**Endogenizing Choice Set Formation Using the Perturbed Utility Model**

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## Short summary

Defining the choice set is a critical challenge in discrete choice models with large sets of alternatives, such as destination choice. This study proposes a novel approach based on perturbed utility maximization, which explicitly represents zero probabilities for certain alternatives, enabling the simultaneous estimation of deterministic consideration sets and model parameters. Tests with synthetic data reveal that ignoring choice set restrictions in multinomial logit models leads to overestimation of choice probabilities and marginal effects, especially for higher values, resulting in biased policy evaluations. In contrast, the proposed method effectively mitigates these biases and enhances estimation accuracy. Application to a real-world destination choice model demonstrates the method's scalability and robustness, confirming its ability to handle large choice sets and adapt consideration set sizes based on contextual factors.
**Keywords**: Choice Set, Destination Choice, Perturbed Utility, Tsallis Entropy

## 1 Introduction

Discrete choice models are widely used in transportation for demand forecasting and policy evaluation. A critical aspect of these models is the definition of the choice set, as inaccuracies can lead to biased estimates and flawed policy recommendations (Swait & Ben-Akiva, 1987; Li et al., 2015). Despite its importance, practical applications often rely on externally constrained choice sets (Fukuda & Ishii, 2024).

Existing methods for modeling consideration sets, the subsets of alternatives that decision-makers actually evaluate, can be categorized into two approaches. Explicit approaches, such as Manski's two-stage model (Manski, 1977), treat consideration sets probabilistically (Swait & Ben-Akiva, 1987; Ben-Akiva & Boccara, 1995), offering theoretical rigor but struggling with computational feasibility for large-scale problems. Implicit approaches, on the other hand, introduce penalty terms into utility functions to approximate consideration set formation through a single-stage model (Swait, 2001; Cascetta & Papola, 2001), enabling their application to large choice sets.

Both approaches share key limitations. They cannot strictly assign zero probabilities to alternatives, leading to biases in estimation and prediction results depending on the definition of the full choice set. Additionally, both require pre-specifying variables for consideration set formation, making results highly sensitive to variable selection.

Motivated by these challenges, this study proposes an implicit approach based on perturbed utility maximization. By explicitly representing zero probabilities within the choice set, the proposed method enables unique estimation of deterministic consideration sets and model parameters using all explanatory variables. It eliminates the need for pre-specified variables, allowing robust estimation even when the full choice set is mechanically defined. This framework is computationally efficient and scalable, making it suitable for large-scale problems.

## 2 Methodology

This section introduces the $\alpha$PUM, a perturbed utility model capable of representing zero probabilities within the choice set, and describes a simultaneous estimation method for consideration sets and model parameters.

### *Perturbed utility model and $\alpha$PUM*

The perturbed utility model (PUM) maximizes perturbed utility, defined as the sum of expected utility and a non-linear perturbation function (McFadden & Fosgerau, 2012; Fudenberg et al., 2015). It encompasses any additive random utility model (ARUM) (Hofbauer & Sandholm, 2002; Fosgerau et al., 2024).

The $\alpha$PUM employs $\alpha$-Tsallis entropy (Tsallis, 1988) as its perturbation function, enabling explicit representation of zero probabilities (Watanabe & Hidaka, 2023). The choice probability vector $\boldsymbol{p}_n \equiv (p_{ni})_{i\in\mathscr{U}_n}$ for a decision-maker $n \in \mathscr{N}$ facing the full set of alternatives $\mathscr{U}_n$ is defined in $\alpha$PUM as:

$$\boldsymbol{p}_n = \arg\max_{\boldsymbol{q}_n\in\Delta_n} \left\{ \boldsymbol{q}_n^T \boldsymbol{v}_n(X_n;\boldsymbol{\beta}) + \frac{1}{\mu} H^\alpha(\boldsymbol{q}_n) \right\}, \tag{1}$$

where $\Delta_n$ is the $|\mathscr{U}_n|$-dimensional probability simplex. The vector $\boldsymbol{v}_n(X_n;\boldsymbol{\beta})$ represents deterministic utilities, which are often assumed to be a linear combination of the explanatory variable matrix $X_n$ and the parameter vector $\boldsymbol{\beta}$. The parameter $\mu$ is a scale parameter, standardized to $\mu = 1$ unless otherwise specified. The term $H^\alpha(\boldsymbol{q}_n)$ is the $\alpha$-Tsallis entropy:

$$H^\alpha(\boldsymbol{q}_n) = \begin{cases} \frac{1}{\alpha(\alpha-1)} \sum_{j\in\mathscr{U}_n} \left(q_{nj} - q_{nj}^\alpha\right), & \alpha \neq 1, \\ -\sum_{j\in\mathscr{U}_n} q_{nj} \ln q_{nj}, & \alpha = 1, \end{cases} \tag{2}$$

where $\alpha$ controls the entropy. As $\alpha \to 1$, $H^\alpha(\boldsymbol{q}_n)$ converges to the Shannon entropy. Thus, $\alpha$PUM includes the multinomial logit (MNL) model as a special case when $\alpha = 1$.

When $\alpha > 1$, the first-order condition reveals a threshold behavior:

$$\boldsymbol{p}_n = \left[(\alpha-1)\boldsymbol{v}_n - v_n^\star \boldsymbol{1}\right]_+^{\frac{1}{\alpha-1}}, \tag{3}$$

where $v_n^\star$ is a unique threshold, $\boldsymbol{1}$ is a vector of ones, and $[x]_+ = \max(x, 0)$. This shows that $\alpha$PUM assigns zero probabilities to alternatives with utilities below $v_n^\star/(\alpha-1)$. The number of alternatives with zero probabilities increases with higher $\alpha$, leading to sparser probability distributions. The set of alternatives with positive choice probabilities can be regarded as the consideration set in the $\alpha$PUM framework.

### *Estimation method*

The parameters $\boldsymbol{\theta} \equiv (\alpha, \boldsymbol{\beta})$ are estimated by maximizing the log-likelihood function, calculated as follows:

$$L_{\boldsymbol{\theta}}(\boldsymbol{p};\boldsymbol{y}) \equiv \sum_{n\in\mathscr{N}} \boldsymbol{y}_n \ln \boldsymbol{p}_n, \tag{4}$$

where $\boldsymbol{y}_n \equiv (y_{ni})_{i\in\mathscr{U}_n}$ represents the observed choice probability vector for decision-maker $n \in \mathscr{N}$. In the estimation process, if the choice probability for an observed alternative becomes zero, the logarithmic term in equation (4) cannot be evaluated. To prevent this issue, small probabilities are assigned to zero-probability alternatives, a technique commonly used in MNL estimation.

Maximizing the log-likelihood in equation (4) benefits from the differentiation of the choice probability $\boldsymbol{p}_n$ with respect to the parameters $\boldsymbol{\theta}$. Although the choice probability in $\alpha$PUM, derived as the solution to the optimization problem in equation (1), does not have a closed-form expression, its derivatives with respect to both $\alpha$ and $\boldsymbol{\beta}$ can be expressed in closed form (Correia et al., 2019; Peters et al., 2019; Blondel et al., 2020). This enables computationally efficient log-likelihood maximization, making the estimation method applicable to large-scale choice problems.

## 3 Results and discussion

This section validates the proposed model and estimation method described in Section 2 using synthetic data and applies the method to a destination choice model with real data to demonstrate its applicability and effectiveness.

### *Validation with synthetic data*

To validate the proposed method, simulations were conducted assuming a two-stage decision-making process: a conjunctive noncompensatory formation of the consideration set followed by compensatory choice within the consideration set (Payne, 1976).

In the experimental setup, data was generated for $N = 1000$ users, each facing a choice set $\mathscr{U}_n$ of size $|\mathscr{U}_n| = 100$. Alternatives were characterized by two explanatory variables sampled from a uniform distribution over $[0, 1]$. Consideration sets were formed by selecting alternatives with explanatory variables below the threshold $\boldsymbol{\tau} = [0.7, 0.7]$. Choices within the consideration set were made using the MNL model with true parameters $\boldsymbol{\beta}^\star = [-2.0, -1.0]$. Both $\alpha$PUM and MNL were estimated on the full set $\mathscr{U}_n$, without information on the true consideration sets. The data generation and estimation process was repeated for $R = 100$ replications.
The estimation results are summarized in Table 1. Here, $M(x_i)$ represents the marginal effect of variable $i$, calculated using probability-weighted sample enumeration (Train, 2009). The $t$-values are evaluated as follows: $\alpha$ against 1, $\boldsymbol{\beta}$ against 0, and $M$ against the true values. It is important to note that direct comparison of true parameter values with estimated values is not meaningful when the data generation and estimation models differ, as is the case with MNL and $\alpha$PUM, due to their fundamentally different model structures.

Table 1: Estimation results in synthetic data

| | TRUE | | $\alpha$PUM | | | MNL | | |
|---|---|---|---|---|---|---|---|---|
| | Val. | S.E. | Est. | S.E. | t-val. | Est. | S.E. | t-val. |
| $\alpha$ | — | — | 1.500 | 0.045 | 11.01 | — | — | — |
| $\beta_1$ | -2.0 | — | -0.357 | 0.077 | -4.64 | -3.189 | 0.110 | -29.07 |
| $\beta_2$ | -1.0 | — | -0.299 | 0.065 | -4.59 | -2.537 | 0.093 | -27.21 |
| $\tau_1$ | 0.7 | — | — | — | — | — | — | — |
| $\tau_2$ | 0.7 | — | — | — | — | — | — | — |
| $M(x_1)$ | -0.048 | 0.000159 | -0.0502 | 0.0034 | -6.46 | -0.0777 | 0.0049 | -60.58 |
| $M(x_2)$ | -0.024 | 0.000079 | -0.0420 | 0.0029 | -62.05 | -0.0618 | 0.0040 | -94.48 |
| $L0$ | | | -4605.17 | — | | -4605.17 | — | |
| $LL$ | | | -4001.20 | 23.22 | | -4043.93 | 24.34 | |
| $\bar{\rho}^2$ | | | 0.1305 | 0.0050 | | 0.1214 | 0.0052 | |
| AIC | | | 8008.39 | 46.44 | | 8091.85 | 48.69 | |

From the perspective of $\bar{\rho}^2$ and AIC, $\alpha$PUM demonstrates superior explanatory and predictive performance compared to MNL. This improvement highlights the value of endogenizing consideration set restrictions in the modeling process. Additionally, MNL tends to overestimate parameters and marginal effects. This overestimation appears to result from artificially increasing parameter sensitivity to mimic sparsity. By contrast, $\alpha$PUM reduces these biases, improving the reliability of the estimated results.
These findings are explored in more detail in Figures 1 and 2, which illustrate, for each alternative, the relationship between the true and estimated values for choice probabilities and marginal effects, respectively. MNL overestimates both measures for alternatives with larger true probabilities, with the discrepancies becoming more pronounced as true values increase. In contrast, $\alpha$PUM mitigates these discrepancies, providing more accurate estimates. Such discrepancies in MNL can significantly affect policy evaluations, underscoring the benefits of using $\alpha$PUM when consideration sets are restricted. The robustness of $\alpha$PUM's performance was confirmed through additional tests varying parameter values and sample sizes.
Figure 3 illustrates how $\alpha$PUM estimates the consideration set in relation to observed alternatives. In the data generation process, only alternatives with explanatory variable values below the threshold $[0.7, 0.7]$ were included in the true consideration set, resulting in observed alternatives being distributed exclusively within this region. $\alpha$PUM estimates consideration sets that expand like a convex hull enclosing the observed alternatives. This behavior arises because assigning zero probability to an observed alternative leads to a significant penalty in the log-likelihood function. Under the current assumption of a linear utility function, $\alpha$PUM tends to overestimate the size of the true consideration set, especially when sufficient observations are available.

***Empirical analysis***

To demonstrate the applicability of the proposed method, it was applied to a destination choice model using real data from an activity survey conducted in March 2014 in Aichi Prefecture, Japan. The survey collected information on respondents' activities, including the type and location of each activity, recorded hourly for their most recent weekday. Focusing on 233 shopping trips originating

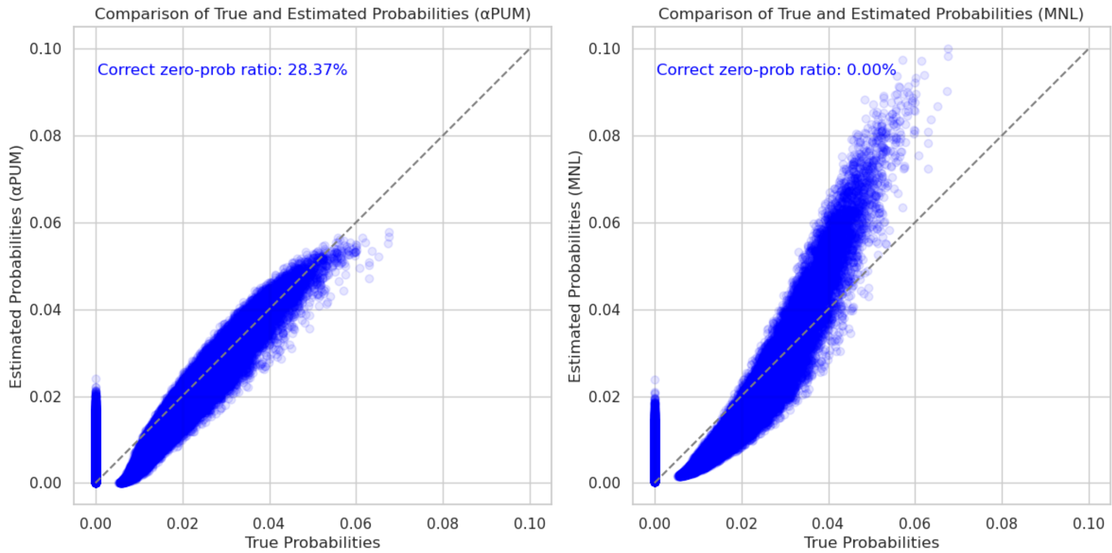


Figure 1: Comparison of true and estimated choice probabilities for each alternative: $\alpha$PUM (left) and MNL (right).

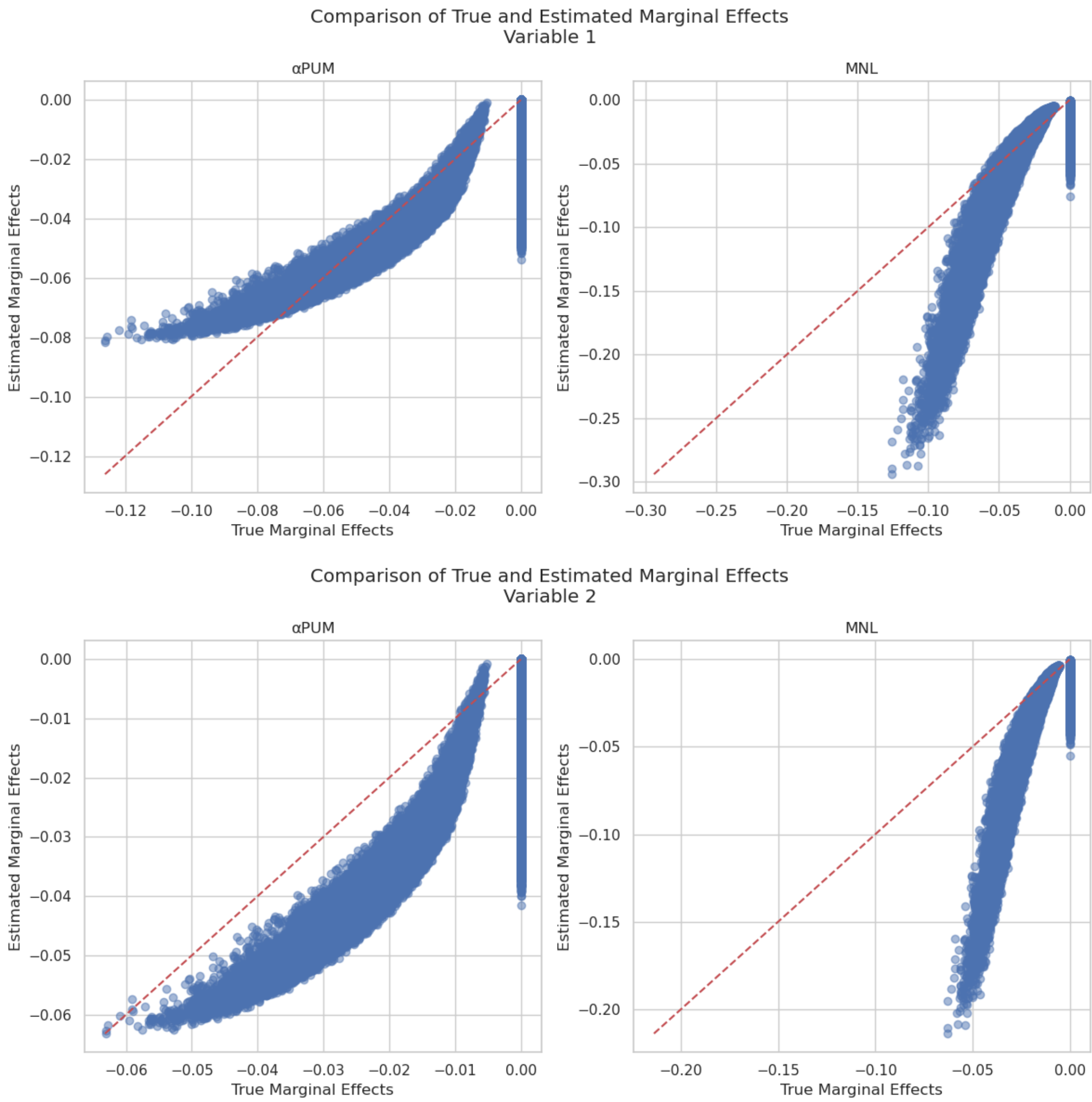


Figure 2: Comparison of true and estimated marginal effects for each alternative: $\alpha$PUM (left) and MNL (right).

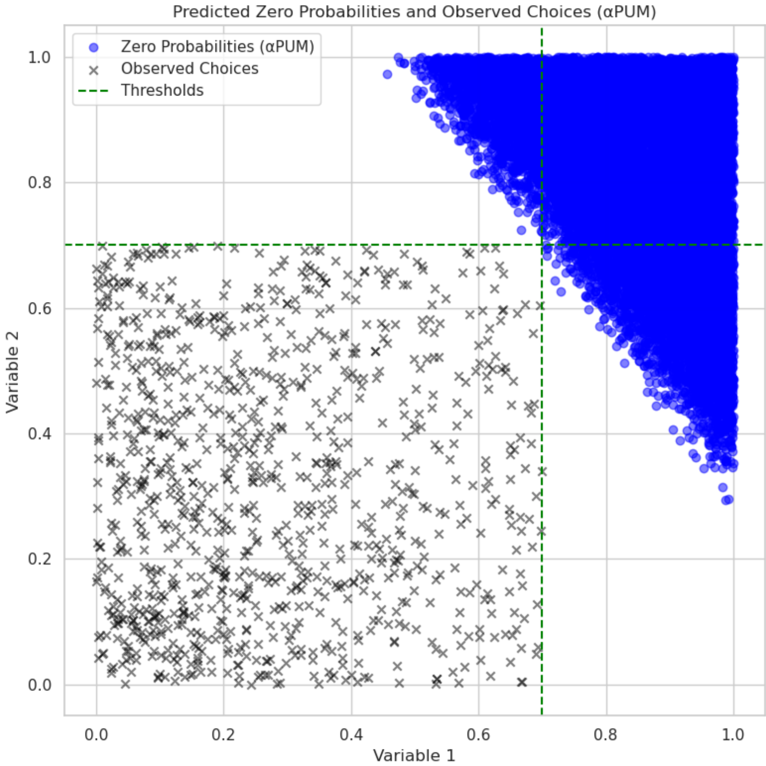


Figure 3: Observed alternatives and estimated consideration set.

from home, the analysis considered a full set of alternatives $\mathscr{U}_n$ comprising 5,334 1-km mesh zones across the prefecture.

The explanatory variables include distance band dummies (nearest: 0–2 km, short: 2–5 km, medium: 5–10 km, long: over 10 km), the number of establishments (log-transformed) obtained from the Economic Census for Business Activity (Statistics Bureau of Japan, 2019), and the sales floor area (log-transformed, in hundreds of square meters) obtained from the Commercial Statistics (Ministry of Economy, Trade and Industry, 2017). To capture the variation in shopping behaviors across distances, the number of establishments and sales floor area were included as interaction terms with the distance band dummies. For instance, routine purchases for daily necessities are more common at shorter distances, while comparison shopping for durable goods tends to occur at longer distances.

The estimation results, presented in Table 2, indicate that $\alpha$PUM outperforms MNL in terms of likelihood and AIC, suggesting better explanatory and predictive performance. The parameter $\alpha$ is estimated to be significantly greater than 1, indicating the existence of restricted consideration sets for shopping trips. Moreover, $\alpha$PUM enhances interpretability by demonstrating that sales floor area is significant for shorter distances, whereas the number of establishments is significant for longer distances—aligning with the intuitive understanding that shopping trips include both routine purchases and comparison shopping.

Figure 4 visualizes the choice probabilities for each mesh when a specific origin mesh is given. Under MNL, consideration set restrictions are not accounted for, resulting in positive probabilities being assigned to all 5,334 alternatives. In contrast, $\alpha$PUM restricts the set of alternatives with positive probabilities to 627 meshes, allowing clearer interpretation of likely destinations. Additionally, as shown in Figure 5, which depicts the choice probabilities for alternatives assigned positive probabilities under $\alpha$PUM, $\alpha$PUM provides sharper distinctions for low-probability alternatives compared to MNL. This behavior demonstrates $\alpha$PUM's ability to limit alternatives to those with meaningful probabilities, thereby enhancing both interpretability and robustness.

## 4 Conclusions

In this study, we demonstrated that $\alpha$PUM, a perturbed utility model capable of explicitly representing zero probabilities, enables the unique estimation of consideration sets and model pa-

Table 2: Destination choice model estimation results (shopping purpose)

| Purpose | shopping | | | | | |
|---|---|---|---|---|---|---|
| Number of samples | 233 | | | | | |
| | $\alpha$PUM | | | MNL | | |
| | Est. | t-val. | | Est. | t-val. | |
| $\boldsymbol{\alpha}$ | 1.181 | 12.37 | $\star$ | — | — | — |
| **Distance band dummies** | | | | | | |
| Inner trip | — | — | — | — | — | — |
| Nearest: 0-2km | -2.861 | -9.94 | $\star$ | -4.202 | -8.49 | $\star$ |
| Short-distance: 2-5km | -4.228 | -22.32 | $\star$ | -8.824 | -5.44 | $\star$ |
| Medium-distance: 5-10km | -6.845 | -18.46 | $\star$ | -19.137 | -17.47 | $\star$ |
| Long-distance: 10km- | -10.273 | -24.97 | $\star$ | -34.359 | -26.86 | $\star$ |
| **Nearest: 0-2km** | | | | | | |
| Number of establishments | -0.084 | -0.83 | | -0.219 | -0.87 | |
| Sales floor area | 0.510 | 6.53 | $\star$ | 0.847 | 2.99 | $\star$ |
| **Short-distance: 2-5km** | | | | | | |
| Number of establishments | 0.003 | 0.04 | | -0.241 | -1.24 | |
| Sales floor area | 0.419 | 5.55 | $\star$ | 1.228 | 5.87 | $\star$ |
| **Medium-distance: 5-10km** | | | | | | |
| Number of establishments | 0.479 | 3.90 | $\star$ | 1.754 | 1.55 | |
| Sales floor area | 0.221 | 1.83 | | 0.498 | 0.46 | |
| **Long-distance: 10km-** | | | | | | |
| Number of establishments | 0.617 | 2.34 | $\star$ | 2.839 | 5.59 | $\star$ |
| Sales floor area | 0.495 | 1.61 | | 1.369 | 3.42 | $\star$ |
| $L0$ | -1999.572 | | | -1999.572 | | |
| $LL$ | -655.420 | | | -658.569 | | |
| AIC | 1336.840 | | | 1341.137 | | |

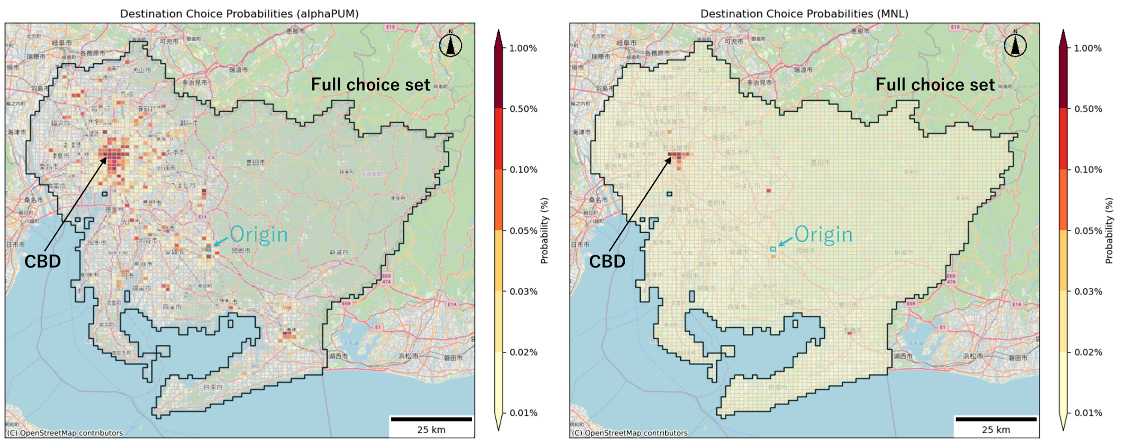


Figure 4: Estimated choice probability for each mesh when the starting mesh is specified: $\alpha$PUM (left) and MNL (right).

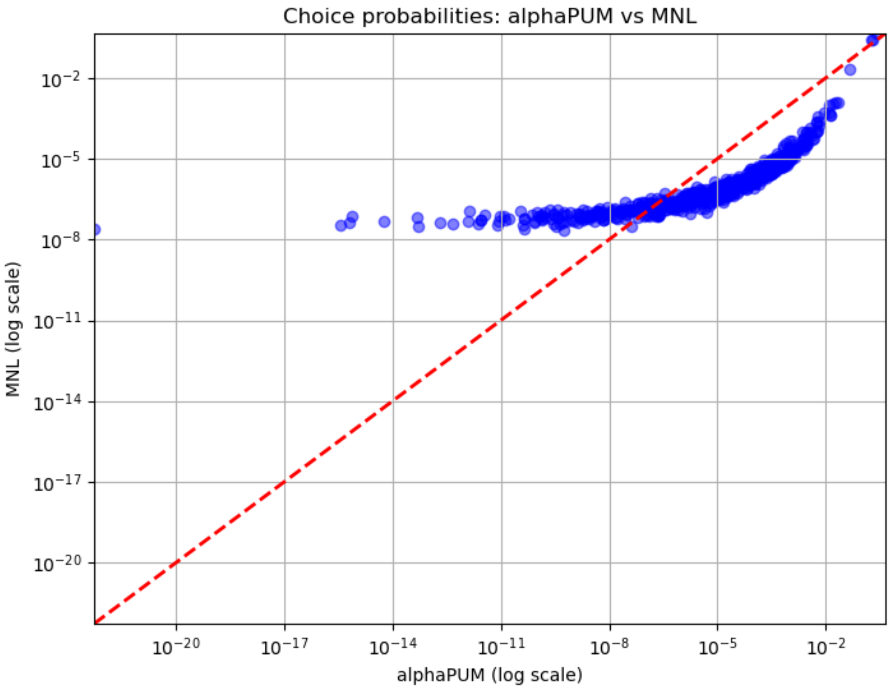


Figure 5: Comparison of predicted choice probabilities between $\alpha$PUM and MNL.

rameters, even when the full set of alternatives is mechanically defined. The proposed method was validated using generated data under the assumption of conjunctive consideration set formation. The results showed that the MNL model, which does not account for consideration set restrictions, tends to overestimate choice probabilities and marginal effects, particularly for higher values, leading to potential biases in policy evaluation. In contrast, $\alpha$PUM mitigated these biases, thereby improving estimation accuracy. Furthermore, the method was applied to a destination choice model using real data with a large number of alternatives, confirming its applicability and effectiveness in practical scenarios.

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