Improving Travel Time Reliability with Variable Speed Limits

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Short summary

This paper investigates how variable speed limits can optimize travel time reliability for commuters. We focus on a traffic corridor with a bottleneck subject to the capacity drop phenomenon. Traffic demand during peak hours is modeled as a stochastic variable, with the optimization objective being a linear combination of the expected value and standard deviation of total travel time. To solve this problem, we develop a three-stage optimization algorithm based on the kinematic wave model of traffic flow (Lighthill & Whitham (1955), Richards (1956)). We demonstrate its practical applicability using a numerical example calibrated with empirical data. In contrast to existing approaches, our method provides an exact solution within the framework of the kinematic wave model without requiring prior discretization of space and time. This enables the determination of actually optimal positions and timing for the implementation of speed limits.
Keywords: Capacity Drop, Kinematic Wave Model, Traffic Control, Travel Time Reliability, Variable Speed Limit.

1 Introduction

Variable Speed Limit (VSL) systems are dynamic traffic management tools that adjust posted speed limits in response to real-time traffic conditions through overhead or roadside display signs. The review paper by Khondaker & Kattan (2015b) summarizes the operational benefits of traditional VSL applications as follows: improved safety, prevention of traffic breakdown, and increase of throughput at bottlenecks. The latter two are possible due to the capacity drop phenomenon, where the maximum flow rate at an active bottleneck can drop below the theoretical capacity once congestion sets in (Banks (1990), Hall & Agyemang-Duah (1991) - a reduction that VSL systems can help prevent by strategically slowing down approaching traffic. Modern VSL control strategies (e.g. Carlson et al. (2010), Carlson et al. (2011), H.-Y. Jin & Jin (2015), Khondaker & Kattan (2015a)) are based on predictive modeling of traffic behavior. They employ a closed-loop feedback control mechanism to continuously update its predictions and control actions based on new traffic data. These predictions typically rely on macroscopic traffic flow models, primarily approximations of Payne's model (Payne (1971)) and the kinematic wave model.

Much less attention has been paid to the potential of VSL systems for improving travel time reliability. The reliability of travel time, commonly measured by its standard deviation or variance, is nearly as crucial to commuters as the expected travel time itself Prato et al. (2014). To our knowledge, the optimization of VSL systems to improve travel time reliability has not been investigated in existing literature. This paper addresses this research gap and develops methods for specifically optimizing VSL strategies for this purpose. Our analysis is based on the assumption that traffic inflow during peak hours follows a known probability distribution, which can be estimated from historical data. Under this premise, we solve the optimization problem

$$J = \alpha \cdot \mathbb{E}[\mathcal{T}] + (1-\alpha) \cdot \mathrm{Std}[\mathcal{T}], \tag{1}$$

$$\min J. \tag{2}$$

where $\alpha \in [0,1]$ is a weighting parameter, and $\mathcal{T}$ represents the total travel time, defined for a given peak flow q_p^0 as $\mathcal{T}(q_p^0) = \int_0^T \int_0^l k(x,t)\,dx\,dt$. Here, T denotes the length of the optimization interval.

2 Results and discussion

The contemporary formulation of the LWR theory can be summarized as follows. The rate of change in the total number of vehicles contained in any road segment $[x_1, x_2]$ where $x_2 > x_1$ is equal to the net flow of vehicles out of the segment, i.e.

$$\frac{d}{dt}\int_{x_1}^{x_2} k(x,t)\,dx = -\left[q(x,t)\right]_{x_1}^{x_2}, \tag{3}$$

Since this relation is valid for any arbitrary road segment $[x_1, x_2]$, by letting $x_2 \to x_1$ and dividing by the segment's length, the expression simplifies to the partial differential equation

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{4}$$

In addition to 3, the LWR theory assumes the existence of a functional relationship between q and k under differentiable conditions:

$$q(x,t) = Q(x, k(x,t)), \tag{5}$$

where Q is a concave, non-negative function that is equal to zero at $k = 0$ and at the *jam density* $k = k_j$. Flow and density are related to the cumulative flow $N(x,t)$ as follows:

$$q(x,t) = \frac{\partial N}{\partial t}(x,t), \quad k(x,t) = -\frac{\partial N}{\partial x}(x,t) \tag{6}$$

In cases where k has a discontinuity at (x,t), known as a shockwave, the shockwave's speed u is specified as:

$$u = \frac{[q]}{[k]} = \frac{q_2 - q_1}{k_2 - k_1}. \tag{7}$$

The third variable, the *average speed*, is defined as $v = \frac{q}{k}$. For the homogeneous LWR model analyzed in this paper, these conditions determine a unique solution for the functions $k(x,t)$ and $q(x,t)$ when initial and boundary conditions are given. Traffic moves along a road segment of length l, which ends in a bottleneck with a maximum capacity q_{bn}. When congestion forms at the bottleneck, its discharge capacity decreases by Δ percent. We model capacity drop according to the approach by W.-L. Jin et al. (2015):

$$q(l,t) = \begin{cases} d(l^-,t), & d(l^-,t) \le s(0^+,t) \\ \min\{s(l^+,t), q_{bn}^*\}, & d(l^-,t) > s(l^+,t) \end{cases} \tag{8}$$

where $d(l^-,t)$ is the upstream demand at the bottleneck location, $s(l^+,t)$ the downstream supply, and $q_{bn}^* = (1-\Delta)\cdot q_{bn}$ the dropped capacity. The travel time $\tau(t)$ for a vehicle entering the segment at time t is described by:

$$\tau(t) = \inf\{T \ge 0 : N(l, t+T) > N(0,t)\}. \tag{9}$$

To model rush hour traffic, the upstream boundary flow $q(0,t)$ is represented as a trapezoidal function with a randomly distributed peak $q_p \sim \phi$, defined as:

$$q(0,t) = \begin{cases} q_b + a \cdot t, & \text{for } 0 \le t \le \frac{q_p - q_b}{a}, \\ q_p - b \cdot (t_e - t), & \text{for } \frac{q_p - q_b}{a} \le t \le \frac{q_p}{q_e \cdot b} + \frac{q_p - q_b}{a}, \\ q_e, & \text{for } \frac{q_p}{q_e \cdot b} + \frac{q_p - q_b}{a} \le t \le \infty. \end{cases} \tag{10}$$

for suitably chosen parameters q_b (initial flow), q_e (end flow), a (flow increase rate at the onset of congestion), and b (flow reduction rate at the offset of congestion). We further assume that $q_e < q_{bn}$, so that the expected travel time for very late departure times, as $t_{\text{dep}} \to \infty$, approaches the free flow travel time $\tau_{\text{free}} = \frac{l}{v(0)}$.

In the first optimization step, we find the minimum total travel time for a given peak flow q_p^0, regardless of the control strategy. This is equivalent to maximizing the downstream flow $q(l,t)$.

We define $N^+(l,t)$ as the number of vehicles that would pass position l by time t without a bottleneck, and $q^+(l,t)$ as the corresponding flow. We calculate $q^+(l,t)$ by identification of the latest emanating kinematic wave from upstream which intersects (l,t) (see Hammerl et al. (2024) for justification and details), and obtain $N^+(l,t)$ through integration:

$$N^+(l,t) = \int_0^t q(l,\tau)\,d\tau. \tag{11}$$

The actual vehicle count $N_{\min}(l,t)$ is determined using $q^+(l,t)$ as arrival rate and $q(l,t)$ as departure rate in a $D/D/1$ queue with service rate q_{bn} (c.f. Newell (1993)), as illustrated in figure 1. The minimum total travel time is then:

$$\mathcal{T}_{\min}(q_p^0) = \int_0^N \left(N_{\min}^{-1}(l,t) - N_{\min}^{-1}(0,t)\right)\, dt. \tag{12}$$

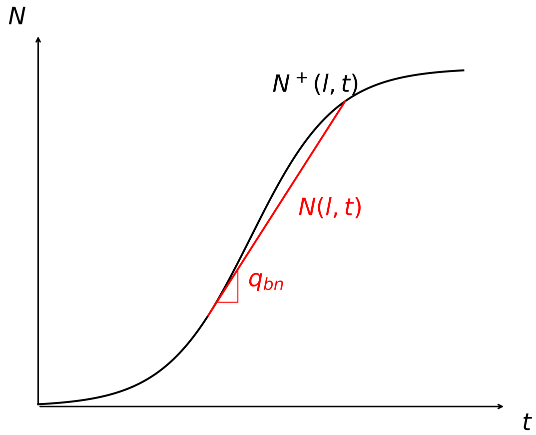

Figure 1: Graphical determination of $N(l,t)$ for given $N^+(l,t)$.

In the second stage, we solve the stochastic variational optimization problem 1 subject to the additional constraint: $\mathcal{T}(q_p) \geq \mathcal{T}_{\min}(q_p) \quad \forall q_p \in \operatorname{supp}(\phi)$. To solve 1, we first discretize the support of the probability distribution, which is then extended to continuous distributions by considering the corresponding limit for $n \to \infty$:

$$\operatorname{supp}(\phi) = \{q_p^0, q_p^1, \ldots, q_p^n\}$$

For this discrete case, the problem can be formulated as a quadratic optimization problem with constraints using Lagrange multipliers:

$$\begin{aligned}\mathcal{L}(\mathcal{T}(q_p^1), \ldots, \mathcal{T}(q_p^n), \lambda_1, \ldots, \lambda_n) = \alpha \cdot & \sum_{i=1}^{n} \left(p(q_p^i) \cdot \mathcal{T}(q_p^i)\right) \\ &+ (1-\alpha) \cdot \sqrt{\sum_{i=1}^{n} p(q_p^i) \cdot \left(\mathbb{E}[\mathcal{T}] - \mathcal{T}(q_p^i)\right)^2} \\ &+ \sum_{i=1}^{n} \lambda_i \cdot \left(\mathcal{T}_{\min}(q_p^i) - \mathcal{T}(q_p^i)\right).\end{aligned}$$

Without loss of generality, let the indices be chosen such that the lower bounds are monotonically increasing, i.e., $\mathcal{T}\min(q_p^i) \leq \mathcal{T}\min(q_p^{i+1})$ for all i. Since both the mean and standard deviation are convex in their components, the unique critical point of the function is a minimum. Therefore, among the Karush-Kuhn-Tucker (KKT) conditions, it is sufficient to only consider the stationarity and complementary slackness conditions for the Lagrangian:
1. **Stationarity:** The partial derivatives of the Lagrangian with respect to $\mathcal{T}(q_p^i)$ must vanish:

$$\frac{\partial \mathcal{L}}{\partial \mathcal{T}(q_p^i)} = \alpha \cdot \frac{\partial \mathbb{E}[\mathcal{T}]}{\partial \mathcal{T}(q_p^i)} + (1-\alpha) \cdot \frac{\partial \mathrm{Std}[\mathcal{T}]}{\partial \mathcal{T}(q_p^i)} - \lambda_i = 0, \quad \forall i. \tag{13}$$

The partial derivative of $\mathbb{E}[\mathcal{T}]$ with respect to $\mathcal{T}(q_p^i)$ is $p(q_p^i)$. For $\frac{\partial \mathrm{Std}[\mathcal{T}]}{\partial \mathcal{T}(q_p^i)}$, we first calculate

$$\begin{aligned}\frac{\partial \mathrm{Var}(\mathcal{T})}{\partial \mathcal{T}(q_p^i)} &= \frac{\partial \mathbb{E}[\mathcal{T}^2]}{\partial \mathcal{T}(q_p^i)} - \frac{\partial (\mathbb{E}[\mathcal{T}])^2}{\partial \mathcal{T}(q_p^i)} \\ &= 2p(q_p^i)\left(\mathcal{T}(q_p^i) - \mathbb{E}[\mathcal{T}]\right) - 2\left(\mathcal{T}(q_p^i) - \mathbb{E}[\mathcal{T}]\right)p(q_p^i) \\ &= 2p(q_p^i)\left(\mathcal{T}(q_p^i) - \mathbb{E}[\mathcal{T}]\right).\end{aligned}$$

Then, using the chain rule, we receive

$$\frac{\partial \mathrm{Std}[\mathcal{T}]}{\partial \mathcal{T}(q_p^i)} = \frac{1}{2\sqrt{\mathrm{Var}[\mathcal{T}]}} \cdot \frac{\partial \mathrm{Var}[\mathcal{T}]}{\partial \mathcal{T}(q_p^i)} = \frac{p(q_p^i)\big(\mathcal{T}(q_p^i) - \mathbb{E}[\mathcal{T}]\big)}{\mathrm{Std}[\mathcal{T}]}.$$

Substituting back into (13), we obtain

$$\frac{\partial \mathcal{L}}{\partial \mathcal{T}(q_p^i)} = \alpha \cdot p(q_p^i) + (1-\alpha) \cdot p(q_p^i) \cdot \frac{\mathcal{T}(q_p^i) - \mathbb{E}[\mathcal{T}]}{\mathrm{Std}[\mathcal{T}]} - \lambda_i = 0, \quad \forall i.$$

2. **Complementary slackness:** The Lagrange multipliers must satisfy:

$$\lambda_i \cdot \big(\mathcal{T}_{\min}(q_p^i) - \mathcal{T}(q_p^i)\big) = 0, \quad \forall i. \tag{14}$$

Combining 13 with 14 yields

$$\begin{aligned} &\alpha p_i\big(\mathcal{T}_{\min}(q_p^i) - \mathcal{T}(q_p^i)\big) \\ &\quad + (1-\alpha)\frac{p_i\big(\mathcal{T}(q_p^i) - \mathbb{E}[\mathcal{T}]\big)}{\mathrm{Std}[\mathcal{T}]}\big(\mathcal{T}_{\min}(q_p^i) - \mathcal{T}(q_p^i)\big) = 0 \\ &\implies \alpha\big(\mathcal{T}_{\min}(q_p^i) - \mathcal{T}(q_p^i)\big) \\ &\quad + (1-\alpha)\frac{\big(\mathcal{T}(q_p^i) - \mathbb{E}[\mathcal{T}]\big)}{\mathrm{Std}[\mathcal{T}]}\big(\mathcal{T}_{\min}(q_p^i) - \mathcal{T}(q_p^i)\big) = 0. \end{aligned}$$

Let i^* be an index for which $\mathcal{T}(q_p^{i^*}) = \mathcal{T}_{\min}(q_p^{i^*})$ holds, i.e., constraint i^* is binding. From the dual feasibility conditions, it follows that

$$\lambda_{i^*} = \alpha + (1-\alpha)\frac{\mathcal{T}(q_p^{i^*}) - \mathbb{E}[\mathcal{T}]}{\mathrm{Std}[\mathcal{T}]} \geq 0.$$

Since

$$\lambda_{i^*+1} \geq \alpha + (1-\alpha)\frac{\mathcal{T}_{\min}(q_p^{i^*+1}) - \mathbb{E}[\mathcal{T}]}{\mathrm{Std}[\mathcal{T}]} \geq \alpha + (1-\alpha)\frac{\mathcal{T}_{\min}(q_p^{i^*}) - \mathbb{E}[\mathcal{T}]}{\mathrm{Std}[\mathcal{T}]} \geq 0$$

holds, $\mathcal{T}(q_p^{i^*+1}) = \mathcal{T}_{\min}(q_p^{i^*+1})$ must necessarily be satisfied.
This implies that the optimal solution exhibits the following structure: When ordering the indices by ascending values of $\mathcal{T}_{\min}(q_p^i)$, there exists a critical index up to which the values $\mathcal{T}(q_p^i)$ may deviate from the lower bound $\mathcal{T}_{\min}(q_p^i)$, while for all larger indices the respective constraint is binding, i.e., $\mathcal{T}(q_p^i) = \mathcal{T}_{\min}(q_p^i)$ holds. Of course, this property remains valid when taking the limit $n \to \infty$.
Let j^* be the maximal index such that $\mathcal{T}(q_p^{j^*}) > \mathcal{T}_{\min}(q_p^{j^*})$. Then

$$\mathcal{T}(q_p^1) = \mathcal{T}(q_p^2) = \cdots = \mathcal{T}(q_p^n) = \mathcal{T}^*.$$

We prove this statement for the case of two decision variables $\mathcal{T}(q_p^1)$ and $\mathcal{T}(q_p^2)$; the generalization to n variables is straightforward. It suffices to show: For $\mathcal{T}(q_p^1) < \mathcal{T}(q_p^2)$,

$$\frac{\partial J}{\partial \mathcal{T}(q_p^1)} > 0 \quad \text{(objective function increases in } \mathcal{T}(q_p^1))$$

always implies

$$\frac{\partial J}{\partial \mathcal{T}(q_p^2)} > 0 \quad \text{(objective function decreases in direction } -\mathcal{T}(q_p^2)).$$

The partial derivative is given by

$$\frac{\partial J}{\partial \mathcal{T}(q_p^1)} = (1-p_2)\left[\alpha + (1-\alpha)\frac{\mathcal{T}(q_p^1) - \mu}{\sigma}\right].$$

Since $0 \leq (1-p_2)$, its sign is determined by the term

$$B_1 = \alpha + (1-\alpha)\frac{\mathcal{T}(q_p^1) - \mu}{\sigma}.$$

Similarly, the sign of $\frac{\partial J}{\partial \mathcal{T}(q_p^2)}$ is determined by

$$B_2 = \alpha + (1-\alpha)\frac{\mathcal{T}(q_p^2) - \mu}{\sigma}.$$

We have

$$B_2 - B_1 = (1-\alpha)\left(\frac{\mathcal{T}(q_p^2) - \mu}{\sigma} - \frac{\mathcal{T}(q_p^1) - \mu}{\sigma}\right) = (1-\alpha)\frac{\mathcal{T}(q_p^2) - \mathcal{T}(q_p^1)}{\sigma} \geq 0,$$

which implies $B_2 \geq B_1$. This completes the proof.
This analysis effectively reduces the functional optimization problem to a two-dimensional problem. For a given distribution ϕ of lower bounds on total travel time $\mathcal{T}_{\min}(q_p)$, the optimization problem can be formulated as:

$$\min_{\mathcal{T}^*, \mathcal{T}^*_{\min}} J[\phi'] \tag{15}$$

where $\phi'(\mathcal{T}_{\min})$ is defined as:

$$\phi'(\mathcal{T}_{\min}) = \begin{cases} \mathcal{T}^* & \text{if } \mathcal{T}_{\min} \leq \mathcal{T}^*_{\min}, \\ \phi(\mathcal{T}_{\min}) & \text{otherwise.} \end{cases}$$

In the third optimization phase, we determine the optimal control of the variable speed limits. We start with a given realization of the peak flow q_p^0 and take into account the optimal total travel time $\mathcal{T}(q_p^0)$ calculated in phase 2. The optimal speed limit v_{ctrl} is implemented as a closed-loop control system that depends on the current system state: $v_{\text{ctrl}} = v_{\text{ctrl}}(x, t, k(\cdot))$. This feedback control enables more precise traffic flow control compared to an open-loop system. We define two additional parameters: a minimum speed $v_{\min}$ that serves as a lower bound for the variable speed limit and the number n of speed limits to be installed along the route.
For an optimal distribution of total travel times, we employ the following control strategy: First, we determine the maximum optimal throughput $q^*_{\text{bn}} \leq q_{\text{bn}}$ using:

$$\mathcal{T}(q_p^0) - \mathcal{T}_{\min}(q_p^0) = \int_0^T \left[N(l,t) - N^*(l,t)\right] dt$$

This equation is derived through the following transformation:

$$\begin{aligned} \int_0^N \left[N^{*-1}(l,n) - N^{*-1}(0,n)\right] dn - \int_0^N \left[N^{-1}(l,n) - N^{-1}(0,n)\right] dn \\ = \int_0^{N^{*-1}(l,n)} dn - \int_0^{N^{-1}(l,n)} dn \\ = \left[T \cdot N - \int_0^T N^*(l,t)\, dt\right] - \left[T \cdot N - \int_0^T N(l,t)\, dt\right] \\ = \int_0^T N(l,t)\, dt - \int_0^T N^*(l,t)\, dt \end{aligned}$$

A graphical representation of the solution method is shown in Figure 2.
To calculate the optimal speed limit v^* for a given peak flow q^*_{bn}, we use the fundamental diagram. There, we determine the larger of the two k^+ values for which the condition $q(k^+) = q^*_{\text{bn}}$ is satisfied. The desired speed limit is then obtained from the quotient

$$v^* = \frac{q^*_{\text{bn}}}{k^+}.$$

The actual implementation of the speed limit depends on the current traffic flow:
If the flow $q(x,t)$ near l exceeds the value q_{bn}, the limit is calculated according to the formula

$$v\left(\frac{l}{i+1}, t\right) = v_f - (v_f - v^*)\frac{i}{n}.$$

If the traffic flow $q(x,t)$ is below q_{bn}, no speed limit is activated.

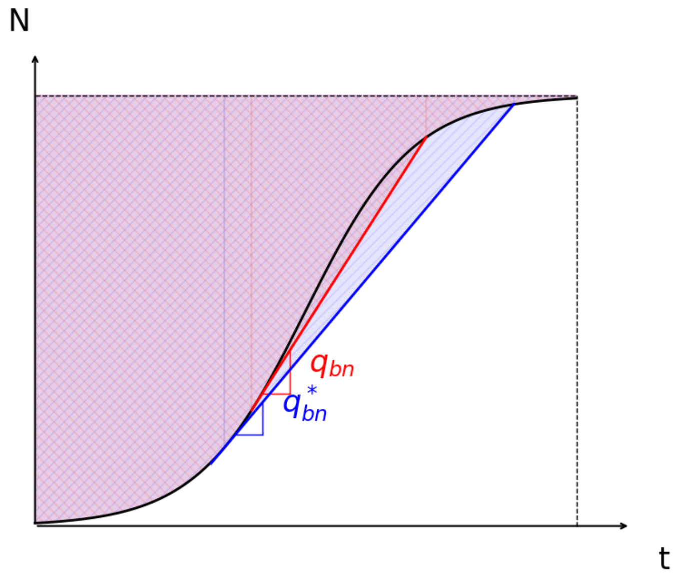

Figure 2: q^*_{bn} is chosen such that the blue area has the size $\mathcal{T}(q_p^0) - \mathcal{T}_{\min}(q_p^0)$

At the upstream boundary, the traffic flow $q(0,t)$ was modeled using a piecewise linear function. This is defined as:

$$q(0,t) = \begin{cases} r_1 \cdot t + 5571.84, & \text{for } 0 \le t \le 2, \\ -r_2 \cdot t + 7708.81, & \text{for } 2 \le t \le 4, \end{cases}$$

where t represents the time in hours starting from 6:00 PM. The peak flow is modeled as a normally distributed random variable q_p with mean 6470 and standard deviation 150. This results in the slopes $r_1 = \frac{q_p - 5571.84}{2}, \quad r_2 = \frac{7708.81 - q_p}{2}$. The fundamental diagram is approximated by a triangular model. The free-flow speed was set to 112 km/h in accordance with the local speed limit. The jam density was estimated at 608 vehicles per kilometer, and the critical density at 121 vehicles per kilometer. The bottleneck has an estimated capacity of 6240 vehicles per hour. During congestion, this capacity drops by 7 percent, resulting in $q_{\text{bn}} = 6240 \quad \text{and} \quad q_{\text{bn}}^- = 5803$ vehicles per hour. The considered road segment was set to a length of approximately 9.34 kilometers—a distance that can be covered in 5 minutes when traveling at free-flow speed. The results are presented in Table **??**.

α	Opt. Value	$\mathcal{T}^*$	v^*
1	99869.54	N/A	27 km/h
0.75	99869.54	N/A	27 km/h
0.5	68948.63	125944.99	24.2 km/h
0.25	36866.05	140935.71	21.9 km/h

3 Conclusions

The investigation of variable speed limits (VSL) for improving travel time reliability presents several significant findings. Our three-stage optimization algorithm successfully demonstrates that VSL systems can be effectively deployed to manage travel time variability during peak hours. The results show that different weightings between expected travel time and reliability (represented by) significantly affect control stragies, with optimal speed limits ranging from 21.9 km/h to 27 km/h. Notably, for values of 0 and 0.25, which place greater emphasis on expected travel time, no effect of the reliability parameter could be measured.. The study's novel contribution lies in its exact solution approach within the kinematic wave model framework, eliminating the need for spatial and temporal discretization. This advancement enables more precise determination of optimal VSL positions and timing. Incorporation of the capacity drop phenomenon and stochastic peak-hour demand provides a realistic framework for practical applications. The findings suggest that VSL systems can be effectively implemented to balance the competing objectives of minimizing expected travel time and improving reliability, offering valuable insights for highway operations optimization.

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