A Continuous Approximation for Demand Responsive Transport with Meeting Points

Richard D. Connors*[1], Haruko Nakao[1], Tai-Yu Ma[2], and Francesco Viti[1]

[1]Faculty of Science, Technology and Communication, University of Luxembourg, Luxembourg
[2]Luxembourg Institute of Socio-Economic Research, Luxembourg

Short summary

Optimisation of demand responsive transport requires solving a vehicle routing problem (VRP), a computationally demanding discrete combinatorial problem that can be challenging to integrate within a bi-level network design framework.
Continuous approximations are therefore appealing, yielding tractable analytic formulae for key performance indicators of discrete routing problems. It is well known that the optimal tour length, V, of the travelling salesman problem for N points randomly distributed in area A has asymptotic formula $V \sim N^{\frac{1}{2}} A^{\frac{1}{2}}$. Established continuous approximations for more complex VRPs have the same, separable, functional form.
In this paper we consider a first mile demand responsive transport problem characterised by the use of meeting points, with customer ride time constrained by a maximum detour factor. We show that the total tour length of this VRP *cannot* be described by existing continuous approximations. We propose a new continuous approximation functional form and demonstrate its effectiveness.
Keywords: Vehicle Routing Problem, Demand Responsive Transport, Stochastic Demand, Continuous Approximation.

1 Introduction

The computational complexity of solving (large) vehicle routing problems (VRPs) presents an obstacle to (i) understanding their performance as a function of input parameters and (ii) integrating them within optimisation frameworks that arise when, for example, a DRT operator wishes to optimise their operational area given a demand density profile, or a network manager seeks to understand DRT's contribution to congestion under different forecast demand scenarios. Continuous approximations (CAs) to these discrete problems provide continuous, analytical models that are tractable - being quick to compute - and hence facilitate the analysis and optimisation of performance indicators of interest.
There is a rich history of research into CAs for vehicle routing problems. The canonical Beardwood et al. (1959) result is that the optimal tour length, V, of the travelling salesman problem (TSP) for N points uniformly and independently distributed in compact area A has asymptotic formula

$$V = \beta N^{\frac{1}{2}} A^{\frac{1}{2}} \tag{1}$$

where the value of the constant β depends on the shape of the area and the distance metric used; see Applegate et al. (2007) for appropriate constant values. For irregular shapes, A is often defined to be the area of the smallest rectangle covering the customer locations. Eilon et al. (1971) provides an empirical formula for the VRP where N points in a square region are served from a centrally located depot by a vehicle that can make at most C stops.

$$V = \beta N^{\frac{1}{2}} A^{\frac{1}{2}} \left[1 + \frac{\sqrt{N}}{C} \right] \tag{2}$$

with $\beta = 0.6876$. Daganzo (1984) extended this problem to the depot being distance D away from clusters of customers distributed within irregular shaped areas. Robuste et al. (1990) offers a good summary of TSP and VRP tour length models. Subsequently, (Kwon et al. (1995)) compared several models with general form $V = \beta N^{\frac{1}{2}} A^{\frac{1}{2}}$ (plus a possible extra term) fitted by both regression and neural networks.

Notably, established CAs for VRPs all adopt the same separable functional form seen in equation (1). In this paper we consider a VRP that underpins a demand responsive feeder service (DRFS). We show that the total tour length deviates systematically from the functional form of equation (1). We propose a new functional form, fit it by regression to our simulation results and show that it captures key features of our DRT system.

2 Methodology

We consider a demand responsive first mile service using meeting points, based on the model detailed in Ma et al. (2024); we will refer to this as DRFS. The phenomenon of interest in this paper does not require all the complexity of the full Ma et al. model; here we describe the simplified version used. Figure 1 illustrates some key features.
The DRFS serves customers throughout a disc shaped operational area, transporting them to a transit station at the centre (which is at the coordinates origin in the figure). Each day N customer requests are generated, with customers located uniformly randomly throughout area radius R. One scenario is therefore characterised by the pair (R, N).
In our simple scenario, all customers wish to catch the same transit service i.e. all customers have the same destination (the transit station at the centre) and the same desired arrival time. Customers are not picked up from their home locations. Instead, they walk to a meeting point to be picked up by a DFRS vehicle. A regular square grid of meeting points (MPs) covers the operational area. By construction, at least one meeting point will be within the maximum walking distance, w_{max}, of each customer; the maximum MP separation is set to be $w_{max}\sqrt{2}$. If multiple MPs are accessible to a customer, the DRFS operator specifies which MP will be used for pick-up. Customers submit their ride requests in advance. The operator communicates the meeting point and pick-up time. Figure 1 shows the operating area, radius $R = 5.0km$ within which customers (red dots) have been generated at random locations. Meeting points (grey +) within walking distance of some customer are shown. Other MPs have been ignored. We do not reject customers. We do not generate customers within walking distance of the transit station, the operational area is therefore $A(R) = \pi(R^2 - w_{max}^2)$.

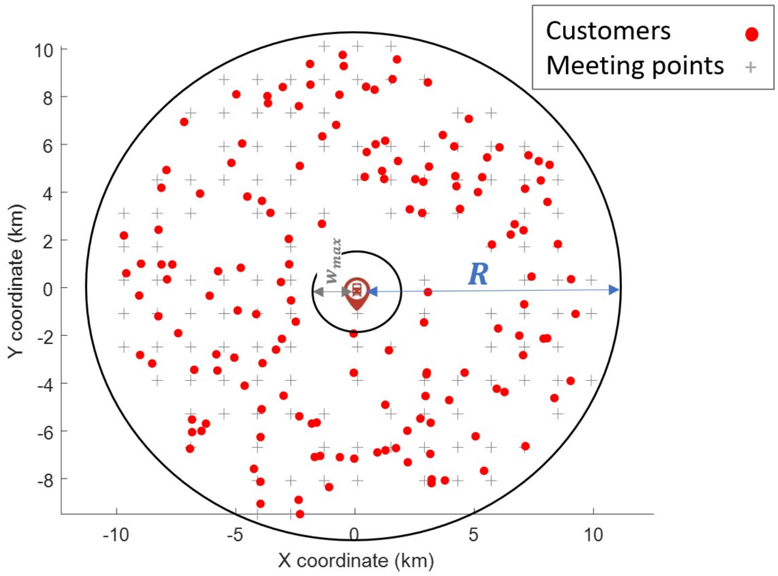

Figure 1: DRFS: Customer locations and (assigned) meeting points within operational area.

A fleet of vehicles start from a depot, which is also located at the transit station at the centre of the study area. Vehicles have unlimited range and passenger capacity. Given a set of customer requests, the DRFS objective function minimises the weighted sum of total vehicle travel time and customers' total walking time, see Ma et al. (2024). We control customer's in-vehicle time by imposing a constraint on the maximum detour factor, γ. We assume vehicles travel at constant speed along straight lines between visited locations, hence time and distance are interchangeable. The detour factor constraint forces the actual in-vehicle time (distance) to be no more than γ times the direct travel time (distance) from the meeting-point pick up, to the destination. In all tests reported $\gamma = 1.5$ so that customers' ride time is at most 50% more than having a direct service. The number of vehicles used, the fleet size, is an output of the DRFS.

Solution Algorithm

We solve this DRFS in two stages: (1) the customer-meeting-point assignment problem minimises the weighted sum of the total customer walking time and bus travel time between the activated meeting points; (2) a Deterministic Annealing (DA) based metaheuristic Braekers et al. (2014) with a post-optimisation procedure solves the routing problem. The performance of the DA-based algorithm was validated by comparison with the solution obtained by a state-of-the-art MILP solver (Gurobi, version 10.0.3).

Table 1: Key model parameters.

Parameter	Value	Description
N	See Table 2	Number of customer requests
R		Radius of operational area, so that $A = \pi(R^2 - w_{max}^2)$
w_{max}	$1\,km$	Maximum walking distance to meeting point
MPS	$1.4\,km$	Meeting point separation distance (grid-size)
γ	1.5	Maximum detour factor

Ensemble Analysis

To understand the characteristics of the DRFS under stochastic demand, for each (R, N) scenario, we generate an ensemble of randomly generated customer demands. In the tests reported here, we generate 20 random demand instances (days) for each (R, N) scenario. On each day, N customer locations are generated from the uniform distribution (constant probability per unit area) and the DRFS is solved providing values of KPIs such as total vehicle kms, empty vehicle kms, fleet size, customer in-vehicle kms and direct customer kms (for each customer the direct distance to the transit station). This results in a distribution of 20 values for each KPI for each (R, N) scenario.

3 Models, Results and Discussion

We run 20 replications (days) of every combination of R and N values shown in Table 2. This comprises 36 scenarios, hence the DRFS is solved $36 \times 20 = 720$ times.

Table 2: (R, N) Scenarios.

$R\,[km]$	3	5	10	15	20	30
N	20	50	100	150	250	500

Following established CAs, we seek to fit a function of the form seen in equation (1). For convenience we rename the (positive) constant $\beta = e^a$ and consider the more general function prototype $V = e^a A^b N^c$, which trivially can be written as

$$\log V = a + b\log(A) + c\log(N)\,. \qquad \text{[Model 1]} \tag{3}$$

We fit this function to the data shown in Figure 2. The goodness of fit is recorded in Table 3.
It would be natural to use 3D plots (with axes V, N, A) but the phenomena of interest are difficult to see in 2D renderings. The key features leading our analysis are best viewed by plotting total vehicle kms against demand density, $D = N/A$.
To explain why this cosmetic choice is also functionally reasonable, consider that each value of R, from Table 2 corresponds to a unique value of A. Given A, each value of N corresponds to a unique value of demand density D. Hence we can consider the independent variables of the model

to be A and D. Rewriting Equation (3) as a function of density does not change its linear form: $\log V = a + (b+c)\log(A) + c\log(D)$. Note that, crucially, for a given value of A (equivalently R), the gradient of $\log V$ with respect to $\log D$ is constant. We now check to see whether this linear trend occurs, or not.

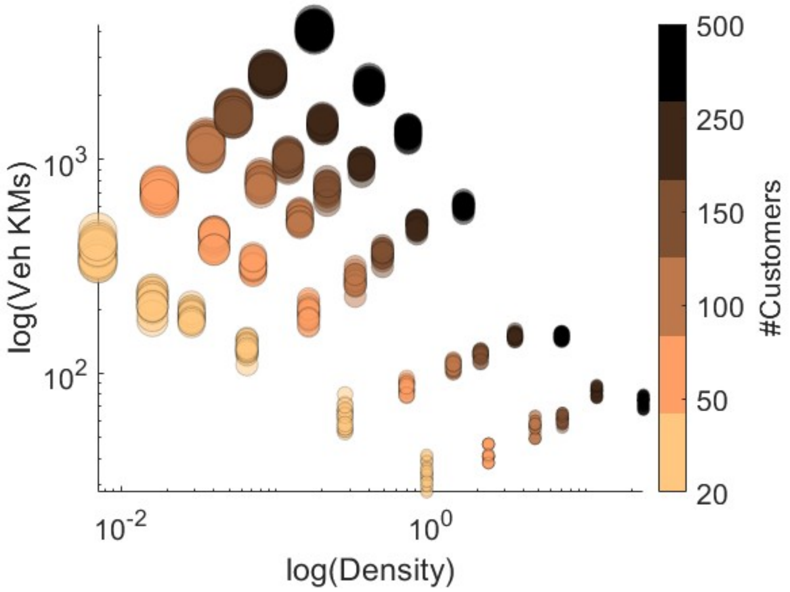

Figure 2: Log-log plot of total vehicle kms vs demand density for 720 runs.

Figure 2 shows the results of 720 runs. Colours represent N, the number of customers. The size of circle represents the radius of the operational area, R, (though not to scale). Each rising diagonal row corresponds to one size of operational area, A. Consider the highest diagonal row of largest radius circles corresponding to $R = 30km$; there are 6 clusters corresponding to $N = 20, 50, 100, 150, 250, 500$ with 20 runs in each. This ensemble of 20 runs have the same values of R and N (hence the same x-coordinate of density) but differ in V, the vehicle kms travelled and hence form a cluster of vertically overlapping circles with the same colour. There appears to be less spread at higher values of R and N, but this is mostly due to the log scale. The lower two rising diagonals show deviations from linear behaviour (for large N) which remains to be investigated.

We fit the linear model in Equation (3) to the log-transformed data to get

$$V = 0.58A^{0.66}N^{0.53} \tag{4}$$

The fit statistics are shown in Table 3. Note that this fitted model is not dissimilar to the established CA square-root model form $V = \beta A^{0.5}N^{0.5}$.
Figure 3 [left] shows this model fit (black dashed lines) alongside the data. The model is linear, as expected, but with the key feature noted above that the gradient of the model is constant. This highlights that in the data, for each value of R the diagonal row has a different gradient. The residuals, shown in Figure 4, also show structure that the model has missed. A separable model of the form Equation (3) cannot capture this changing gradient which reveals an interaction between the two terms of the model.

Table 3: Model 1 Fit

Model	Parameter	Estimate	SE	tStat	RMSE	Adjusted R^2
1: Eq (3)	(Intercept)	-0.552	0.0476	-11.608	0.222	0.969
	$\log(A)$	0.661	0.0049	133.88		
	$\log(N)$	0.529	0.0079	67.15		

We therefore consider a model with an additional interaction term, to see if this can capture the structure we see in the data. Instead of being separable, the exponents of the two variables A and

N need to be functions of the other variate:

$$V = e^a A^{b(N)} N^{c(A)} . \tag{5}$$

Taking a parsimonious approach, we add a single simple interaction term to the log formula of the separable CA model above to get:

$$\log V = a + b\log(A) + c\log(N) + d\log(A)\log(N) . \qquad \text{[Model 2]} \tag{6}$$

To give some intuition about why this model might better fit the data plotted in Figure 2 we once again write the model (now Equation 6) in terms of density. Fixed R corresponds to a fixed operational area A, which is one single rising diagonal row in the Figure. The gradient of each row is the derivative of V with respect to density. Hence we consider A and $D = N/A$ as the two independent variables and write

$$\begin{aligned} V &= e^a A^{b-(c+d\log(A))} D^{c+d\log(A)} \\ \log V &= a + (b - c - d\log(A))\log(A) + (c + d + \log(A))\log(D) . \end{aligned} \tag{7}$$

In log-log space, for fixed A, the intercept is $e^a A^{b-(c+d\log(A))}$ and the gradient of the "row" for this fixed size of operational area is $(c + d + \log(A))$. Crucially this gradient is *different for each value of* A, which is exactly what we are seeking.
With one additional parameter, we fit Model 2 to the same data and find that

$$V = 11.82A^{0.14} N^{0.11\log(A)-1.01} . \tag{8}$$

The model fit statistics reveal an improved fit (RMSE of Model 2 in Table 4 is half that of Model 1). More starkly, plotting Model 2 against the data in Figure 3 [right] clearly shows that this model captures the structure of changing gradients within the data. The residuals, shown in Figure 5, also have a much better distribution.

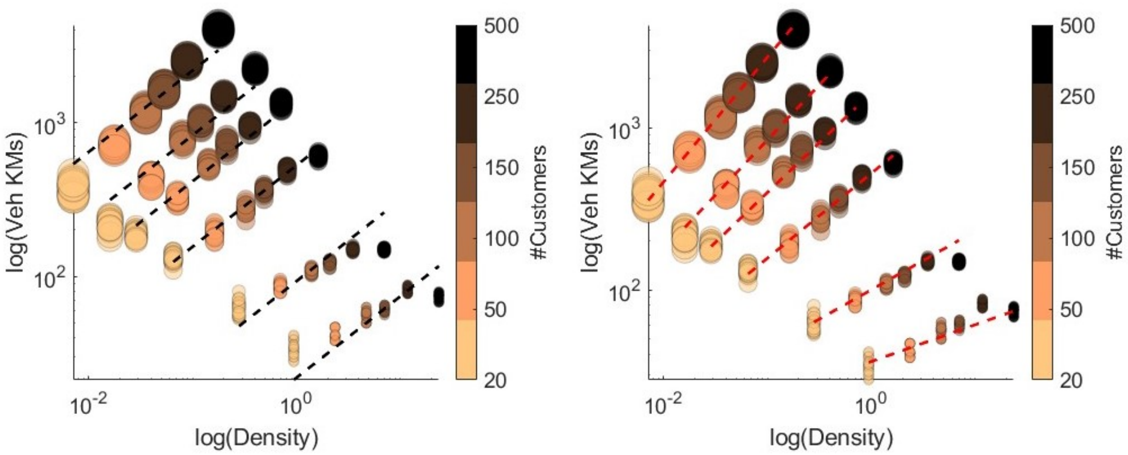

Figure 3: Log-log plot of total vehicle kms vs demand density for 720 simulation runs. [left] with Model 1 (parallel black dashed lines) and [right] with Model 2 (red dashed lines).

Table 4: Model 2 Fit

Model	Parameter	Estimate	SE	tStat	pValue	RMSE	Adjusted R^2
2: Eq (6)	(intercept)	2.47	0.065	38.17	0	0.105	0.993
	$\log(A)$	0.138	0.011	12.8	0		
	$\log(N)$	-0.113	0.013	-8.4	0		
	$\log(A)\log(N)$	0.111	0.0022	49.8	0		

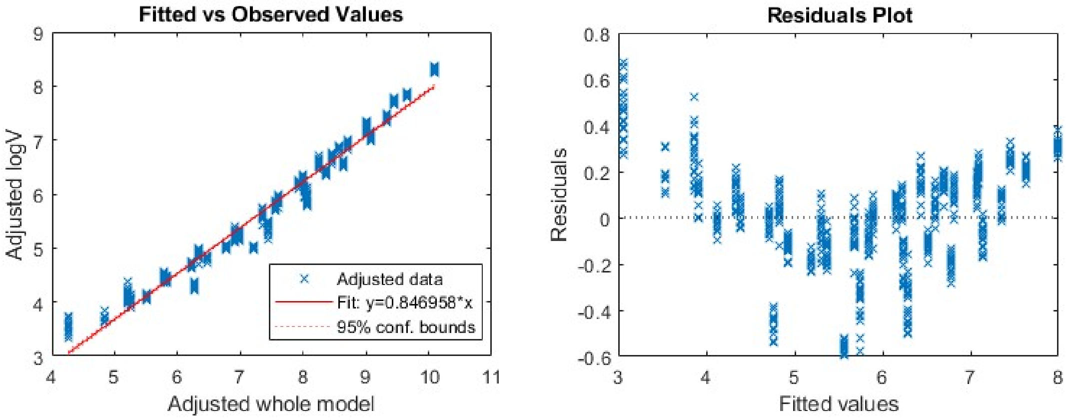

Figure 4: Model 1: fitted vs observed and residuals with structural deviations.

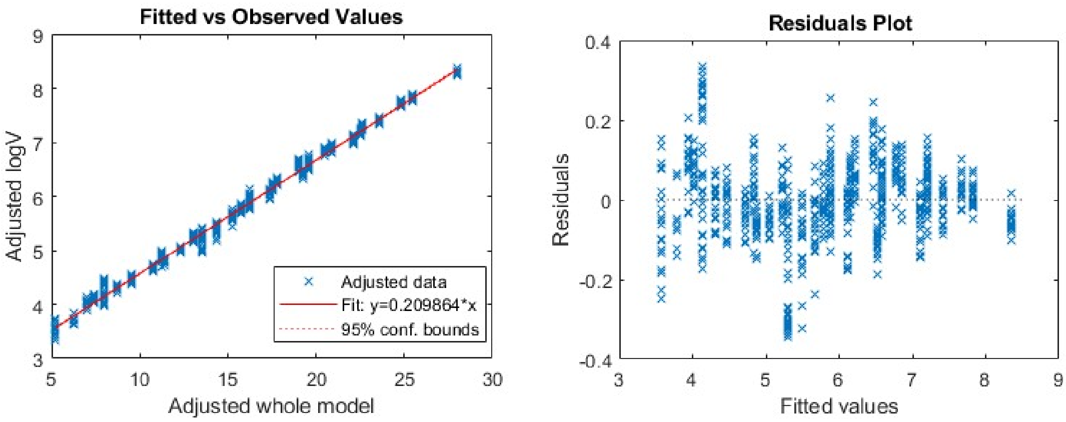

Figure 5: Model 2: fitted vs observed and residuals with better distribution.

The Role of Meeting Points in the DRFS Continous Approximation

A question naturally arises: what aspects of this DRFS give rise to the need for a more complex model than the *traditional* form of $V = \beta N^{\frac{1}{2}} A^{\frac{1}{2}}$. An obvious candidate is that it is due to the presence of meeting points. We re-run the same 720 simulations, but without meeting points i.e., every customer is picked up at their 'home' location. The results are shown in Figure 6. By construction, Model 1 shows as parallel lines which in this case fit the data well. By visual inspection, Model 2 is no better than Model 1. This is verified by the fit statistics; both models have the same RMSE (0.115) and Adjusted R^2 (0.989). This suggests that for DFRS without meeting points, Model 1 is preferable since it has fewer terms. The BIC confirms this with Model 1 (BIC = -1050.90) slightly outperforming Model 2 (BIC = -1048.81).

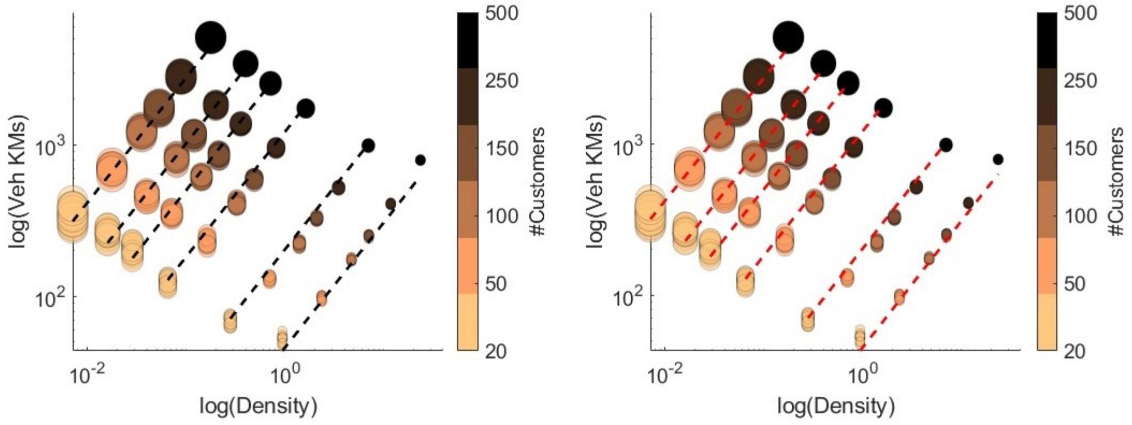

Figure 6: DRFS without meeting points fitted [left] to Model 1 and [right] to Model 2

4 Conclusions

Simulation of DRFS with meeting points reveals systematic structure in the vehicle kms of the underlying DRFS that cannot be captured by any established CA models from the literature. We propose a new, non-separable model form for the CA of this DRFS and show that it does captures the structure in the data from our simulations. The model is parsimonious, easy to fit and trivial to evaluate.

We explicitly identify that it is the presence of meeting points in this DRFS that gives rise to the need for a more complex model than the *traditional* form of $V = \beta N^{\frac{1}{2}} A^{\frac{1}{2}}$. At the hEART 2024 conference we will share additional insights into the causal roots of this previously unobserved interaction phenomenon, and applications of the CA. In this paper we only seek a CA for the total vehicle kms. DRFS has many other performance indicators e.g. number of vehicles used. We have conducted additional tests that show the same model form with interaction term (Equation 6) is necessary to formulate a CA for the fleet size of DRFS with meeting points, and for other performance indicators of interest.

Acknowledgements

The work was supported by the Luxembourg National Research Fund (C20/SC/14703944).

References

Applegate, D. L., Bixby, R. E., Chvátal, V., & Cook, W. J. (2007). *The traveling salesman problem, a computational study.* Princeton: Princeton University Press. doi: doi:10.1515/9781400841103

Beardwood, J., Halton, J. H., & Hammersley, J. M. (1959, October). The shortest path through many points. *Mathematical Proceedings of the Cambridge Philosophical Society*, *55*(4), 299–327. (Publisher: Cambridge University Press) doi: 10.1017/S0305004100034095

Braekers, K., Caris, A., & Janssens, G. K. (2014). Exact and meta-heuristic approach for a general heterogeneous dial-a-ride problem with multiple depots. *Transportation Research Part B: Methodological*, *67*, 166-186. doi: https://doi.org/10.1016/j.trb.2014.05.007

Daganzo, C. F. (1984, November). The Distance Traveled to Visit N Points with a Maximum of C Stops per Vehicle: An Analytic Model and an Application. *Transportation Science*, *18*(4), 331–350. doi: 10.1287/trsc.18.4.331

Eilon, S., Watson-Gandy, C., & Christofides, N. (1971). Distributed management. *Hafner, New York*.

Kwon, O., Golden, B., & Wasil, E. (1995, December). Estimating the length of the optimal TSP tour: An empirical study using regression and neural networks. *Computers & Operations Research*, *22*(10), 1039–1046. doi: 10.1016/0305-0548(94)00093-N

Ma, T.-Y., Fang, Y., Connors, R. D., Viti, F., & Nakao, H. (2024, January). *A hybrid metaheuristic to optimize electric first-mile feeder services with charging synchronization constraints and customer rejections.* arXiv. (arXiv:2401.03838 [math]) doi: 10.48550/arXiv.2401.03838

Robuste, F., Daganzo, C. F., & Souleyrette, R. R. (1990, August). Implementing vehicle routing models. *Transportation Research Part B: Methodological*, *24*(4), 263–286. doi: 10.1016/0191-2615(90)90002-G