Rationally Inattentive Route Choice by Link-Based Segments

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SHORT SUMMARY

In this paper, we consider a rational inattentive (RI) route choice problem. Since a computational challenge, called curse of dimensionality, always arises in the existing route-based RI choice models. We establish a RI choice model by link-based segments, where the RI choice behaviour is learnt along the links instead of the routes. We introduce the background information into the RI choice model, which ensures its optimal solution locates within the interior of the feasible region. We then analytically characterize the closed-form expression of the optimal solution, which resembles a nested logit model. Finally, a numerical example is illustrated, showing that the solution algorithm can identify the RI route choice behaviour within a reasonable computation timeframe.

Keywords: Closed-form expression; Curse of dimensionality; Discrete choice modelling; Nested logit model; Rational inattentive route choice

1. INTRODUCTION

In the information-rich society, attention is a finite resource that is easily consumed by the wealth of available information. Since attention is valuable and limited, its consumption on acquiring information has to be taken into consideration across the whole decision-making process. The rationally inattentive (RI) choice modelling framework, established in the pioneering works of Sims (2003), quantifies the expenditure of attention as the information cost using the Shannon entropy (Shannon 1948). Under this framework, Matějka and McKay (2015) is the pioneering work that links the RI framework with the discrete choice modelling by characterizing the RI choice behaviour as a generalized multinomial logit (GMNL) model.

Applying the concept of rationally inattention in transportation, Fosgerau and Jiang (2019) develops a theoretical model that features the RI traveller who aims to acquire information on the traffic conditions so as to optimally choose the departure time of their daily commuting. Expanding upon this work, Jiang et al. (2020) further extend the RI framework to address the route choice problem within a stochastic network, wherein the traveller acquires information on the traffic conditions so as to optimally choose the routes for their journeys. In these two works, the set of candidate routes is considered as an input, notwithstanding the demanding task of generating all these routes, particularly in a large-scale transportation network.

The route-based RI choice models would grapple with one computational challenge - the curse of dimensionality, i.e., the dimension of the model grows exponentially with the cardinality of the sets of states for individual links, which adversely impacts the computational efficiency of the solution algorithm. The source for the curse of dimensionality can be traced back to the assumption of independence of irrelevant alternatives (IIA) for the candidate routes. In order to overcome the IIA assumption, Fosgerau et al. (2020) proves that the nested logit (NL) model structure could overcome the IIA restriction.

In this paper, we establish a RI route choice model by link-based segments, where learning the RI route choice behaviour is decomposed into learning the RI choice behaviour on the links that comprise the candidate routes. This link-based RI choice model is served to control the dimensionality of the choice model to a manageable amount, thereby helps overcome the curse of dimensionality. We introduce the background information in the link-based RI choice model, which reflects the environment confronted by the RI traveller prior to commencing the information acquisition process. Subsequently, we prove a crucial property of our link-based RI choice model, called interior-preserving property, which ensures that its optimal solution always locates within the interior of the feasible region. Based on the interior-preserving-property, we analytically characterize the closed-form expression of the RI choice behaviours using KKT conditions, which resembles a nested logit model. A numerical examples are illustrated to validate the theoretical analysis. The outcomes show that the solution algorithm can identify the optimal RI choice behaviour in a reasonable computation timeframe.

2. A LINK-BASED RI CHOICE MODEL

We consider a general transportation network with a directed acyclic graph (without self-loop) graph $\mathcal{G}=(\mathcal{N},\mathcal{A})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ the set of links, respectively. A directed link ℓ is represented by an ordered pair of two distinct nodes, i.e., $\ell=(\mathfrak{h}_\ell^s,\mathfrak{h}_\ell^t)$, where $\mathfrak{h}_\ell^s$ is the source node and $\mathfrak{h}_\ell^t$ is the sink node. We denote the set of in-going links to node $\mathfrak{h}$ by $\mathcal{A}_\mathfrak{h}^-=\{(\mathfrak{h}',\mathfrak{h})\in\mathcal{A}|\mathfrak{h}'\in\mathcal{N}\}$, and the set of out-going links from node $\mathfrak{h}$ by $\mathcal{A}_\mathfrak{h}^+=\{(\mathfrak{h},\mathfrak{h}')\in\mathcal{A}|\mathfrak{h}'\in\mathcal{N}\}$. We denote the out-degree of node $\mathfrak{h}$ be $|\mathcal{A}_\mathfrak{h}^+|$. The links in $\mathcal{A}_{\mathfrak{h}_\ell^s}^-$ and $\mathcal{A}_{\mathfrak{h}_\ell^t}^+$ are called in-neighbours and out-neighbours of link ℓ, respectively. Without loss of generality, we assume that $\mathcal{G}$ has a single origin-destination pair, where o and d stand for the origin and destination nodes, respectively. Then, a route i from origin to destination is defined as a sequence of nodes $(\mathfrak{h}_1,\mathfrak{h}_2,\dots,\mathfrak{h}_M)$ with $\mathfrak{h}_1=o$ and $\mathfrak{h}_M=d$. With the assumption that the choice decisions on links satisfy the Markov property, the unconditional choice probability for a route i can be disassembled into the unconditional choice probabilities for the links that constitute route i, i.e., $p(i)=\prod_{m=1}^{M-1}p(\ell_m)$ with $\ell_m=(\mathfrak{h}_m,\mathfrak{h}_{m+1})$, which implies that the choice among the routes can be decomposed into the choice along the links that constitute each route.

In this paper, we use the multi-parameter approach provided in Huettner et al. (2019) and Walker-Jones (2023) to quantify the information cost. When $\mathcal{A}_\mathfrak{h}^+\neq\emptyset$, i.e., $|\mathcal{A}_\mathfrak{h}^+|>0$, we denote background information be $\Omega_{\mathfrak{h};-1}=\mathcal{A}_\mathfrak{h}^+$, habitual information be $\Omega_{\mathfrak{h};0}=\{p(\ell)\}_{\ell\in\mathcal{A}_\mathfrak{h}^+}$ and the set of states be $\mathbf{\Omega}_\mathfrak{h}=\Omega_{\mathfrak{h};1}\times\cdots\times\Omega_{\mathfrak{h};|\mathcal{A}_\mathfrak{h}^+|}$, where $\Omega_{\mathfrak{h};k}$ is the set of possible states for the k-th link in $\mathcal{A}_\mathfrak{h}^+$. We define $\mathbf{\Omega}_{\mathfrak{h};-1,\dots,k}=\Omega_{\mathfrak{h};-1}\times\Omega_{\mathfrak{h};0}\times\mathbf{\Omega}_{\mathfrak{h};1,\dots,k}$, where $\mathbf{\Omega}_{\mathfrak{h};1,\dots,k}=\Omega_{\mathfrak{h};1}\times\cdots\Omega_{\mathfrak{h};k}$. The prior belief on the occurrence of partial state $\boldsymbol{\omega}_{\mathfrak{h};1,\dots,k}$ is denoted by $g(\boldsymbol{\omega}_{\mathfrak{h};1,\dots,k})\in\mathscr{P}(\mathbf{\Omega}_{\mathfrak{h};1,\dots,k})$ for all $k=1,2,\dots,|\mathcal{A}_\mathfrak{h}^+|$, where $\mathscr{P}(\mathbf{\Omega}_{\mathfrak{h};1,\dots,k})$ denotes the set of probability distributions on $\mathbf{\Omega}_{\mathfrak{h};1,\dots,k}$.

Prior to commencing the information acquisition process, the RI traveller is confronted solely with the background information provided by the transportation network. In this context, the RI traveller is only aware of the existence of $|\mathcal{A}_{\mathfrak{h}}^{+}|$ candidate links available for choosing at the outset. Consequently, prior to commencing the information acquisition process, the amount of background information can be measured by the entropy associated with the uniform distribution.

$$\mathscr{H}(\Omega_{\mathfrak{h};-1}) = -\sum_{\ell \in \mathcal{A}_{\mathfrak{h}}^{+}} \frac{1}{|\mathcal{A}_{\mathfrak{h}}^{+}|} \ln \frac{1}{|\mathcal{A}_{\mathfrak{h}}^{+}|} = \ln |\mathcal{A}_{\mathfrak{h}}^{+}|, \tag{1}$$

which is independent of the RI route choice behaviour.

Then, the RI traveller acquires habitual information $\Omega_{\mathfrak{h};0} = \{p(\ell)\}_{\ell \in \mathcal{A}_{\mathfrak{h}}^{+}}$ from the background. Once this information is obtained, the amount of information possesses by the RI traveller can be measured by

$$\mathscr{H}(\boldsymbol{p}|\boldsymbol{\Omega}_{\mathfrak{h};-1,0}) = -\sum_{\ell \in \mathcal{A}_{\mathfrak{h}}^{+}} p(\ell) \ln p(\ell). \tag{2}$$

Thus, the amount of acquired habitual information can be measured by

$$\mathscr{I}(\boldsymbol{p}; \Omega_{\mathfrak{h};1}|\Omega_{\mathfrak{h};0}) = \mathscr{H}(\boldsymbol{p}|\Omega_{\mathfrak{h};-1}) - \mathscr{H}(\boldsymbol{p}|\boldsymbol{\Omega}_{\mathfrak{h};-1,0}).$$

Then, the RI traveller acquires information on the first sub-component of states. Once this information is obtained, they formulate the RI route choice behaviour $\{p(i|\omega_{\mathfrak{h};1})\}$. The amount of information possessed by the RI traveller can be measured by

$$\mathscr{H}(\boldsymbol{p}|\boldsymbol{\Omega}_{\mathfrak{h};-1,0,1}) = -\sum_{\ell \in \mathcal{A}_{\mathfrak{h}}^{+}} \sum_{\omega_{\mathfrak{h};1} \in \Omega_{\mathfrak{h};1}} g(\omega_{\mathfrak{h};1}) p(\ell|\omega_{\mathfrak{h};1}) \ln p(\ell|\omega_{\mathfrak{h};1}). \tag{3}$$

Then, the amount of acquired information on the first sub-component $\Omega_{\mathfrak{h};1}$ can be measured by the conditional mutual information using (2) and (3)

$$\mathscr{I}(\boldsymbol{p}; \Omega_{\mathfrak{h};1}|\boldsymbol{\Omega}_{\mathfrak{h};-1,0}) = \mathscr{H}(\boldsymbol{p}|\boldsymbol{\Omega}_{\mathfrak{h};-1,0}) - \mathscr{H}(\boldsymbol{p}|\boldsymbol{\Omega}_{\mathfrak{h};-1,0,1}). \tag{4}$$

Repeat the above information acquisition process, the total information cost can be sequentially quantified according to (7) across all the sub-components of $\boldsymbol{\Omega}$

$$C_{\text{info}}(\boldsymbol{p}) = \sum_{k=0}^{|\mathcal{A}_{\mathfrak{h}}^{+}|} \lambda_{\mathfrak{h};k} \mathscr{I}(\boldsymbol{p}; \Omega_{\mathfrak{h};k}|\boldsymbol{\Omega}_{\mathfrak{h};-1,\ldots,k-1}), \tag{5}$$

where $\lambda_{\mathfrak{h};k} > 0$ is the marginal information cost for acquiring each unit information from $\Omega_{\mathfrak{h};k}$.

On the other hand, we dente $c(\ell, \boldsymbol{\omega}_{\mathfrak{h}})$ be the travel cost for link $\ell \in \mathcal{A}_{\mathfrak{h}}^{+}$ in state $\boldsymbol{\omega}_{\mathfrak{h}}$. When the RI choice behaviour $\{p(\ell|\boldsymbol{\omega}_{\mathfrak{h}})\}$ has been learnt, the expected travel cost can be calculated by

$$C_{\mathfrak{h};\text{travel}}(\boldsymbol{p}) = \sum_{\ell \in \mathcal{A}_{\mathfrak{h}}^{+}} \sum_{\boldsymbol{\omega}_{\mathfrak{h}} \in \boldsymbol{\Omega}_{\mathfrak{h}}} c(\ell, \boldsymbol{\omega}_{\mathfrak{h}}) g(\boldsymbol{\omega}_{\mathfrak{h}}) p(\ell|\boldsymbol{\omega}_{\mathfrak{h}}). \tag{6}$$

Based on (5) and (6), the link-based RI choice model can be recursively formulated as follows

$$\begin{aligned} \min_{\{p(\ell|\boldsymbol{\omega}_{\mathfrak{h}})\}} C(\boldsymbol{p}_{\mathfrak{h}}) = & \sum_{\ell \in \mathcal{A}_{\mathfrak{h}}^{+}} \sum_{\boldsymbol{\omega}_{\mathfrak{h}} \in \boldsymbol{\Omega}_{\mathfrak{h}}} c(\ell, \boldsymbol{\omega}_{\mathfrak{h}}) g(\boldsymbol{\omega}_{\mathfrak{h}}) p(\ell|\boldsymbol{\omega}_{\mathfrak{h}}) \\ & + \sum_{k=0}^{|\mathcal{A}_{\mathfrak{h}}^{+}|} \lambda_{\mathfrak{h};k} \mathscr{I}(\boldsymbol{p}_{\mathfrak{h}}; \Omega_{\mathfrak{h};k}|\boldsymbol{\Omega}_{\mathfrak{h};-1,\ldots,k-1}) \\ & + \sum_{\ell \in \mathcal{A}_{\mathfrak{h}}^{+}} p(\ell) V(\mathcal{A}_{\mathfrak{h}_{\ell}^{t}}^{+}), \end{aligned} \tag{7}$$

$$1 = \sum_{\ell \in \mathcal{A}_{\mathfrak{h}}^{+}} p(\ell|\boldsymbol{\omega}_{\mathfrak{h}}), \text{ for all } \boldsymbol{\omega}_{\mathfrak{h}} \in \Omega_{\mathfrak{h}}, \tag{8}$$

$$0 \leq p(\ell|\boldsymbol{\omega}_{\mathfrak{h}}), \text{ for all } \ell \in \mathcal{A}_{\mathfrak{h}}^{+}, \ \boldsymbol{\omega}_{\mathfrak{h}} \in \boldsymbol{\Omega}_{\mathfrak{h}}, \tag{9}$$

where $V(\mathcal{A}_{\mathfrak{h}_\ell^t}^{+}) \triangleq \min C(\boldsymbol{p}_{\mathfrak{h}_\ell^t})$ is the value function that represents the optimal total cost for learning the RI choice behaviour among the out-neighbours of link ℓ. For the sake of completeness, we denote $V(\mathcal{A}_{\mathfrak{h}}^{+}) = 0$, if $\mathcal{A}_{\mathfrak{h}}^{+} = \emptyset$. Note that a link has no out-neighbour if and only if it sinks at the destination node, i.e. $\mathfrak{h} = d$. The last term signifies the anticipated total cost for learning the RI choice behaviour from the out-neighbours of each link in $\mathcal{A}_{\mathfrak{h}}^{+}$.

3. OPTIMAL LINK-BADED RI CHOICE BEHAVIOUR

In order to analytically characterize the optimal RI choice behaviour described by link-based RI choice model (7)-(9), we should first prove its optimal solution always locates within the interior of the feasible region, as long as $\mathcal{A}_{\mathfrak{h}}^{+}$ contains at least two feasible links.

Interior-preserving property

We make the following assumption on the marginal information costs.

Assumption 1. *The marginal information costs satisfy* $0 < \lambda_{\mathfrak{h};0} < \lambda_{\mathfrak{h};1} < \cdots < \lambda_{\mathfrak{h};|\mathcal{A}_{\mathfrak{h}}^{+}|} < +\infty$ *for all* $\mathfrak{h} \in \mathcal{N} \backslash \{d\}$.

Then, we can derive the following interior-preserving property for the link-based RI choice model (7)-(9).

Theorem 1. *Let* $\{p^*(\ell|\boldsymbol{\omega}_{\mathfrak{h}})\}$ *be the optimal solution of link-based RI choice model (7)-(9). For any* $\mathfrak{h} \in \mathcal{N} \backslash \{d\}$ *and* $|\mathcal{A}_{\mathfrak{h}}^{+}| \geq 2$, *under Assumption 1, we have* $p^*(\ell|\boldsymbol{\omega}_{\mathfrak{h}}) > 0$ *for all* $\ell \in \mathcal{A}_{\mathfrak{h}}^{+}$ *and* $\boldsymbol{\omega}_{\mathfrak{h}} \in \boldsymbol{\Omega}_{\mathfrak{h}}$.

Closed-form expression of the optimal conditional links choice probabilities

Based on the interior-preserving property established in Theorem 1, wen can now proceed to analytically characterize the closed-form expression of the optimal conditional link choice probabilities, which is given as follows

Theorem 2. *For any* $\mathfrak{h} \in \mathcal{N} \backslash \{d\}$ *and* $|\mathcal{A}_{\mathfrak{h}}^{+}| \geq 2$. *Under Assumption 1,* $\boldsymbol{p}_{\mathfrak{h}} = \{p(\ell|\boldsymbol{\omega}_{\mathfrak{h}})\}$ *is an optimal solution of link-based RI choice model (7)-(9), if and only if* $p(\ell|\boldsymbol{\omega}_{\mathfrak{h}}) > 0$ *and*

$$p(\ell|\boldsymbol{\omega}_{\mathfrak{h}}) = \frac{e^{-\frac{c(\ell,\boldsymbol{\omega}_{\mathfrak{h}}) + V\left(\mathcal{A}_{\mathfrak{h}_\ell^t}^{+}\right)}{\lambda_{\mathfrak{h};|\mathcal{A}_{\mathfrak{h}}^{+}|}}}\, \Gamma(p(\ell|\boldsymbol{\omega}_{\mathfrak{h}}))}{\sum_{\ell' \in \mathcal{A}_{\mathfrak{h}}^{+}} e^{-\frac{c(\ell',\boldsymbol{\omega}_{\mathfrak{h}}) + V\left(\mathcal{A}_{\mathfrak{h}_{\ell'}^t}^{+}\right)}{\lambda_{\mathfrak{h};|\mathcal{A}_{\mathfrak{h}}^{+}|}}}\, \Gamma(p(\ell'|\boldsymbol{\omega}_{\mathfrak{h}}))}, \tag{10}$$

for all $\ell \in \mathcal{A}_{\mathfrak{h}}^{+}$ *and* $\boldsymbol{\omega}_{\mathfrak{h}} \in \boldsymbol{\Omega}_{\mathfrak{h}}$, *where* $\Gamma(p(\ell|\boldsymbol{\omega}_{\mathfrak{h}})) = \prod_{k=1}^{|\mathcal{A}_{\mathfrak{h}}^{+}|} p\,(\ell|\boldsymbol{\omega}_{\mathfrak{h};1,\ldots,k-1})^{\frac{\lambda_{\mathfrak{h};k} - \lambda_{\mathfrak{h};k-1}}{\lambda_{\mathfrak{h};|\mathcal{A}_{\mathfrak{h}}^{+}|}}}$.

Similar to the iterative algorithm proposed in Huettner et al. (2019), we can devise a backward induction algorithm to seek the optimal RI choice behaviour described by link-based RI choice model (7)-(9) for all $\mathfrak{h} \in \mathcal{N} \backslash \{d\}$ using the closed-form expression (10).

4. NUMERICAL EXPERIMENTS

We take a simple transportation network, given in Jiang et al. (2020), as an illustration. The illustrated network is shown in Figure 1, which contains 6 nodes and 9 links. The origin is Node 1 and the destination is Node 6. The RI traveller chooses among 5 feasible routes from origin to destination, as listed in Table 1. In Figure 1, the set of states encompasses 9 sub-components, i.e., $\mathbf{\Omega} = \Omega_1 \times \Omega_2 \cdots \times \Omega_9$.

Table 1. Enumeration of routes.

Route No.	Node sequence
1	(1,2,3,6)
2	(1,2,5,6)
3	(1,5,6)
4	(1,4,5,6)
5	(1,2,6)

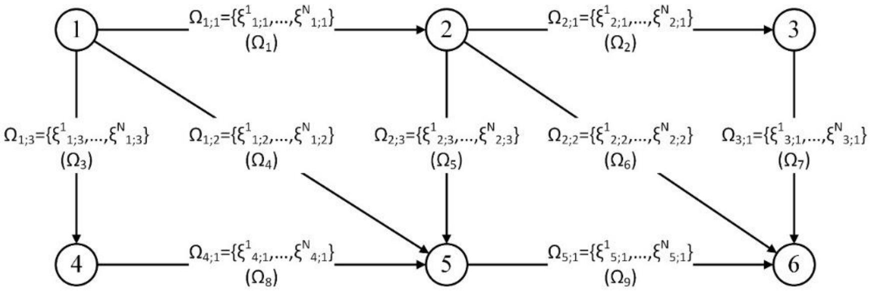


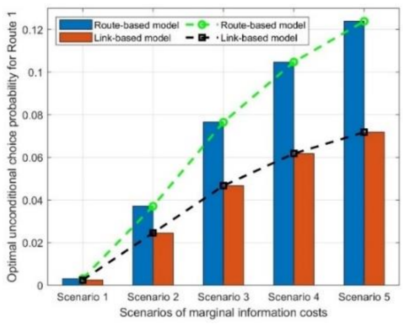
Figure 1. A simple network for illustration.

We compare the computational performances for solving the route-based and link-based RI choice model under four different quantities of possible states for each link. For each specific quantity, we consider five different scenarios of marginal information costs. Specifically, for route-based RI choice model, we take the marginal information costs as $\lambda_k = (k+1) \times w$ for all $k = 0, 1, ..., 9$ in each specified scenario $w = 1, 2, ..., 5$. Subsequently, the marginal information costs for the sub-components in link-based RI choice model are identical to the corresponding sub-component in route-based RI choice model. The value of possible states for each link are randomly selected, but keep constant as long as the quantity of possible states is specified.

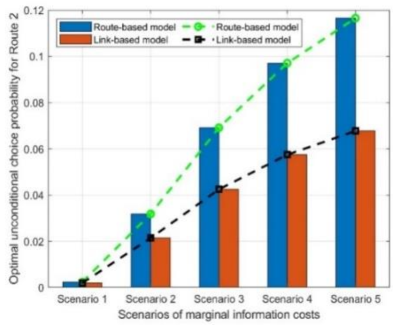
The numerical results are illustrated in Table 3 and Figure 3. As observed in Table 3, a larger quantity of possible states for each link notably amplifies the dimension of the route-based RI choice model, consequently leading to a substantial increase in the runtime required for its solution. In comparison, the link-based RI choice model ensures a more manageable increase in its dimension as the quantity of possible states for each link grows. Consequently, this modelling framework maintains a high level of computational efficiency for its solution. It can be observed from Figure 3 that the RI route choice behaviour described by the link-based RI choice model markedly deviates from which described by the route-based counterpart. Specifically, there are notable discrepancies in the unconditional choice probabilities for the five routes between the link-based RI choice model and the route-based counterpart in each scenario of marginal information costs.

Table 3. Runtime for route-based model and link-based model.

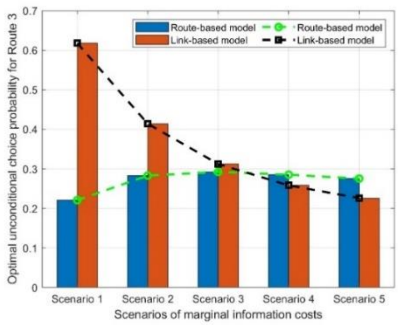
Quantity of possible states for each link	Scenario No.	Route-based model		Link-based model	
		Dimension (in num.)	Runtime (in sec.)	Dimension (in num.)	Runtime (in sec.)
2	1	2560	2.74	48	0.02
	2		2.93		0.03
	3		3.11		0.02
	4		2.98		0.02
	5		3.06		0.02
3	1	98415	107.82	162	0.07
	2		110.22		0.07
	3		109.80		0.07
	4		115.56		0.06
	5		107.98		0.07
4	1	1310720	1676.78	384	0.15
	2		1747.73		0.16
	3		1667.07		0.18
	4		1595.86		0.16
	5		1661.44		0.16



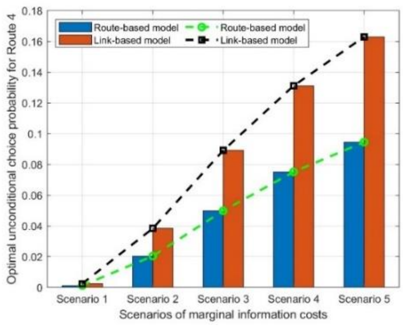
(a)



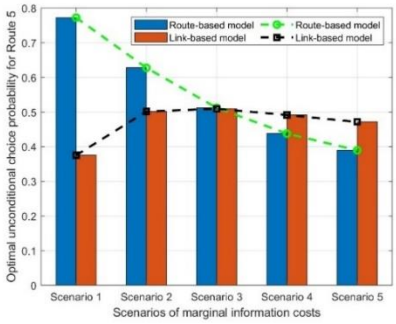
(b)



(c)



(d)



(e)

Figure 3. Unconditional choice probabilities for the five routes when each link has two possible states: Route-based model versus Link-based model. (a) Route 1; (b) Route 2; (c) Route 3; (d) Route 4; (e) Route 5.

5. CONCLUSIONS

In this paper, we propose a link-based RI choice model to describe the RI choice behaviour over the transportation network. The contribution of this paper is to overcome the curse of dimensionality by the link-based segments, which is a common computational challenge involved in the existing route-based RI choice models. By incorporating the background information, we find that our RI choice model ensures that the optimal solution of our RI choice model always locates within the interior of the feasible region, i.e., every candidate is assigned a positive choice probability. Using the interior-preserving-property, we analytically characterize the closed-form expression of the optimal RI choice behaviour, which resemble a nested logit model. A numerical example is illustrated for two networks. The results show that the link-based RI choice model is more computational effective than the route-based counterpart.

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