Analysis of a Maximum Throughput Ramp-metering Control

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Short summary

Ramp metering is critical for managing freeway traffic, yet no optimal control with theoretical guarantees exists in the current literature. With increasing market penetration rates of autonomous vehicles (AVs), methods lacking theoretical guarantees risk unpredictable traffic conditions. Max-pressure control has proven throughput-maximizing properties at signalized intersections with strong theoretical and experimental validation. However, it cannot be directly applied to ramp meters, particularly in multi-class traffic with evolving AV penetration. This paper develops a new max-pressure-based ramp metering control that ensures maximum throughput while considering multi-class traffic. We prove that if any theoretical controller can achieve maximum throughput, our controller can do so as well. Using a multi-class cell transmission model, we show how to approximate parameters for practical implementation while preserving the controller's throughput-maximizing properties.
Keywords: Ramp metering, Multi-class traffic flow approximation, max pressure algorithm.

1 Introduction

The growing integration of autonomous vehicles (AVs) in transportation networks brings about significant challenges and transformative opportunities for traffic management, particularly in complex scenarios such as highway on-ramp merges (H. Wang et al., 2021). Therefore, as the market penetration rate (MPR) of AVs grows on streets, developing robust control algorithms that can effectively manage interactions in mixed autonomy traffic of AVs and human-driven vehicles (HVs) becomes crucial for improving traffic safety and throughput.
In previous studies, traffic simulations were used to evaluate the impacts of partially automated vehicles with driver-assist features as well as fully automated vehicles (e.g., SAE Level 5) (Stern et al., 2018; Talebpour & Mahmassani, 2016; Shang & Stern, 2021) with differing net impacts to traffic capacity depending on the specific type of AVs being considered. Moreover, several ramp metering strategies have been proposed in the literature. Fixed time strategies are planned offline for a particular time of day based on historical demands (Wattleworth, 1965; Yuan & Kreer, 1971). This type of control is not adaptive to real-time traffic which raises the question of effectiveness compared to adaptive controls. Reactive ramp metering strategies contain demand-capacity and occupancy strategies (Masher et al., 1975), which are vulnerable to unmeasurable disturbances. The closed-loop strategy, ALINEA (Papageorgiou et al., 1991), has shown improvements in capacity, and a reduction in congestion time and travel times compared to previous strategies. Similarly to ALINEA, significant progress has been made in the ramp metering control strategy with the development of the corridor-coordinated version, which is known as METALINE (Papageorgiou et al., 1990). The effectiveness of METALINE has been tested in simulated studies as well as real-world applications (Taale et al., 1994). Building on the capabilities of ALINEA, Wang et. al (Y. Wang et al., 2014) introduced the proportional integral ALINEA (PI-ALINEA) to better identify congestion further downstream.
While ALINEA and its variants have demonstrated efficacy in addressing congestion issues at on-ramps, the exploration of alternative control algorithms has continued to evolve, meeting diverse

practical needs. Among these are the bottleneck algorithm (Jacobson et al., 1989), the HELPER algorithm (Lipp et al., 1991), the Zone algorithm (Stephanedes, 1994; Levinson & Zhang, 2006), and the fuzzy logic algorithm (Taylor et al., 1998), each contributing unique solutions to the challenges of ramp metering. Further strategies include nonlinear optimal ramp metering, which considers nonlinear traffic dynamics, limited ramp storage capacity, and long-term demand forecasting (Kotsialos et al., 2002). However, these studies do not provide theoretical proof of controls for maximum throughput and fail to account for the characteristics of mixed autonomy in their control strategies.

Varaiya has shown that max-pressure (MP) control can achieve maximum throughput across various networks with desirable theoretical guarantees (Varaiya, 2013a). While studies such as Sun and Yin (Sun & Yin, 2018), Barman and Levin (Barman & Levin, 2022), Robbennolt et al. (Robbennolt et al., 2022) have shown favorable results for different MP controls as traffic signal controls, its application to ramp metering remains unexplored. Traffic signal control and ramp metering deal with different network topologies and traffic dynamics. Freeway-specific traffic conditions are also different from traffic conditions near signalized intersections. Ramp metering controls need to balance how much flow from different movements can be allowed simultaneously. Therefore, we cannot use MP control directly for ramp metering based on the results for signalized intersections. This paper aims to develop a ramp metering control based on the max pressure algorithm, that provides theoretical proof of maximum throughput while accommodating a multi-class traffic flow. Moreover, recent studies by Shang, Wang, and Stern (Shang et al., 2023) have extended a theoretical control strategy to address mixed autonomy traffic. They have developed and simulated a ramp metering control algorithm to enhance highway throughput and travel time for both mainline and ramps with mixed autonomy traffic. While the simulation results indicate that the proposed control algorithm can improve traffic flow, it is still unclear whether the proposed algorithm theoretically ensures the maximum throughput for a multi-class traffic flow.

Therefore, we emphasize the following contributions of this study:

- Proposing a novel ramp metering strategy based on the max pressure algorithm that accounts for multi-class traffic.
- Presenting the theoretical proof of the efficiency, stability, and sensitivity of the network.

The remainder of this article is outlined as follows. First, we present the methodological approach to model traffic dynamics in the presence of multiple classes of vehicles. Then, we discuss the conditions under which a network can be stabilized or maximum throughput can be ensured. Next, we present the control strategy that ensures the maximum throughput of traffic. Finally, we show how we use a multi-class CTM model to approximate the traffic flow on ramps.

2 Methodology

We model the traffic network of off-ramps, on-ramps and freeways as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of the nodes and the set of all links or edges is given by $\mathcal{E} = \mathcal{E}_{\mathrm{r}} \cup \mathcal{E}_{\mathrm{h}} \cup \mathcal{E}_{\mathrm{s}}$. Here $\mathcal{E}_{\mathrm{r}}$, $\mathcal{E}_{\mathrm{h}}$ and $\mathcal{E}_{\mathrm{s}}$ are the sets of on-ramp, freeway and off-ramp links. For node v, denote the sets of incoming and outgoing links with $\Gamma^{-}(v) = \{(i,j) : v = j \ \forall (i,j) \in \mathcal{E}\}$ and $\Gamma^{+}(v) = \{(i,j) : v = i \ \forall (i,j) \in \mathcal{E}\}$, respectively. For link $e = (i,j)$, denote the sets of incoming and outgoing links with $\mathcal{I}(i,j) = \Gamma^{-}(i)$ and $\mathcal{O}(i,j) = \Gamma^{+}(j)$, respectively. Let $\mathcal{M} = \bigcup_{v \in \mathcal{V}} \mathcal{M}_v$, where the set of turning movements for a node v is $\mathcal{M}_v = \Gamma^{-}(v) \times \Gamma^{+}(v)$. We denote the set of entry links by $\mathcal{E}_{+} = \mathcal{E}_r \cup \mathcal{E}_h\{1\}$. Vehicles enter the network $\mathcal{G}$ using these entry links. Each entry link has an incoming virtual link that we model using the PQ link model. The set of virtual links is denoted by $\mathcal{E}^{\mathrm{src}}$ and $|\mathcal{E}^{\mathrm{src}}| = |\mathcal{E}_{+}|$. Vehicles exit the network $\mathcal{G}$ using the off-ramp links $\mathcal{E}_s$.

Multi-class traffic dynamics

Now we will discuss the traffic dynamics in the presence of multiple classes of vehicles. Denote the length of each link $i \in \mathcal{E}$ by l_i. Therefore, the highest upstream and lowest downstream positions of the link are $x = 0$ and $x = l_i$, respectively. We consider m different classes of vehicles in the set $C = \{c_1, c_2, \cdots, c_m\}$. Let $k^c_{ab}(x,t)$ and $q^c_{ab}(x,t)$ denote the density and flow rate for vehicle class $c \in C$ at time t along the link a at the position x destined to the link b. Now the combined density for all the different types of vehicles can be written as $k_{ab}(x,t) = \sum_{c \in C} k^c_{ab}(x,t)$. According to (Levin & Boyles, 2016) the class specific flow rate can be defined as the following:

$$q^c_{ab}(x,t) = u_a(k_1, k_2, \cdots, k_m)k^c_{ab}(x,t) \tag{1}$$

$$u_a(k_1, k_2, \cdots, k_m) = min\left\{u_f, \frac{q^{\max}\left(\frac{k_1}{k}, \cdots, \frac{k_m}{k}\right)}{k}, w\left(\frac{k_1}{k}, \cdots, \frac{k_m}{k}\right)\frac{k_j - k}{k}\right\} \tag{2}$$

Here capacity $q^{\max}\left(\frac{k_1}{k}, \cdots, \frac{k_m}{k}\right)$ and backward wave speed $w\left(\frac{k_1}{k}, \cdots, \frac{k_m}{k}\right)$ depend on the proportion of density in each class. The jam density k_j remains unchanged because it depends on the physical characteristics like length, maximum acceleration etc. of the vehicles. We assume that the different classes of vehicles all have similar physical characteristics.
The combined flow rates for movement $(a, b) \in \mathcal{M}$ and link $a \in \mathcal{E}$ now can be written as the following respectively.

$$q_{ab}(x,t) = \sum_{c \in C} q^c_{ab}(x,t) = u_a(k_1, k_2, \cdots, k_m)\sum_{c \in C} k^c_{ab}(x,t) = u_a(k_1, k_2, \cdots, k_m)k_{ab}(x,t) \tag{3}$$

$$q_a(x,t) = \sum_{b \in \mathcal{O}(a)} q_{ab}(t) \tag{4}$$

Here the freeflow speed for a link depends on the class proportions of that link. The densities for each class are also different, resulting in different flow rates for different classes of vehicles. Therefore, the fundamental diagram changes based on the proportion of different types of vehicles. The combined fundamental diagram describes the relationship between the total flow rate, free flow speed and total density for a link.
We denote the transition flow with $y_{ab}(t)$ for all nodes $v \in \mathcal{V}$, for all turning movements $(a, b) \in M_v$. This flow $y_{ab}(t)$ represents the number of vehicles that are waiting to leave the link a to enter the link b at time t. At node v at time t the flow $y_{ab}(t)$ where $(a, b) \in M_v$ is given by equation (5).

$$y_{ab}(t) = \begin{cases} \min\{S_{ab}(t), R_b(t)\} & \text{if } |\Gamma^-(v)| = |\Gamma^+(v)| = 1 \\ \phi \cdot P_{ab}(t) \cdot S_{ab}(t) & \text{if } 1 = |\Gamma^-(v)| < |\Gamma^+(v)| \\ \psi_{ab}(t) & \text{if } |\Gamma^-(v)| > |\Gamma^+(v)| = 1 \end{cases} \tag{5}$$

The first case considers the scenario when node v has only one incoming and one outgoing link. In this case, the flow rate is the minimum of the sending flow $S_{ab}(t)$ and the receiving flow $R_b(t)$. The second and third cases deal with diverging and merging nodes respectively. Details about the merge and diverge models can be found in (Daganzo, 1995) and (Boyles et al., 2020). We consider the merge and diverge conditions of two links: 1. freeway link and 2. either an on-ramp or an off-ramp link depending on merge or diverge conditions. For a divergegence of link a into two links b and b' we have $\phi = \min\left\{1, \frac{R_b(t)}{P_{ab}(t) \cdot S_{ab}(t)}, \frac{R_{b'}(t)}{P_{ab'}(t) \cdot S_{ab'}(t)}\right\}$ according to the diverge model (Boyles et al., 2020).
On-ramps entering a freeway can be modeled using a single merge of two incoming links into one outgoing link. Therefore, a two way merge model can capture this behavior without needing to consider n-way merges where $n > 2$. The transition flow for a merge of two links a' and a into link b is just $\psi_{ab}(t) = S_{ab}(t)$ if $S_{ab}(t) + S_{a'b}(t) \leq R_b(t)$ (uncongested merge condition). This is because for an uncongested merge, we can send all the sending flow from both links. However, for the congested merge case when $S_{ab}(t) + S_{a'b}(t) > R_b(t)$ the transition flow can be written as the following:

$$\psi_{ab}(t) = \text{med}\left\{\frac{q_a^{\max}\left(\frac{k_1}{k}, \cdots, \frac{k_m}{k}\right)R_b(t)}{q_a^{\max}\left(\frac{k_1}{k}, \cdots, \frac{k_m}{k}\right) + q_{a'}^{\max}\left(\frac{k_1}{k}, \cdots, \frac{k_m}{k}\right)}, R_b(t) - S_{a'b}(t), S_{ab}(t)\right\} \tag{6}$$

We focus on the congested condition for ramp metering. Denote $\delta_a^{\swarrow}(t)$ and $\delta_a^{\nearrow}(t)$ to indicate the ramp metering rates at the upstream and downstream of the link a at time t respectively. The functions $q_{ab}^{\swarrow}$ and $q_{ab}^{\nearrow}$ are the influx and outflux functions for link a destined to link $b \in \mathcal{O}(a)$.

$$q_{ab}^{\swarrow}(\delta_a^{\swarrow}(t)) = P_{ab}(t)\sum_{z \in \mathcal{I}(a)} q_{za}(\delta_a^{\swarrow}(t)) \tag{7}$$

$$q_{ab}^{\nearrow}(\delta_a^{\nearrow}(t)) = q_{ab}(\delta_a^{\nearrow}(t)) \tag{8}$$

Here eq. (7) describes the fact that, the metering rate upstream of link a will affect the flow rate entering a destined to b. Similarly, eq. (8) describes that the metering rate at downstream of link a will influence how much flow can exit from the queueing movement (a, b).

At the link boundaries, the conservation law is given by the following for the links $a \in \mathcal{E}$.

$$\frac{\partial k_{ab}(x,t)}{dt} = -\frac{\partial q_{ab}(x,t)}{dx} = \begin{cases} q^{\swarrow}_{ab}(\delta^{\swarrow}_{a}(t)) - y_{ab}(0,t) \text{ if } x = 0 \\ y_{ab}(l_a,t) - q^{\nearrow}_{ab}(\delta^{\nearrow}_{a}(t)) \text{ if } x = l_a \end{cases} \tag{9}$$

For all links $a \in \mathcal{E}^{\text{src}}$

$$\frac{\partial k_{ab}(x,t)}{dt} = -\frac{\partial q_{ab}(x,t)}{dx} = \frac{dA_{ab}(t)}{dt} - q^{\nearrow}_{ab}(\delta^{\nearrow}_{a}(t)) \tag{10}$$

The dependence on position for densities is not modeled because we model the virtual links with point queues. In the next section, we discuss the how vehicles enter the traffic network.

Exogenous Arrivals

Exogenous demand enters the network through the virtual links in $\mathcal{E}^{\text{src}}$. Let $A_{ab}(t) = \sum_{c\in C} A^{c}_{ab}(t)$ be a random arrival process with instantaneous rate $\lambda_{ab}(t) = E\left\{\frac{dA_{ab}(t)}{dt}\right\}$. The exogenous inflow rates to the virtual links are therefore $\frac{dA_{ab}(t)}{dt}$ for all $a \in \mathcal{E}^{\text{src}},\ b \in \mathcal{O}(a)$. The turning proportion from link i to link j at time t is given by $P_{ij}(t)$ such that $\sum_{j'\in\mathcal{I}(i)} P_{ij'}(t) = 1$ and $P_{ij}(t) \geq 0$ for all $(i,j) \in M_v$ for all nodes $v \in \mathcal{V}$. Like previous literature (Varaiya, 2013b; Levin et al., 2020; Barman & Levin, 2022), we also assume that the turn proportions $P(t)$ are independent and identically distributed (iid) random variables that depend on the route choices of the vehicles.

$$\bar{P}_{ij} = E\left\{P_{ij}(t)\right\} = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} P_{ij}(t) \tag{11}$$

Here $\bar{P}_{ij}$ is the average exogenous demand and the turning proportions. The next section will discuss the conditions under which a network can be stabilized or maximum throughput can be ensured.

Feasible Demand Region

Even the best theoretical ramp metering cannot support or serve unlimited demand. Demands or exogenous arrival processes that can be served by a controller, therefore, need to be defined. In this section, we define the demand that can possibly be served by some theoretical controller, which is also known as the feasible demand region in the literature. The feasible demand region consists of arrival processes with certain rates. These arrival processes with certain rates have this property where the best theoretical controller guarantees limited queue lengths in expectation. However, beyond these arrival processes or demands outside the feasible demand region, this guarantee no longer holds. Meaning that no guarantee exists for whether any controller can stop the queue lengths from growing to infinity, which implies congested queues.
Formally, we can write that for each network-wide metering rate vector $\boldsymbol{\delta} \in \{\underline{\boldsymbol{\delta}}, 1\}^{|\mathcal{E}_{\text{r}}|}$, the network can support arrival rate vector $\boldsymbol{\lambda}(\boldsymbol{\delta}) \in \mathbb{R}^{\mathcal{E}^{\text{src}}}_{+}$ if

$$\lim_{T\to\infty} \sum_{i\in\mathcal{E}\cup\mathcal{E}^{\text{src}}} \frac{1}{T} \int_{0}^{T} \left(\mathbb{1}_{\{i\in\mathcal{E}^{\text{src}}\}}\lambda_i(\boldsymbol{\delta}) + \mathbb{1}_{\{i\in\mathcal{E}\}} q^{\swarrow}_{i}(\boldsymbol{\delta}) - q^{\nearrow}_{i}(\boldsymbol{\delta}) \right) dt = 0 \tag{12}$$

Equation (12) implies that with network-wide metering rate vector $\boldsymbol{\delta}$ active, the total inflow is the sum of inflow due to exogenous arrivals for virtual links and inflow $q^{\swarrow}_{i}(\boldsymbol{\delta})$ for all other links. Furthermore, the left hand side evaluating to zero indicates that the total link outflow can accomodate the total link inflow in the long run. Therefore, any network-wide metering $\boldsymbol{\delta}$ for which eq. (12) is true is an admissible metering. Now if the initial conditions do not influence the performance because initially the network is not congested and therefore queue lengths are bounded then eq. (12) can be equivalently written as

$$\lim_{T\to\infty} \sum_{i\in\mathcal{E}^{\text{src}}} \frac{1}{T} \int_{0}^{T} \left(\lambda_i(\boldsymbol{\delta}) - q^{\nearrow}_{i}(\boldsymbol{\delta}) \right) dt = 0 \tag{13}$$

Now for each network-wide metering vector $\boldsymbol{\delta}$ for which eq. (12) and eq. (13) holds, define the set of all possible arrival rate vectors as $\Omega(\boldsymbol{\delta})$. Then the feasible demand region can be written as the union of all such arrival rates.

$$\mathcal{D} = \left\{ \Omega(\boldsymbol{\delta}) \quad \text{s.t.} \ \lim_{T\to\infty} \sum_{i\in\mathcal{E}^{\text{src}}} \frac{1}{T} \int_0^T \Big(\lambda_i(\boldsymbol{\delta}) - q_i^{\nearrow}(\boldsymbol{\delta}) \Big)\, dt = 0 \quad \forall \boldsymbol{\delta} \in \{\underline{\delta}, 1\}^{|\mathcal{E}_{\text{r}}|} \right\} \tag{14}$$

Here the feasible demand region contains all servable arrival rates.

Network stability

Let the state of the system be defined by the vector of class-specific traffic densities. Therefore, we can write the system state as $\mathbf{k}(t) = \{k_{ab}^c(\cdot, t) \ \forall c \in C \ \forall (a,b) \in M\}$. We assume that all entry links to the network has an infinite jam density and all internal links have finite jam densities.

Definition 1 (Stability)**.** *The network is stable if and only if there exists $\kappa < \infty$ such that*

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T E\left\{ \sum_{a\in\mathcal{E}^{\text{src}}, b\in\mathcal{O}(a)} k_{ab}(t) + \sum_{a\in\mathcal{E}, b\in\mathcal{O}(a)} \int_{x=0}^{l_a} k_{ab}(x,t) \right\} dt \leq \kappa \tag{15}$$

Equation (15) implies that the average densities from the ramp links $\mathcal{E}_{\text{r}}$ will be bounded if it holds and the queuing network will be stable. Now we define the Lypunov function that takes the system states to the positive reals in eq. (16).

$$\mathcal{L}(\mathbf{k}(t)) = \frac{1}{2} \sum_{a\in\mathcal{E}_+, b\in\mathcal{O}(a)} c_{ab} k_{ab}^2(t) + \frac{1}{2} \sum_{a\in\mathcal{E}_i, b\in\mathcal{O}(a)} c_{ab} \int_{x=0}^{l_a} \int_{x'=0}^{l_a} \left| \frac{l_a - x - x'}{l_a} \right| k_{ab}(x',t) k_{ab}(x,t) dx' dx \tag{16}$$

Proof of Stability

Each non-source link $a \in \mathcal{E}$ in the network $\mathcal{G}$ has a finite capacity by the condition $0 \leq \bar{q}_a < \infty$ and a finite jam density by the condition $0 \leq \bar{k}_a < \infty$. In real life, vehicles have finite speeds and positive lengths which imply finite time and space headways. Finite space and time headways in turn naturally imply finite capacities and jam densities. Lemma 1 formally presents the finiteness of capacities and the jam densities.

Lemma 1. *Let $a \in \mathcal{E}$ and suppose there exists constants $0 \leq \bar{q}_a < \infty$ and $0 \leq \bar{k}_a < \infty$ such that $\Pr\{q_{ab}(x,t) \leq \bar{q}_a\} = 1$ and $\Pr\{k_{ab}(x,t) \leq \bar{k}_a\} = 1$ for any $(a,b) \in \mathcal{M}$, any $x \in [0, l_a]$ and any $t \geq 0$. Then, there exist constants $0 \leq K_a^1 < \infty$ and $0 \leq K_a^2 < \infty$ such that, with probability 1,*

(i) for any $(x_1, x_2) \subseteq [0, l_a]$

$$E\left\{ q_{ab}(l_a, t) \int_{x_1}^{x_2} k_{ab}(x,t) dx \mid \boldsymbol{k}(t) \right\} \leq K_a^1 \tag{17}$$

(ii) for any $(x_1, x_2) \subseteq (0, l_a)$ and any $(x_3, x_4) \subseteq [0, l_a]$

$$-E\left\{ \int_{x_3}^{x_4} \int_{x_1}^{x_2} k_{ab}(x,t) \frac{\partial q_{ab}(x',t)}{\partial x} dx' dx \mid \boldsymbol{k}(t) \right\} \leq K_a^2 \tag{18}$$

Proof. The proof is similar to Lemma 2 in (Li & Jabari, 2019). □

A direct corollary from Lemma 1 is given in Corollary 1

Corollary 1. *Let $a \in \mathcal{E}$ and assume the probabilistic bounds of Lemma 1. Then, there exists a constant $0 \leq K < \infty$ such that*

$$\begin{aligned} K \geq & \sum_{a\in\mathcal{E}} \sum_{b\in\mathcal{O}(a)} c_{ab} \left[E\left\{ q_{ab}(l_a,t) \int_0^{l_a} \left| \frac{x}{l_a} \right| k_{ab}(x,t) dx \mid \boldsymbol{k}(t) \right\} \right] \\ & - E\left\{ \int_0^{l_a} \int_{0^+}^{l_a^-} \left| \frac{l_a - x - x'}{l_a} \right| k_{ab}(x,t) \frac{\partial q_{ab}(x',t)}{\partial x} dx' dx \mid \boldsymbol{k}(t) \right\} \end{aligned} \tag{19}$$

Theorem 1. *Assuming finite capacities and densities for each link $a \in \mathcal{E}$, arrival rates or demand in $\mathcal{D}$ and $E\left\{\mathcal{L}(\mathbf{k}(0))\right\} < k_c$ where $k_c < \infty$ network-wide metering control δ^* that can ensure strong stability of the network as per Equation* (15).

Proof. Lyapunov drift

$$
\begin{aligned}
E\left\{\frac{d\mathcal{L}(\mathbf{k}(t))}{dt} \;\Big|\; \mathbf{k}(t)\right\} = E&\left\{\sum_{a\in\mathcal{E}^{\mathrm{src}}}\sum_{b\in\mathcal{O}(a)} c_{ab}k_{ab}(t)\frac{dk_{ab}(t)}{dt} \;\Big|\; \mathbf{k}(t)\right\} \\
&+\frac{1}{2}E\left\{\sum_{a\in\mathcal{E}}\sum_{b\in\mathcal{O}(a)} c_{ab}\int_{x=0}^{l_a}\int_{x'=0}^{l_a}\left|\frac{l_a-x-x'}{l_a}\right|\frac{d}{dt}\left(k_{ab}(x',t)k_{ab}(x,t)\right)dx'dx \;\Big|\; \mathbf{k}(t)\right\}
\end{aligned}
\tag{20}
$$

Now

$$
\int_{x=0}^{l_a}\int_{x'=0}^{l_a}\left|\frac{l_a-x-x'}{l_a}\right|\frac{d}{dt}\left(k_{ab}(x',t)k_{ab}(x,t)dx'dx\right) \tag{21}
$$

$$
=\int_{x=0}^{l_a}\int_{x'=0}^{l_a}\left|\frac{l_a-x-x'}{l_a}\right|\left(k_{ab}(x',t)\frac{\partial k_{ab}(x,t)}{\partial t}+k_{ab}(x,t)\frac{\partial k_{ab}(x',t)}{\partial t}\right)dx'dx \tag{22}
$$

$$
=\int_{x=0}^{l_a}\int_{x'=0}^{l_a}\left|\frac{l_a-x-x'}{l_a}\right|\left(k_{ab}(x',t)\left(-\frac{\partial q_{ab}(x,t)}{\partial x}\right)+k_{ab}(x,t)\left(-\frac{\partial q_{ab}(x',t)}{\partial x}\right)\right)dx'dx \tag{23}
$$

$$
\begin{aligned}
&=-\int_{x=0}^{l_a}\int_{x'=0}^{l_a}\left|\frac{l_a-x-x'}{l_a}\right|\left(k_{ab}(x',t)\frac{\partial q_{ab}(x,t)}{\partial x}\right)dx'dx \\
&\quad-\int_{x=0}^{l_a}\int_{x'=0}^{l_a}\left|\frac{l_a-x-x'}{l_a}\right|\left(k_{ab}(x,t)\frac{\partial q_{ab}(x',t)}{\partial x}\right)dx'dx
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&=-\int_{x'=0}^{l_a}\int_{x=0}^{l_a}\left|\frac{l_a-x-x'}{l_a}\right|\left(k_{ab}(x,t)\frac{\partial q_{ab}(x',t)}{\partial x}\right)dx'dx \\
&\quad-\int_{x=0}^{l_a}\int_{x'=0}^{l_a}\left|\frac{l_a-x-x'}{l_a}\right|\left(k_{ab}(x,t)\frac{\partial q_{ab}(x',t)}{\partial x}\right)dx'dx
\end{aligned}
\tag{25}
$$

$$
=-2\int_{x=0}^{l_a}\int_{x'=0}^{l_a}\left|\frac{l_a-x-x'}{l_a}\right|k_{ab}(x,t)\frac{\partial q_{ab}(x',t)}{\partial x}dx'dx \tag{26}
$$

$$
\begin{aligned}
&=-2\left[\int_{x=0}^{l_a}\left|\frac{l_a-x}{l_a}\right|k_{ab}(x,t)\frac{\partial q_{ab}(0,t)}{\partial x}dx+\int_{x=0}^{l_a}\left|\frac{-x}{l_a}\right|k_{ab}(x,t)\frac{\partial q_{ab}(l_a,t)}{\partial x}dx\right. \\
&\quad\left.+\int_{x=0}^{l_a}\int_{x'=0^+}^{l_a^-}\left|\frac{l_a-x-x'}{l_a}\right|\left(k_{ab}(x,t)\frac{\partial q_{ab}(x',t)}{\partial x}\right)\right]
\end{aligned}
\tag{27}
$$

From Equation (20) we get

$$
\begin{aligned}
&E\left\{\frac{d\mathcal{L}(\mathbf{k}(t))}{dt} \;\Big|\; \mathbf{k}(t)\right\} &(28)\\
&= E\left\{\sum_{a\in\mathcal{E}^{\mathrm{src}}}\sum_{b\in\mathcal{O}(a)} c_{ab}k_{ab}(t)\frac{dk_{ab}(t)}{dt} \;\Big|\; \mathbf{k}(t)\right\}\\
&\quad+\frac{1}{2}E\left\{\sum_{a\in\mathcal{E}}\sum_{b\in\mathcal{O}(a)} c_{ab}\int_{x=0}^{l_a}\int_{x'=0}^{l_a}\left|\frac{l_a-x-x'}{l_a}\right|\frac{d}{dt}\left(k_{ab}(x',t)k_{ab}(x,t)\right)dx'dx \;\Big|\; \mathbf{k}(t)\right\} &(29)\\
&= E\left\{\sum_{a\in\mathcal{E}^{\mathrm{src}}}\sum_{b\in\mathcal{O}(a)} c_{ab}k_{ab}(t)\left[\frac{dA_{ab}(t)}{dt}-q^{\nearrow}_{ab}(\delta^{\nearrow}_a(t))\right] \;\Big|\; \mathbf{k}(t)\right\}\\
&-E\left\{\sum_{a\in\mathcal{E}}\sum_{b\in\mathcal{O}(a)} c_{ab}\left[\int_{x=0}^{l_a}\left|\frac{l_a-x}{l_a}\right|k_{ab}(x,t)\frac{\partial q_{ab}(0,t)}{\partial x}dx+\int_{x=0}^{l_a}\left|\frac{-x}{l_a}\right|k_{ab}(x,t)\frac{\partial q_{ab}(l_a,t)}{\partial x}dx\right.\right.\\
&\quad\left.\left.+\int_{x=0}^{l_a}\int_{x'=0^+}^{l_a^-}\left|\frac{l_a-x-x'}{l_a}\right|\left(k_{ab}(x,t)\frac{\partial q_{ab}(x',t)}{\partial x}\right)\right] \;\Big|\; \mathbf{k}(t)\right\} &(30)\\
&= E\left\{\sum_{a\in\mathcal{E}^{\mathrm{src}}}\sum_{b\in\mathcal{O}(a)} c_{ab}k_{ab}(t)\left[\frac{dA_{ab}(t)}{dt}-q^{\nearrow}_{ab}(\delta^{\nearrow}_a(t))\right] \;\Big|\; \mathbf{k}(t)\right\}\\
&-E\left\{\sum_{a\in\mathcal{E}}\sum_{b\in\mathcal{O}(a)} c_{ab}\left[\int_{x=0}^{l_a}\left|\frac{l_a-x}{l_a}\right|k_{ab}(x,t)\left(-q^{\swarrow}_{ab}(\delta^{\swarrow}_a(t))+y_{ab}(0,t)\right)\right.\right.\\
&+\int_{x=0}^{l_a}\left|\frac{-x}{l_a}\right|k_{ab}(x,t)\left(-y_{ab}(l_a,t)+q^{\nearrow}_{ab}(\delta^{\nearrow}_a(t))\right)+\left.\left.\int_{x=0}^{l_a}\int_{x'=0^+}^{l_a^-}\left|\frac{l_a-x-x'}{l_a}\right|\left(k_{ab}(x,t)\frac{\partial q_{ab}(x',t)}{\partial x}\right)\right] \;\Big|\; \mathbf{k}(t)\right\} &(31)\\
& &(32)
\end{aligned}
$$

Using Corollary 1 we have that there exists a constant $0<K<\infty$ such that

$$
\begin{aligned}
&E\left\{\frac{d\mathcal{L}(\mathbf{k}(t))}{dt} \;\Big|\; \mathbf{k}(t)\right\} &(33)\\
&\le K+\sum_{a\in\mathcal{E}^{\mathrm{src}}}\sum_{b\in\mathcal{O}(a)} E\left\{c_{ab}k_{ab}(t)\frac{dA_{ab}(t)}{dt}-c_{ab}k_{ab}(t)q^{\nearrow}_{ab}(\delta^{\nearrow}_a(t)) \;\Big|\; \mathbf{k}(t)\right\}\\
&\quad-\sum_{a\in\mathcal{E}}\sum_{b\in\mathcal{O}(a)} c_{ab}E\left\{\int_{x=0}^{l_a}\left|\frac{x}{l_a}\right|k_{ab}(x,t)q^{\nearrow}_{ab}(\delta^{\nearrow}_a(t))dx-\int_0^{l_a}\left|\frac{l_a-x}{l_a}\right|k_{ab}(x,t)q^{\swarrow}_{ab}(\delta^{\swarrow}_a(t)) \;\Big|\; \mathbf{k}(t)\right\} &(34)
\end{aligned}
$$

The terms without the control variables can be reduced to a constant whose value is acounted for in K.

$$
\begin{aligned}
E\left\{\frac{d\mathcal{L}(\mathbf{k}(t))}{dt} \;\Big|\; \mathbf{k}(t)\right\} \le K&-\sum_{a\in\mathcal{E}^{\mathrm{src}}}\sum_{b\in\mathcal{O}(a)} k_{ab}(t)E\left\{c_{ab}q^{\nearrow}_{ab}(\delta^{\nearrow}_a(t))-c_{ab}\frac{dA_{ab}(t)}{dt} \;\Big|\; \mathbf{k}(t)\right\}\\
&-\sum_{a\in\mathcal{E}}\sum_{b\in\mathcal{O}(a)}\int_{x=0}^{l_a}k_{ab}(x,t)\left[c_{ab}\left|\frac{x}{l_a}\right|q^{\nearrow}_{ab}(\delta^{\nearrow}_a(t))-c_{ab}\left|\frac{l_a-x}{l_a}\right|q^{\swarrow}_{ab}(\delta^{\swarrow}_a(t))\right]dx &(35)
\end{aligned}
$$

K is a constant so the Lyapunov drift can be minimized by selecting a control that can minimize the right hand side of eq. (35). If demand is in the feasible demand region then by definition we have for each $t \geq 0$ that there exists $\epsilon^* > 0$ such that

$$\epsilon^* \left(\sum_{a\in\mathcal{E}^{\text{src}}} \sum_{b\in\mathcal{O}(a)} E\left\{k_{ab}(t)\right\} + \sum_{a\in\mathcal{E}} \sum_{b\in\mathcal{O}(a)} E\left\{ \int_0^{l_a} k_{ab}(x,t)dx \right\} \right) \tag{36}$$

$$\leq \max_{\boldsymbol{\delta}\in[\underline{\boldsymbol{\delta}},1]^{|\mathcal{E}_r|}} \sum_{a\in\mathcal{E}^{\text{src}}} \sum_{b\in\mathcal{O}(a)} E\left\{k_{ab}(t)\right\} E\left\{ c_{ab} q_{ab}^{\nearrow}(\delta_a^{\nearrow}(t)) - c_{ab}\frac{dA_{ab}(t)}{dt} \mid \mathbf{k}(t) \right\}$$

$$+ \sum_{a\in\mathcal{E}} \sum_{b\in\mathcal{O}(a)} E\left\{ \int_{x=0}^{l_a} k_{ab}(x,t) \left[c_{ab}\left|\frac{x}{l_a}\right| q_{ab}^{\nearrow}(\delta_a^{\nearrow}(t)) - c_{ab}\left|\frac{l_a - x}{l_a}\right| q_{ab}^{\swarrow}(\delta_a^{\swarrow}(t)) \right] \right\} \tag{37}$$

In other words, if demand is within the feasible demand region, it is possible to select network-wide metering δ^* so that, the difference of the outflow and inflow is always non-negative. The control will be discussed in more detail in the next section. Therefore, eq. (35) can be written as follows where $0 < K^* < \infty$.

$$E\left\{ \frac{d\mathcal{L}(\mathbf{k}(t))}{dt} \mid \mathbf{k}(t) \right\} \leq K^* - \epsilon^* \left(\sum_{a\in\mathcal{E}^{\text{src}}} \sum_{b\in\mathcal{O}(a)} E\left\{k_{ab}(t)\right\} + \sum_{a\in\mathcal{E}} \sum_{b\in\mathcal{O}(a)} E\left\{ \int_0^{l_a} k_{ab}(x,t)dx \right\} \right) \tag{38}$$

Now we integrate both sides from 0 to T and take the expectation of both sides to get

$$E\left\{ \int_0^T E\left\{ \frac{d\mathcal{L}(\mathbf{k}(t))}{dt} \mid \mathbf{k}(t) \right\} dt \right\}$$

$$\leq K^*T - \epsilon^* \int_0^T E\left\{ \left(\sum_{a\in\mathcal{E}^{\text{src}}} \sum_{b\in\mathcal{O}(a)} E\left\{k_{ab}(t)\right\} + \sum_{a\in\mathcal{E}} \sum_{b\in\mathcal{O}(a)} E\left\{ \int_0^{l_a} k_{ab}(x,t)dx \right\} \right) \right\} dt \tag{39}$$

$$\implies E\left\{\mathcal{L}(\mathbf{k}(T))\right\} - E\left\{\mathcal{L}(\mathbf{k}(0))\right\}$$

$$\leq K^*T - \epsilon^* \int_0^T E\left\{ \left(\sum_{a\in\mathcal{E}^{\text{src}}} \sum_{b\in\mathcal{O}(a)} E\left\{k_{ab}(t)\right\} + \sum_{a\in\mathcal{E}} \sum_{b\in\mathcal{O}(a)} E\left\{ \int_0^{l_a} k_{ab}(x,t)dx \right\} \right) \right\} dt \tag{40}$$

$$\tag{41}$$

Now dividing both sides by $T\epsilon^*$ we get

$$\frac{1}{T}\int_0^T E\left\{ \left(\sum_{a\in\mathcal{E}^{\text{src}}} \sum_{b\in\mathcal{O}(a)} E\left\{k_{ab}(t)\right\} + \sum_{a\in\mathcal{E}} \sum_{b\in\mathcal{O}(a)} E\left\{ \int_0^{l_a} k_{ab}(x,t)dx \right\} \right) \right\} dt \leq \frac{K^*}{\epsilon^*} + \frac{1}{T\epsilon^*} E\left\{\mathcal{L}(\mathbf{k}(0))\right\} \tag{42}$$

Since $E\left\{\mathcal{L}(\mathbf{k}(0))\right\}$ is bounded because initially the network contains bounded queues, eq. (42) implies stability as per Equation (15).

□

3 Maximum Throughput Max-pressure Ramp Metering (MTMPR) Control

As mentioned before to ensure stability we have maximize the following

$$\max_{\boldsymbol{\delta}\in[\underline{\boldsymbol{\delta}},1]^{|\mathcal{E}_\text{r}|}} \sum_{a\in\mathcal{E}^{\text{src}}} \sum_{b\in\mathcal{O}(a)} c_{ab} k_{ab}(t) q_{ab}^{\nearrow}(\delta_a^{\nearrow}(t))$$

$$+ \sum_{a\in\mathcal{E}} \sum_{b\in\mathcal{O}(a)} c_{ab} \int_{x=0}^{l_a} k_{ab}(x,t) \left[\left|\frac{x}{l_a}\right| q_{ab}^{\nearrow}(\delta_a^{\nearrow}(t)) - c_{ab}\left|\frac{l_a - x}{l_a}\right| q_{ab}^{\swarrow}(\delta_a^{\swarrow}(t)) \right] \tag{43}$$

Since the metering rates directly affect the transition flows moving from one link to the next we can replace $q_{ab}(\cdot)$ by $y_{ab}(\cdot)$ indicating that $y_{ab}(\cdot)$ flow will be moved by the controller. Therefore, the optimal control becomes

$$\boldsymbol{\delta}^* = \underset{\boldsymbol{\delta}\in[\underline{\delta},1]^{|\mathcal{E}_r|}}{\arg\max} \sum_{a\in\mathcal{E}^{\rm src}} \sum_{b\in\mathcal{O}(a)} c_{ab}k_{ab}(t)y_{ab}(\delta_a^{\nearrow}(t))$$

$$+\sum_{a\in\mathcal{E}_i}\sum_{b\in\mathcal{O}(a)} c_{ab}\int_{x=0}^{l_a} k_{ab}(x,t)\left[\left|\frac{x}{l_a}\right| y_{ab}(\delta_a^{\nearrow}(t)) - c_{ab}\left|\frac{l_a-x}{l_a}\right|\sum_{c\in\mathcal{I}(a)} P_{ca}(t)y_{ca}(\delta_a^{\swarrow}(t))\right] \quad (44)$$

Given the network state $\mathbf{k}(t)$ the transition flows can be calculated accurately. Therefore, we can rewrite the optimal control as follows

$$\boldsymbol{\delta}^*(t) = \underset{\boldsymbol{\delta}\in[\underline{\delta},1]^{|\mathcal{E}_r|}}{\arg\max} \sum_{a\in\mathcal{E}_r}\sum_{b\in\mathcal{O}(a)} w_{ab}(t)E\left\{y_{ab}(\delta_a)\middle|\boldsymbol{k}(t)\right\} \quad (45)$$

where $y_{ab}(t) = E\left\{y_{ab}(\delta_a(t))\middle|\boldsymbol{k}(t)\right\}$ and

$$w_{ab}(t) = \begin{cases} \left|c_{ab}k_{ab}(t) - \int_0^{l_b}\left|\frac{l_b-x}{l_b}\right|\sum_{c\in\mathcal{O}(b)} c_{bc}P_{bc}(t)k_{bc}(x,t)dx\right| & \forall a\in\mathcal{E}^{\rm src}, b\in\mathcal{O}(a) \\ \left|c_{ab}\int_0^{l_a}\left|\frac{x}{l_a}\right|k_{ab}(x,t)dx - \int_0^{l_b}\left|\frac{l_b-x}{l_b}\right|\sum_{c\in\mathcal{O}(b)} c_{bc}P_{bc}(t)k_{bc}(x,t)dx\right| & \forall a\in\mathcal{E}, b\in\mathcal{O}(a) \end{cases} \quad (46)$$

In the next section, we will discuss how to approximate the weights using a multi-class CTM model for practical implementation. We will also show that using the approximation will not affect stability properties.

4 Weight Approximation using CTM

Multi-class CTM

We use the multi-class CTM model described in (Levin & Boyles, 2016). We discretize time into timesteps of $dt = 1$ unit and link into cells labeled by $i = 1, \cdots, F$. Let $n_a^c(i,t)$ be the number of class c vehicle entering cell i from cell $i-1$ at time t for link $a\in\mathcal{E}$. The cell occupancy can now be written as

$$n_a^c(i,t+1) = n_a^c(i,t) + y^c(i,t) - y^c(i+1,t) \quad (47)$$

According to (Levin & Boyles, 2016), the class-specific transition flows can be written as

$$y_a^c(i,t) = \min\left\{n_a^c(i-1,t), \frac{n_a^c(i-1,t)}{n_a(i-1,t)}q_a^{\max}\left(\frac{k_1}{k},\cdots,\frac{k_m}{k}\right), \frac{n_a^c(i-1,t)}{n_a^c(i-1,t)}\frac{w_a(i,t)}{u_f}\left(N - \sum_{c\in C} n_a^c(i,t)\right)\right\} \quad (48)$$

The sending flow from a to $b\in\mathcal{O}(a)$ at time t is

$$S_{ab}(t) = \begin{cases} \min\left\{\delta_{ab}(t)\cdot q_a^{\max}, n_a(F,t)\right\} & \text{if } a\in\mathcal{E}_{\rm r} \\ \min\left\{q_a^{\max}, n_a(F,t)\right\} & \text{if } a\in\mathcal{E}_{\rm h} \end{cases} \quad (49)$$

where $\delta_{ab}(t)\in[\underline{\delta},1]$ is the rate of activation of the ramp meter. The $\delta_{ab}(t)$ represents how much of the maximum flow q_a can be allowed to enter the freeway link b during a time step starting at time t. The ramp metering control $\delta_{ab}(t)$ has a lower bound $\underline{\delta}$ since the ramp meter control is required to be not completely deactivated. The receiving flow for b at time t is

$$R_b(t) = \begin{cases} q_b^{\max} & \text{if } b\in\mathcal{E}_{\rm s} \\ \min\left\{q_b^{\max}, \frac{w_b(0,t)}{u_{\rm f}}\left[N_b - n_b^c(0,t)\right]\right\} & \text{if } b\in\mathcal{E}_{\rm h} \end{cases} \quad (50)$$

Here N_b is the maximum number of vehicles that can physically fit in link b and $n_b(0,t)$ is the total number of vehicles in the first cell of link b. $w_b(0,t)$ and $u_{\rm f}$ are the backward wave speed for the first cell at time t and the free flow speed respectively. Given $\boldsymbol{k}(t)$ we can calculate the transition flows using the CTM model. Therefore, we can write $y_{ij}(t) = E\{y_{ij}(t)|\boldsymbol{k}(t)\}$. However, the usage of CTM for the approximation requires updating the weight function.

$$\hat{w}_{ab}(t) = \begin{cases} \left| c_{ab}k_{ab}(t) - \sum_{i=0}^{F} \left(\int_i^{i+\Delta x} \left| \frac{l_b - x}{l_b} \right| dx \right) \sum_{c \in \mathcal{O}(b)} c_{bc}P_{bc}(t)k_{bc}(i,t) \right| & \forall a \in \mathcal{E}^{\text{src}}, b \in \mathcal{O}(a) \\ \left| c_{ab} \sum_{i=0}^{F} \left(\int_i^{i+\Delta x} \left| \frac{x}{l_a} \right| dx \right) k_{ab}(x,t)dx - \hat{w}_b(t) \right| & \forall a \in \mathcal{E}, b \in \mathcal{O}(a) \end{cases} \tag{51}$$

where

$$\hat{w}_b(t) = \sum_{i=0}^{F} \left(\int_i^{i+\Delta x} \left| \frac{l_b - x}{l_b} \right| dx \right) \sum_{c \in \mathcal{O}(b)} c_{bc}P_{bc}(t)k_{bc}(x,t)dx \tag{52}$$

Now the control for the congested case can be written as the following optimization program.

$$\max \quad y_{ij}(t)\hat{w}_{ij}(t) + y_{i'j}(t)\hat{w}_{i'j}(t) \tag{53a}$$

$$\text{s.t.} \quad y_{ij}(t) = \text{med}\left\{ \lambda_1^{ij}(t), \lambda_2^{ij}(t), \lambda_3^{ij}(t) \right\} \tag{53b}$$

$$y_{i'j}(t) = \min\left\{ q_j, \frac{w}{u_{\text{f}}}(N_j - x_j(t)) \right\} - y_{ij}(t) \tag{53c}$$

$$\lambda_1^{ij}(t) = \frac{\delta_{ij}(t)q_i}{\delta_{ij}(t)q_i + q_{i'}} \cdot \min\left\{ q_j, \frac{w}{u_{\text{f}}}(N_j - x_j(t)) \right\} \tag{53d}$$

$$\lambda_2^{ij}(t) = \min\left\{ q_j, \frac{w}{u_{\text{f}}}(N_j - x_j(t)) \right\} - \min\{x_{i'j}(t),\ q_{i'}\} \tag{53e}$$

$$\lambda_3^{ij}(t) = \min\{x_{ij}(t),\ \delta_{ij}(t)q_i\} \tag{53f}$$

$$\delta_{ij}(t) \in [\underline{\delta}, 1] \tag{53g}$$

Theorem 2. *There exists $\kappa < \infty$ such that*

$$\Big| \sum_{(a,b)\in M} \hat{w}_{ab}(t)y_{ab}(\hat{\delta}(t)) - \sum_{(a,b)\in M} w_{ab}(t)E\left\{ y_{ab}(\delta^*(t)) \Big| \boldsymbol{k}(t) \right\} \Big| \leq \kappa \tag{54}$$

Proof. ***Case 1:***

For all $a \in \mathcal{E}^{\text{src}}$

$$\Bigg| \left(-\sum_{(b,c)\in M} c_{bc}P_{bc}^2(t) \right) \sum_{i=0}^{F} \left(\int_i^{i+\Delta x} \left| \frac{l_b - x}{l_b} \right| dx \right) \hat{k}_b(i,t)y_{ab}(\hat{\delta}(t)) - \left(-\sum_{(b,c)\in M} c_{bc}P_{bc}^2(t) \right) \int_0^{l_b} \left| \frac{l_b - x}{l_b} \right| k_b(x,t)y_{ab}(\delta^*(t))dx \Bigg| \tag{55}$$

To simplify, let's consider the worst case where $q_{ab}(\delta(\hat{t})) = 0$

$$\Bigg| 0 - \left(-\sum_{(b,c)\in M} c_{bc}P_{bc}^2(t) \right) \int_0^{l_b} \left| \frac{l_b - x}{l_b} \right| k_b(x,t)y_{ab}(\delta^*(t))dx \Bigg| \tag{56}$$

$$\leq \Bigg| \sum_{(b,c)\in M} c_{bc}P_{bc}^2(t) \int_0^{l_b} \left| \frac{l_b - x}{l_b} \right| k_b^{\text{jam}}(x,t)dx\, q_{ab}^{\max} \Bigg| \leq \kappa_1 \tag{57}$$

Here $y_{ab}(\cdot) \leq q_{ab}^{\max}$, $\hat{k}_b(i,t) \leq k_b^{\text{jam}}(i,t)$, $k_b(x,t) \leq k_b^{\text{jam}}(x,t)$, $P_{bc}(t) \in [0,1]$ and $c_{bc} < \infty$. Here all the variables have a finite upper bound. Therefore, the left hand side terms of eq. (57) can be upper bounded by some finite constant κ_1.

Case 2:

Similarly, each variable has a finite upper bound. Therefore, the difference is bounded. □

Theorem 2 shows that even if the weight functions are approximated using the CTM model, the difference between the objective functions remains bounded. Therefore, using the approximated objective function still keeps the Lyapunov drift bounded implying that the network will be stable under this control.

5 Conclusions

This study employs the max pressure algorithm to develop a novel ramp metering control strategy designed to manage mixed autonomy traffic effectively. We show that the proposed control strategy successfully stabilizes the network if demand or exogenous arrivals are within the feasible demand region. Similarly to other methods based on max pressure (Varaiya, 2013b; Wongpiromsarn et al., 2012; Li & Jabari, 2019; Levin, 2023; Barman & Levin, 2023), if the demand is not in the feasible demand region, then no controller can stabilize the network. This implies that our controller can achieve maximum throughput. For practical implementation of this ramp metering control, it is essential to incorporate parameters such as turning proportions, capacities of turning movements, and real-time queue length data. Turning proportions and movement capacities can typically be derived from historical traffic data, while queue length information can be gathered through loop detectors. Moreover, including real-time adjustments based on dynamic traffic conditions enhances the adaptability and efficiency of the control strategy.

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