Static Equilibrium of Vehicle Dispatching in Multi-regional Ride-Hailing Markets

Ran Chen*[1] and Nikolaos Geroliminis[1]

[1]Urban Transport Systems Laboratory, EPFL, Switzerland

Short summary

This study introduces a comprehensive model to analyze a ride-hailing market in which multiple companies allocate their vehicles as decision variables across various regions to maximize their profits. The problem is formulated as a classical multi-player, non-cooperative game with coupled constraints on the actions of each company. Although the game is non-convex, and therefore lacks guarantees for the uniqueness of the Nash equilibrium, we propose an iterative algorithm to compute the equilibrium and check whether multiple equilibria occur for further analysis. Utilizing this algorithm, we conduct a numerical study to illustrate the model in a duopoly market with two regions characterized by distinct demand profiles.
Keywords: Ride-hailing, Vehicle dispatch, Non-cooperative game.

1 Introduction

Ride-hailing, referring to a service where customers request personalized rides for commuting via an online platform, has gained significant popularity due to its efficiency, affordability, and technological advancements. Compared to traditional street-hailing transportation, such as taxis, established companies like Uber, Lyft, and Didi have made it far more convenient for smartphone users to book rides through dedicated apps as they substantially reduce the matching friction between drivers and customers (Zha et al. (2016)). Furthermore, ride-hailing contributes significantly to sustainability by reducing parking demand and minimizing empty kilometers driven (Tirachini (2020)). These benefits have also spurred advancements in the shared mobility sector, driving the growth of existing companies and promoting the emergence of new players, ultimately offering improved solutions for urban transportation.

Nonetheless, maintaining a balance between fair competition and market regulation is critical to ensuring the long-term sustainability of the ride-hailing ecosystem with multiple companies. Game theory, as an effective tool for analyzing the behaviors of competing agents, has been extensively applied in the study of ride-hailing markets. For example, static competition among ride-hailing platforms has been explored in Bernstein et al. (2021) and Zhang & Nie (2021), while Cai et al. (2024) introduces an evolutionary game model to capture dynamic decision adjustments. Furthermore, the potential for game-based control mechanisms in ride-hailing companies has been discussed in works such as Maljkovic et al. (2022) and Maljkovic et al. (2023).

Vehicle dispatch problems in ride-hailing markets, On the other hand, involve optimizing fleet allocation across regions with varying demand patterns to maximize profits. Despite its importance, this topic has received limited attention in the literature. To address this gap, this study presents a game-theoretical model to describe how companies influence one another through their dispatch strategies in a multi-regional ride-hailing market. Each company strategically allocates its fleet across regions to maximize the number of customers served, while accounting for the strategies of competitors. The problem is formulated as a multi-player non-cooperative game, and a fundamental analysis of the equilibrium properties is conducted. This work provides a foundation for further exploration and the potential improvement of efficiency in ride-hailing markets.

The remainder of this paper is organized as follows: Section 2 defines the game and outlines an algorithm for computing the Nash equilibrium. Section 3 presents a numerical study conducted on a simplified example system. Finally, Section 4 concludes the paper and discusses potential directions for future research.

2 Methodology

This section provides a general formulation of the problem. Consider a ride-hailing market with n competing companies and m regions, each characterized by distinct properties such as area, average speed, and demand. Each company $i \in \mathcal{N}$ operates $\bar{N}_i$ vehicles and manages its fleet distribution $\bar{N}_i^j$, representing the number of vehicles dispatched to Region $j \in \mathcal{M}$. Since the fleet size $\bar{N}_i^j$ consists of discrete variables, we instead introduce the fleet density $N_i^j = \bar{N}_i^j / A^j$ in [veh/m^2] to create a continuous and differentiable decision space, where A^j denotes the area of Region j. The fleet size $\bar{N}_i$ can be alternatively expressed as $\sum_{j\in\mathcal{M}} A^j \cdot N_i^j$. Similarly, we define the empty fleet density V_i^j in [veh/m^2] and the accumulated density of empty vehicles in Region j as $V^j = \sum_{i\in\mathcal{N}} V_i^j$. The trip flux Q_i^j in [veh/(min $\cdot$ m^2)] represents the trip flow per unit area. At steady state, N_i^j, V_i^j and Q_i^j should satisfy the following the conservation equations:

$$N_i^j = V_i^j + w^j Q_i^j + T^j Q_i^j \tag{1}$$

$$D^j = \sum_{i\in\mathcal{N}} Q_i^j \tag{2}$$

Here $w^j = w^j_{\text{match}} + w^j_{\text{pickup}}$ denotes the expected waiting time of a customer in Region j after submitting a request, comprising the matching time w^j_{match} and the pickup time w^j_{pickup}. The travel time T^j represents the average duration a customer spends in a vehicle. Consequently, N_i^j is divided into three groups: the idling group V_i^j, the deadheading group $w^j Q_i^j$, and the traveling group $T^j Q_i^j$. Moreover, D^j represents the realized demand in Region j and should be expressed as a function of the fleet size. To simplify the problem formulation, we make the following assumptions about the market:

(1) **Dedicated drivers**: $\bar{N}_i$ remains constant $\forall i$ during the operation.

(2) **Zero matching time**: Matching operates on the first-come-first-served (FCFS) and the nearest-first (NF) principles, and is perfectly efficient, i.e. $w^j_{\text{match}} = 0 \Rightarrow w^j = w^j_{\text{pickup}}$.

(3) **Regional Isotropy**: T^j, w^j and D^j are time-invariant $\forall j$ but may differ between regions.

We further assume that the pickup time w^j_{pickup} is a convex and decreasing function of V^j satisfying the following conditions:

$$w^j(V^j \to 0) \to +\infty \quad \text{and} \quad w^j(V^j \to +\infty) \to 0 \tag{3}$$

Additionally, the regional realized demand is assumed to be a concave and increasing function of V^j such that

$$D^j(V^j = 0) = 0 \quad \text{and} \quad D^j(V^j \to +\infty) \to D^j_{\max}, \tag{4}$$

where $D^j_{\max}$ is the maximum realizable demand flux in Region j. These assumptions reflect that a higher density of idling vehicles results in shorter distances for drivers to reach customers and increases the probability of customers opting for ride-hailing services. Following the FCFS and NF assumptions, the regional realized demand flux is proportionally distributed among companies based on their empty fleet densities. Specifically, the realized demand for Company i in Region j is expressed as

$$D_i^j = \frac{V_i^j}{V^j} \cdot D^j(V^j) = \frac{V_i^j}{V_i^j + V_{-i}^j} \cdot D^j(V_i^j, V_{-i}^j), \tag{5}$$

where $V_i^j = \sum_{l\in\mathcal{N}, l\neq i} V_l^j$ represents the aggregated empty fleet density in Region j excluding Company i. In addition, inspired by the concept of supply flux S_i^j introduced in Xu et al. (2019), we rearrange Eq.(1) as,

$$S_i^j = \frac{N_i^j - V_i^j}{w^j(V^j) + T^j} \tag{6}$$

For conciseness, we omit the argument $V^j = V_i^j + V_{-i}^j$ in $D^j(V^j)$ and $w^j(V^j)$ in the following sections. At steady state, the supply equals the realized demand, thus $Q_i^j = S_i^j = D_i^j$. This yields:

$$\frac{N_i^j - V_i^j}{w^j + T^j} = \frac{V_i^j}{V^j} \cdot D^j \quad \Rightarrow \quad N_i^j = V_i^j \cdot \left(1 + \frac{w^j + T^j}{V^j} D^j\right). \tag{7}$$

The right part of Eq.(7) provides an explicit relation between N_i^j and V_i^j, which will be elaborated in the next subsection.

Formulation of the game

As discussed previously, all the quantities are expressed or assumed as functions of the empty fleet density V_i^j, thus it is convenient to formulate the problem with V_i^j as the decision variable. This change of variables is valid since the one-to-one relation between N_i^j and V_i^j for $N_i^j,\ V_i^j \in \mathbb{R}_+^m$ given fixed V_{-i}^j can be easily proved by constructing the derivative of N_i^j w.r.t. V_i^j and proving its continuity and positivity. Once V_i^j is optimized, N_i^j can be computed based on Eq.(7). The goal of company i is then to maximize its aggregate traffic flow $Q_i = \sum_{j \in \mathcal{M}} Q_i^j$ over its empty fleet densities V_i^j given fixed $V_{-i}^j\ \forall j$,

$$Q_i(\mathbf{V}_i, \mathbf{V}_{-i}) = \sum_{j \in \mathcal{M}} A^j D_i^j = \sum_{j \in \mathcal{M}} \frac{A^j V_i^j}{V^j} \cdot D^j\ , \tag{8}$$

subject to the conservation equation,

$$\bar{N}_i = \sum_{j \in \mathcal{M}} A^j N_i^j = \sum_{j \in \mathcal{M}} A^j V_i^j \cdot \left(1 + \frac{w^j + T^j}{V^j} D^j\right). \tag{9}$$

Without loss of generality, we assume $A^j = 1\ \forall j$ as D^j already incorporates regional properties including the area. Therefore, the objective of each company can be rewritten as minimizing the negative flow over $\mathbf{V}_i$ given fixed $\mathbf{V}_{-i}$ as below,

$$J_i(\mathbf{V}_i, \mathbf{V}_{-i}) = -\sum_{j \in \mathcal{M}} \frac{V_i^j}{V^j} \cdot D^j, \tag{10}$$

where $\mathbf{V}_i = \mathrm{vec}\{V_i^j\}_{j \in \mathcal{M}} \in \mathbb{R}_+^m$ and $\mathbf{V}_{-i} = \mathrm{vec}\{V_{-i}^j\}_{j \in \mathcal{M}} \in \mathbb{R}_+^m$. The feasible region of $\mathbf{V}_i$ is determined by the intersection of $\mathbf{V}_i \geq \mathbf{0}$ and the following coupling constraint,

$$g_i(\mathbf{V}_i, \mathbf{V}_{-i}) = \bar{N}_i - \sum_{j \in \mathcal{M}} V_i^j \cdot \left(1 + \frac{w^j + T^j}{V^j} D^j\right). \tag{11}$$

According to the conservation equation, $g_i = 0$ when all the available vehicles from company i are dispatched. In practical operations, however, certain vehicles or drivers may temporarily suspend their operations or withdraw from the fleet, for reasons such as off-peak hours, an increase in the commission rate, and so on. While these factors are not addressed in this study, it is admissible to relax the equality constraint into an inequality, namely $g_i \leq 0$, since the conservation constraint is active in this problem thus the optimal solution remains unchanged. Therefore, the competition can be formulated as a game where each Company $i \in \mathcal{N}$ solves an optimization problem $\mathcal{P}_i$ given the fixed strategies $\mathbf{V}_{-i}$ of other companies as follows,

$$\begin{aligned} \mathcal{P}_i : \min_{\mathbf{V}_i \in \mathbb{R}_+^m} \quad & J_i(\mathbf{V}_i, \mathbf{V}_{-i}) \\ \text{s.t.} \quad & g_i(\mathbf{V}_i, \mathbf{V}_{-i}) \leq 0 \end{aligned} \tag{12}$$

The generalized Nash equilibrium (GNE) of the game is then defined as a collective strategy $\mathbf{V}^* = [\mathbf{V}_1^\top, \cdots, \mathbf{V}_n^\top]^\top$ such that for each $i \in \mathcal{N}$ the following inequality holds:

$$J_i(\mathbf{V}_i^*, \mathbf{V}_{-i}^*) \leq \inf_{\mathbf{V}_i \in \mathbb{R}_+^m} \left\{ J_i(\mathbf{V}_i, \mathbf{V}_{-i}^*) \mid g_i(\mathbf{V}_i, \mathbf{V}_{-i}^*) \leq 0 \right\} \tag{13}$$

In other words, no company can improve its profit by unilaterally changing its dispatch strategy from $\mathbf{V}_i^*$ to another feasible one.

Iterative algorithm

An iterative algorithm is provided below to compute and evaluate the convergence of the GNE of the game. In each iteration, $\mathcal{P}_i$ is used to compute $\mathbf{V}_i$ as the best response to the strategies (assumed to be known to all companies) of other companies, continuing until $\mathbf{V}_i$ converges for all i. It is worth noting that due to the coupling constraint, every update of $\mathbf{V}_i$ will change the feasible regions of $\mathbf{V}_{-i}$. Therefore, instead of updating the strategies simultaneously for all i, we update the strategies sequentially in descending order of fleet size during each iteration.

Algorithm 1 Iterative computation of the GNE

Input: T^j, $D^j(\cdot) : \mathbb{R} \to \mathbb{R}$, $w^j(\cdot) : \mathbb{R} \to \mathbb{R}$, $\mathbf{N}_{\text{init}}$, max_iter, tol
Output: $\mathbf{N}^* \in \mathbb{R}^{nm}$

```
k ← 0
V^k ← getVfromN(T^j, D^j(·), w^j(·), N_init)        ▷ V^k: the k-th iteration of V_i^j ∀i, j
while k ≤ max_iter or ΔJ ≥ tol do
    k ← k + 1
    for i = 1 : n do
        V^k ← V^{k−1}
        V_i^k ← P_i : arg min_{V_i ∈ R_+^m} {J_i(V_i, V_{−i}^k) | g_i(V_i, V_{−i}^k) ≤ 0}
    end for
    ΔJ = Σ_{i∈N} ‖J_i(V_i^k, V_{−i}^k) − J_i(V_i^{k−1}, V_{−i}^{k−1})‖
end while
N* ← getNfromV(T^j, D^j(·), w^j(·), V^k)
```

The computation of $\mathbf{V}_i^k$ based on $\mathcal{P}_i$ relies on non-convex optimization algorithms. In this study, the MATLAB function `fmincon` is applied with different initial conditions $\mathbf{N}_{\text{init}}$. Future work could focus on implementing more efficient non-convex optimization algorithms tailored for $\mathcal{P}_i$. It can be shown that J_i is convex in $\mathbf{V}_i$ given fixed $\mathbf{V}_{-i}$, while $g_i \leq 0$ specifies non-convex but *proximally smooth* sets (Clarke et al. (1995)), also referred to as *weakly convex* sets (Vial (1983)). These properties allow for the efficient computation of the so-called *local generalized Nash equilibrium* (LGNE) using *quasi-variational inequality* (QVI) proposed in Scarabaggio et al. (2024). The theorem of LGNE guarantees both the existence and the convergence to the (locally) unique LGNE under mild conditions. For further research, it would be valuable to analytically investigate the existence and uniqueness of the GNE, as well as the difference between the GNE and the LGNE.

3 Results and discussion

To demonstrate and analyze the properties of the ride-hailing market model, we consider an example system with $n = 2$ ride-hailing companies and $m = 2$ regions, denoted by $i \in \mathcal{N} = \{1, 2\}$ and $j \in \mathcal{M} = \{A, B\}$, respectively. Specifically, we use a concave and monotonically increasing function that involves an exponential component to represent the changes in the realized demand w.r.t. V^j, i.e.,

$$D^j(V^j) = D^j_{\max} \cdot \left(1 - \exp\left(-\alpha^j V^j\right)\right) \tag{14}$$

Meanwhile, a conventional function of w^j for a two-dimensional space (Xu et al. (2019)) is applied as follows,

$$w^j(V^j) = \beta^j \cdot \left(V^j\right)^{-\frac{1}{2}} \tag{15}$$

For simplicity, we set $\alpha^j = \alpha = 0.12$ and $\beta^j = \beta = 4$ to be identical across all regions and remain constant throughout the simulation. To differentiate the regions, Region A is characterized by higher demand and heavier congestion leading to a longer traveling time. In details, we set $[D^A_{\max}, D^B_{\max}] = [16, 9]$ and $[T^A, T^B] = [8, 6]$. The fleet size N_i^j (since the area A^j is assumed to be 1, we use N in place of $\bar{N}$) is varied across different scenarios to reveal supply surpluses and shortages. Several simulations are performed to illustrate the convergence of Algorithm 1, the uniqueness of the GNE, and how the demand is shared between the companies relative to their fleet sizes.

Convergence test

We first perform a convergence test for the iterated Nash equilibria of both companies. By setting $N_1 = 120$, $N_2 = 80$ and the initial conditions $N^A_{1,\text{init}} = N^B_{1,\text{init}} = 60$, $N^A_{2,\text{init}} = N^B_{2,\text{init}} = 40$, we plot the iterations of N^j_i and the corresponding realized Q^j_i in Fig.1. The results show that the equilibria converge after 7 iterations of the best response problem $\mathcal{P}_i$. Note $\mathbf{N}_{\text{init}}$ is converted to $(\mathbf{V}_{i,\text{init}}, \mathbf{V}_{-i,\text{init}})$ $\forall i$ in order to iterate the equilibria $(\mathbf{V}^*_i, \mathbf{V}^*_{-i})$, before computing $(\mathbf{N}^*_i, \mathbf{N}^*_{-i})$.

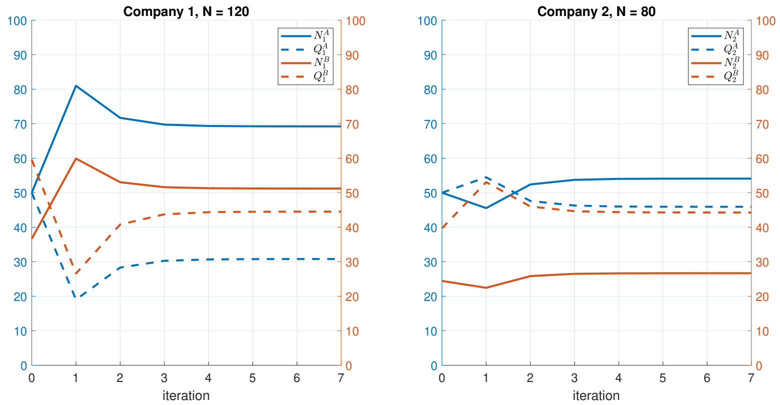


Figure 1: Convergence of the Nash equilibria

We also plot the variation of the optimization variable V^j_i in Fig.2 for both companies. The initial coupling conservation equation $g(\mathbf{V}_{i,\text{init}}, \mathbf{V}_{-i,\text{init}}) = 0$ and that at equilibria $g(\mathbf{V}^*_i, \mathbf{V}^*_{-i}) = 0$ are plotted in blue dashed and solid curves respectively. For Company 1, the conservation constraint hardly changes since the values of $[V^A_2, V^B_2]$ thus $[N^A_2, N^B_2]$ exhibit little variation during iteration. In constrast, the constraint of Company 2 significantly changes due to substantial updates in $[V^A_1, V^B_1]$. The iteration trajectories of V^j_i are plotted in red solid curves with circle markers indicating the intermediate values and a pentagram marking the equilibrium.

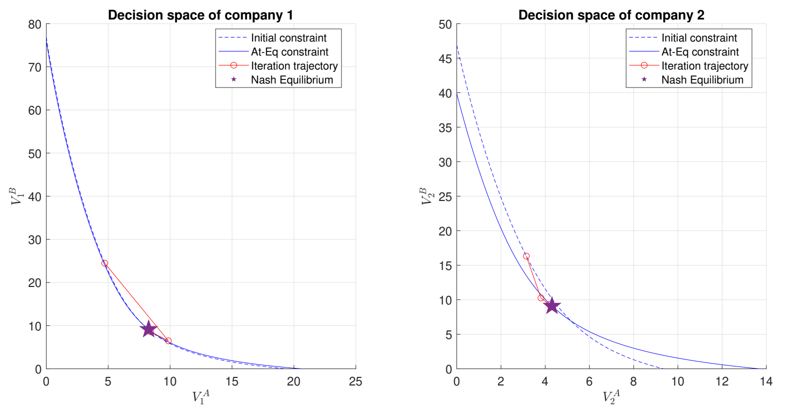


Figure 2: Iteration of V^j_i and the visualization of coupling constraints

Uniqueness test

As mentioned previously, the optimization problem $\mathcal{P}_i$ is non-convex due to the coupling constraints derived from the conservation equation. The non-convexity of these constraints has also been revealed in Fig.2, where the feasible region specified by $g \leq 0$ forms a triangle with its hypotenuse bent inwards. To check whether the equilibrium is unique, we consider multiple initial conditions and plot the trajectories of $[N_1^A, N_2^A]$, i.e. the dispatch strategy of both companies in Region A, in Fig.3. A total of 81 different initial conditions are considered, with $N_{i,\text{init}}^A$ ranging from 10% to 90% in steps of 10%. The trajectories are displayed in different colors while the black pentagram marks the equilibrium. It can be seen that all the trajectories eventually converge to the same point, indicating the equilibrium is empirically unique in this setting.

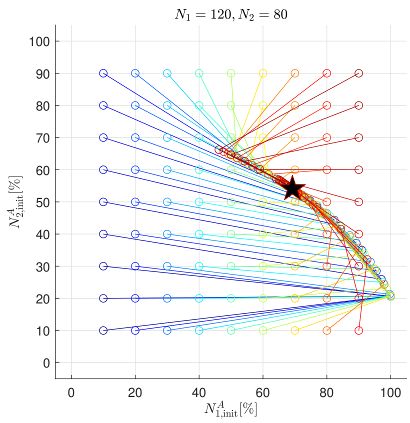
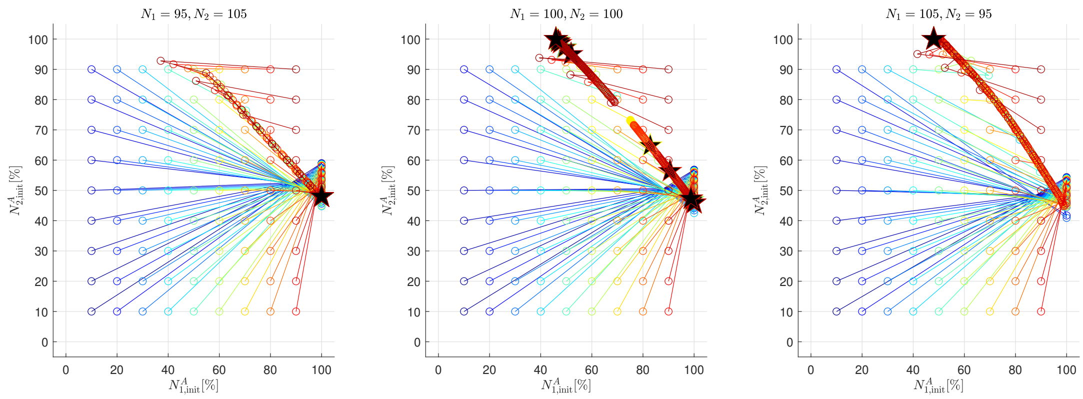


Figure 3: Convergence of N_i^A with different initial conditions

We then demonstrate another case where the equilibrium is not unique, and the results are given in Fig.4. For clarity, we set $T^A = T^B = 8$ in this case to help visualize multiple equilibria. By comparing Fig.4(a) and (c), we observe that the Nash equilibrium of $[N_1^A, N_2^A]$ in [%] shifts symmetrically from $[100, 48]$ to $[48, 100]$ as the fleet size $[N_1, N_2]$ changes from $[95, 105]$ to $[105, 95]$. However, as shown in (b), multiple equilibria occur when two companies have equal fleet sizes. More specifically, there exists a curve that contains infinitely many Nash equilibria.



(a) (b) (c)

Figure 4: Multiple Nash equilibria occur with different values of N_i

Impact of fleet fraction on dispatch strategy and demand

The last simulation explores how capital influences the dispatch strategy. With a fixed $N_{\text{tot}} = N_1 + N_2$, we visualize the changes in Nash equilibria of the dispatch strategy $[N_1^A, N_1^B, N_2^A, N_2^B]$ and the corresponding realized demand $[Q_1^A, Q_1^B, Q_2^A, Q_2^B]$, as functions of the fleet fraction of company 1 N_1/N_{tot} in percentage. Specifically, $N_i^j[\%] = N_i^j/N_i \times 100\%$, $Q_i^j[\%] = Q_i^j/D_{\max}^j \times 100\%$ and $Q_{\text{tot}}[\%] = \sum_{i,j} Q_i^j / \sum_j D_{\max}^j \times 100\%$. For simplicity, we consider $N_{\text{tot}} = 270$, 200 and 120, all of which yield unique Nash equilibria.

Fig.5 shows how the equilibrium and the realized demand change when $N_{\text{tot}} = 270$. The left plot depicts the trends of N_i^j, while the right shows the corresponding Q_i^j. When Company 1's fleet fraction is less than 5%, it focuses only on the lower-demand Region B. On the contrary, the larger company's strategy remains unaffected by the small company's presence and primarily follows the regional demand profile. As Company 1 increases its fleet fraction past 10%, it starts allocating more vehicles to Region A with $N_1^A > N_1^B$, until Company 1 becomes a monopoly. The demand share for each company is roughly linear to its fleet size, and nearly all demand is satisfied, as Q_{tot} approaches 100%.

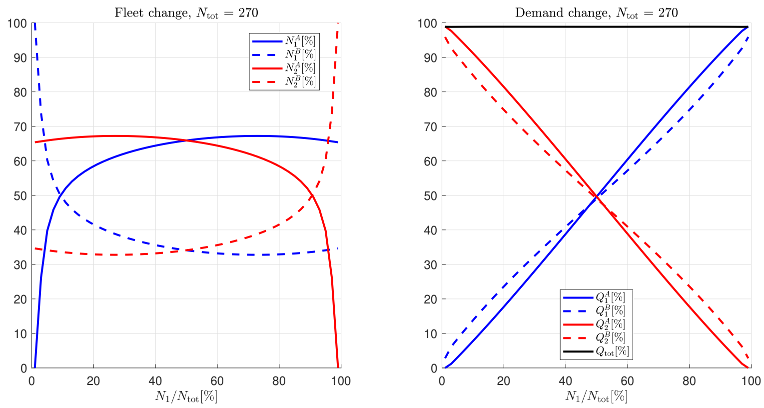


Figure 5: The Nash equilibria and regional demand share w.r.t. N_1 with $N_{\text{tot}} = 270$

Next, We reduce the total fleet size to $N_{\text{tot}} = 200$, with the results displayed in Fig.6. In this scenario, about 80% of the total demand is served (as seen in Q_{tot} in the right plot). When N_1/N_{tot} is below 20%, Company 2 as the larger company captures all the demand of Region A. As N_1/N_{tot} exceeds 35%, Company 1 increases both its fleet fraction and its share of demand in Region A, until $N_1/N_{\text{tot}} \approx 80[\%]$ when it pushes Company 2 out of Region A and begins shifting its the fleet to Region B. During this period, competition between the two companiesis is primarily concentrated in Region A, as Q_1^A changes significantly w.r.t. N_1, while Q_1^B remains constant. After $N_1/N_{\text{tot}} > 80[\%]$, Company 1's demand share in Region B rises rapidly as more vehicles are allocated to that region.

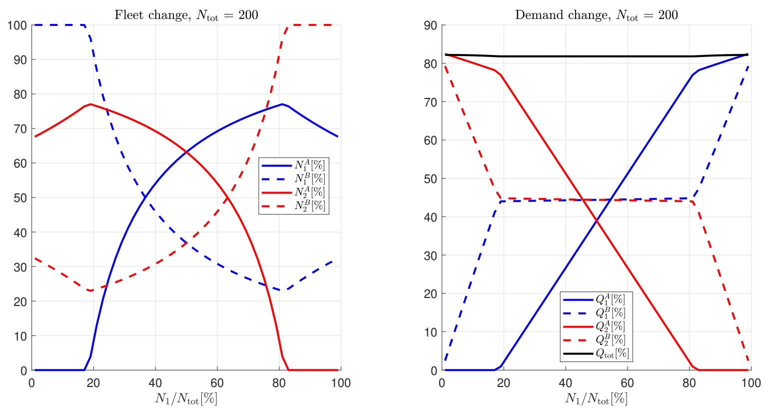


Figure 6: Nash equilibria and demand share with $N_{\text{tot}} = 200$

The last scenario we simulate is for $N_{\text{tot}} = 120$, where the limited supply leads to only 50% of the total potential demand being captured by the ride-hailing market. In this case, the smaller company focuses its operation solely in Region B until its fleet fraction reaches 40% of the total fleet. Interestingly, both companies choose to send a significant portion of their vehicles to Region B, despite Region A having a higher demand, which can be attributed to Region B's lower average travel time. It is also notable that when the two companies are of similar size, i.e. when $N_1/N_{\text{tot}} \approx$ 50%, they both allocate about 50% of their total fleet to each region. However, this dispatch strategies result in unhealthy competition as Q_{tot} at $N_1/N_{\text{tot}} = 50\%$ is lower than at $N_1/N_{\text{tot}} = 0\%$

or 100% (monopoly). Such unnecessary competition, though not immediately obvious, undeniably reduces the efficiency of the market.

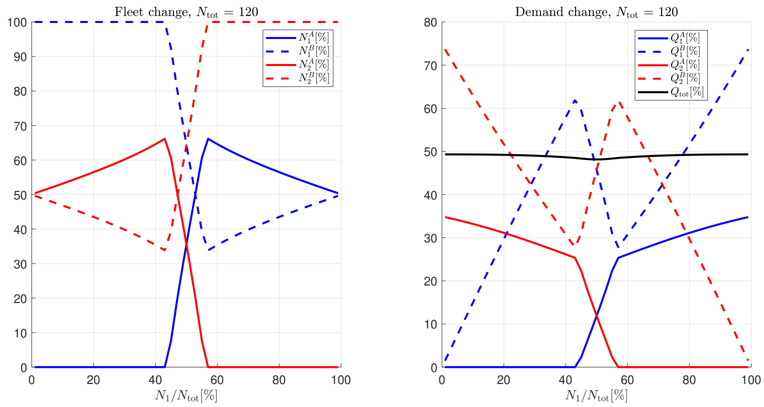


Figure 7: Nash equilibria and demand share with $N_{\text{tot}} = 120$

4 Conclusions

This study develops a comprehensive model to describe a ride-hailing (RH) market, where companies allocate their vehicle fleets across multiple regions to maximize the number of customers served. The problem is formulated as a classical multi-player non-cooperative game, with the actions of each agent (RH company) expected to reach a generalized Nash equilibrium. This equilibrium is established based on an aggregated flow conservation equation, which partitions the total vehicle flow into idling, deadheading, and traveling states. The conservation equation introduces a complicated coupling constraint, contributing to the non-convex nature of the optimization problem. Due to this non-convexity, there is no guarantee of a unique Nash equilibrium. Nevertheless, we propose an iterative algorithm to compute the equilibrium and test its uniqueness by initializing the algorithm with different conditions.

A numerical study using various simulations across different scenarios demonstrates the dynamics of a simple duopoly ride-hailing market, consisting of two regions with varying demand levels and average travel times. We empirically verify the convergence, discuss the uniqueness of the equilibria, and explore how the dispatch strategy of each company is affected by its fleet fraction in the market. The result show that the larger company typically dominates the high-demand region. while the smaller company tends to operate in the low-demand region, unless its fleet fraction surpasses a certain threshold. This threshold is found to increase as the overall fleet size in the market decreases. Eventually, we observe that unnecessary competition arises when the two companies have similar fleet sizes, as the total demand served in this case is lower than in a monopoly market.

For future work, We suggest first conducting a complete theoretical analysis of the generalized Nash equilibrium, proving its existence and (conditional) uniqueness. Furthermore, while the algorithm proposed here works decently for a simple two-company, two-region system, it becomes impractical for high-dimensional problems due to the non-convex optimization involved. Therefore, it is essential to explore more efficient methods tailored to this problem, avoiding direct solutions to the non-convex optimization. Additionally, further studies could consider dynamic or uncertain demand patterns and extend the static equilibrium analysis into a control problem, incorporating vehicle repositioning in markets with asymmetric demand distributions across time and space.

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