho^t$

In Cases 1 and 2, (59) becomes

$$\rho^{T-H} S - \frac{\Delta t}{\eta^D} \overline{P}^D \left( \sum_{t=0}^{T-H-1} \rho^t \right) \leq \underline{S}, \quad (61)$$

which can be rearranged as

$$S \leq \rho^{H-T} \underline{S} + \frac{\Delta t}{\eta^D} \overline{P}^D \rho^{H-T} \left( \sum_{t=0}^{T-H-1} \rho^t \right). \quad (62)$$

In Cases 3 and 4, (59) becomes

$$\rho^{T-H} S - \frac{\Delta t}{\eta^D} \overline{P}^D \left( \sum_{t=0}^{T-H-1} \rho^t \right) \leq \rho^T S^{\text{init}} - \frac{\Delta t}{\eta^D} \overline{P}^D \sum_{t=0}^{T-1} \rho^t, \quad (63)$$

which can be rearranged as

$$S \leq \rho^H S^{\text{init}} - \frac{\Delta t}{\eta^D} \overline{P}^D \sum_{t=0}^{H-1} \rho^t. \quad (64)$$

With (57), we get that

$$S = \rho^H S^{\text{init}} - \frac{\Delta t}{\eta^D} \overline{P}^D \sum_{t=0}^{H-1} \rho^t. \quad (65)$$

In Cases 1 and 3, (60) becomes

$$\rho^{T-H} S + \Delta t \eta^C \overline{P}^C \left( \sum_{t=0}^{T-H-1} \rho^t \right) \geq \overline{S}, \quad (66)$$

which can be rearranged as

$$S \geq \rho^{H-T} \overline{S} - \Delta t \eta^C \overline{P}^C \rho^{H-T} \left( \sum_{t=0}^{T-H-1} \rho^t \right). \quad (67)$$

In Cases 2 and 4, (60) becomes

$$\rho^{T-H} S + \Delta t \eta^C \overline{P}^C \left( \sum_{t=0}^{T-H-1} \rho^t \right) \geq \rho^T S^{\text{init}} + \Delta t \eta^C \overline{P}^C \sum_{t=0}^{T-1} \rho^t, \quad (68)$$

which can be rearranged as

$$S \geq \rho^H S^{\text{init}} + \Delta t \eta^C \overline{P}^C \sum_{t=0}^{H-1} \rho^t. \quad (69)$$

With (58), we get that

$$S = \rho^H S^{\text{init}} + \Delta t \eta^C \overline{P}^C \sum_{t=0}^{H-1} \rho^t. \quad (70)$$

**Case 1:** In the first case, (62) and (67) apply. This is possible only if  $\rho^{H-T}\bar{S} - \Delta t \eta^C \bar{P}^C \rho^{H-T} \left( \sum_{t=0}^{T-H-1} \rho^t \right) \leq \rho^{H-T} \underline{S} + \frac{\Delta t}{\eta^D} \bar{P}^D \rho^{H-T} \left( \sum_{t=0}^{T-H-1} \rho^t \right)$ . We can rearrange as

$$\bar{S} - \underline{S} - \left( \sum_{t=0}^{T-H-1} \rho^t \right) \left( \Delta t \eta^C \bar{P}^C + \frac{\Delta t}{\eta^D} \bar{P}^D \right) \leq 0, \quad (71)$$

which is not possible because of (54).

**Case 2:** In the second case, (62) and (70) apply. Using (70), we can replace in (62):

$$\rho^H S^{\text{init}} + \Delta t \eta^C \bar{P}^C \sum_{t=0}^{H-1} \rho^t \leq \rho^{H-T} \underline{S} + \frac{\Delta t}{\eta^D} \bar{P}^D \rho^{H-T} \left( \sum_{t=0}^{T-H-1} \rho^t \right). \quad (72)$$

Rearranging gives

$$\rho^T S^{\text{init}} - \underline{S} + \Delta t \eta^C \bar{P}^C \left( \sum_{t=T-H}^{T-1} \rho^t \right) - \frac{\Delta t}{\eta^D} \bar{P}^D \left( \sum_{t=0}^{T-H-1} \rho^t \right) \leq 0, \quad (73)$$

which is not possible because of (55).

**Case 3:** In the third case, (65) and (67) apply. Using (65), we can replace in (67):

$$\rho^H S^{\text{init}} - \frac{\Delta t}{\eta^D} \bar{P}^D \sum_{t=0}^{H-1} \rho^t \geq \rho^{H-T} \bar{S} - \Delta t \eta^C \bar{P}^C \rho^{H-T} \left( \sum_{t=0}^{T-H-1} \rho^t \right). \quad (74)$$

Rearranging gives

$$\bar{S} - \rho^T S^{\text{init}} - \Delta t \eta^C \bar{P}^C \left( \sum_{t=0}^{T-H-1} \rho^t \right) + \frac{\Delta t}{\eta^D} \bar{P}^D \left( \sum_{t=T-H}^{T-1} \rho^t \right) \leq 0, \quad (75)$$

which is not possible because of (56).

**Case 4:** In the fourth case, (65) and (70) apply. We thus have:

$$\rho^H S^{\text{init}} - \frac{\Delta t}{\eta^D} \bar{P}^D \sum_{t=0}^{H-1} \rho^t = \rho^H S^{\text{init}} + \Delta t \eta^C \bar{P}^C \sum_{t=0}^{H-1} \rho^t, \quad (76)$$

which means that

$$-\frac{1}{\eta^D} \bar{P}^D = \eta^C \bar{P}^C, \quad (77)$$

which is not possible since all are strictly positive.

We showed that in all the cases there is a contradiction, therefore  $T$  is not a forecast horizon.

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