Stochastic departure time user equilibrium with heterogeneous trip profile

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SHORT SUMMARY

Recently, the generalized bathtub model extended the classic bathtub model to capture various distributions of the trip length by introducing a new state variable: the number of active trips with remaining distances greater than or equal to a threshold. The traffic dynamics are reformulated by four partial differential equations that track the distribution of the remaining trip lengths. This study aims to formulate and solve stochastic user equilibrium for the dynamic departure time choice model based on the generalized bathtub model with heterogeneous trip attributes. In particular, the proposed framework is able to address any distribution for desired arrival time and trip length. We first formulate the problem in continuous form as a fixed-point problem. Then we apply a discretization method to address the trip-based setting. The proposed framework is applied to the real demand profile of the large-scale network of Lyon North for the morning peak hour.

Keywords: traffic congestion, peak-hour traffic dynamics, macroscopic model, stochastic user equilibrium, generalized bathtub model

1. INTRODUCTION

Rapid growth in the demand for the urban transportation networks has given rise to the critical need for mature management of the services in the transportation networks. In this respect, a deep understanding of users' travel behavior in the system is crucial. In order to demonstrate the users' behavior, we can assume that users are rational and tend to optimize their own utilities. However, the discrepancy in the various goals that users have in comparable situations in the network can be attributed to the violation of the user equilibrium (UE) principles, defined by (Wardrop, 1952)). This heterogeneity in the decision-making process of users can be addressed by defining stochasticity in the mathematical model (Kawakami & Xu, 1964). In the context of departure time choice models, the solution is defined as stochastic user equilibrium (De Palma, Ben-Akiva, Lefevre, & Litinas, 1983). This study focused on modeling and solving the stochastic departure time UE problem.

In addition to the users' decision-making process, the traffic congestion should be captured based on the characteristic of the transportation network. Macroscopic models aim to rule out urban traffic congestion by one main assumption that states the speed is fixed for all internal users of a single traffic zone at each time step with respect to its traffic density and topological characteristics (Mahmassani & Herman, 1984). This assumption is validated by the study of (Geroliminis & Daganzo, 2008), wherein they showed that single-unit macroscopic models could represent the congestion of an urban area independently of the demand. One of the advanced macroscopic models is the generalized bathtub model, recently proposed by (Jin, 2020). The main advantage of this model compared to the classic models such as bottleneck models (Vickrey, 1991) is that it can capture the traffic dynamics for demand profiles with any distribution of trip length by introducing a new state variable: the number of active trips with remaining distances greater than or equal to a threshold. (Ameli, Faradonbeh, Lebacque, Abouee-Mehrizi, & Leclercq, 2021) discussed the other advantage of the generalized bathtub model compared to the other macroscopic models.

The SUE model is addressed on the other macroscopic models (see, e.g., (Lim & Heydecker, 2005; Wang, Szeto, Han, & Friesz, 2018; Long, Yang, & Szeto, 2021; Lamotte, 2018)). In contrast, SUE is not yet formulated by the generalized bathtub model. The main advantage of using the generalized bathtub model will be the ability to consider any type of heterogeneity for the demand profile. Therefore, we aim to formulate and solve SUE for the departure time choice model based on the generalized bathtub model with heterogeneous trip attributes. In particular, we propose continuous and discrete models that are able to address any distribution for desired arrival time and trip length.

2. METHODOLOGY

The model.

Let us recall the main elements of the model.

- $H(t)$: the number of travellers in the system at time t, $t \in [O,T] \subset \mathscr{T}$.
- This number is dis-aggregated wrt to desired arrival time $t_a \in \mathscr{T}_a$ and remaining travel distance $x \in \mathscr{X}$. Thus $K(t_a,x,t)$ denotes the density at time t wrt to desired arrival time $t_a \in \mathscr{T}_a$ of travellers with remaining travel distance greater than $x \in \mathscr{X}$. By this definition

$$H(t) = \int_{\mathscr{X}_a} K(t_a,0,t)\,dt_a \tag{1}$$

- $h(x)$: the initial number of travellers with remaining travel distance greater than $x \in \mathscr{X}$) and $k(t_a,x)$ (the initial distribution K) denote the initial conditions for H and K. Thus:

$$h(x) = \int_{\mathscr{X}_a} k(t_a,x)\,dt_a \tag{2}$$

- The total demand for trips has a density $m(t_a,x)dt_a dx$. A standing hypothesis is that the distribution of travellers wrt to trip length x is defined by an integrable bounded density function. On the other hand there may be a finite number of desired arrival times.
- The traffic assignment yields the distribution of traveller demand, which is denoted as $f(t_a,x,t)\,dt_a\,dx\,dt$, and expresses the departure time choice t of travellers, given their desired arrival time t_a and their trip length x. f constitutes the main unknown of the assignment problem. f satisfies to the following convex set of constraints $(\mathscr{K})$:

$$(\mathscr{K}) \quad \left| \begin{array}{l} \int_{\mathscr{T}} f(t_a,x,t)dt = m(t_a,x) \\ f(t_a,x,t) \geq 0 \end{array} \right. \tag{3}$$

- The speed of travellers $v(t)$ is uniform in the network, time-dependant and a function of the total number of travellers $H(t)$. Thus

$$v(t) = V\,(H(t)) \tag{4}$$

  A standing hypothesis of the model is that the fundamental diagram V is bounded from above and below: $0 < V_{min} \leq V \leq V_{max}$.

Let us now describe the dynamics of K. They are described by the following system

$$\left| \begin{array}{ll} z(t) \overset{def}{=} \int_0^t dt\, V(H(t)) & (5.1) \\ K(t_a,x,t) = k(t_a,x+z(t)) + \int_0^t ds\, F(t_a,x+z(t)-z(s),s) & (5.2) \\ F(t_a,x,t) = \int_x^\infty d\xi\, f(t_a,\xi,t) & (5.3) \end{array} \right. \tag{5}$$

(5.2) expresses that K has initial value k, and has inflow rate at time t

$$\int_x^\infty dx\, f(t_a,x,t).$$

The outflow of the system results from travellers terminating their trip, when their remaining trip length is zero. By differentiating (5) the following system results:

$$\left| \begin{array}{ll} \partial_t K - v(t)\partial_x K = f & (6.1) \\ v(t) = V(H(t)) \text{ and } H(t) = \int_{\mathcal{T}_a} dt_a\, K(t_a,0,t) & (6.2) \\ K(t_a,x,0) = k(t_a,x) \text{ the given initial condition} & (6.3) \end{array} \right. \tag{6}$$

Let us now describe the dynamics of H. They are described by the following system

$$\left| \begin{array}{ll} z(t) = \int_0^t dt\, V(H(t)) & (7.1) \\ H(t) = h(z(t)) + \int_0^t ds\, \bar{F}(z(t)-z(s),s) & (7.2) \\ \bar{F}(x,t) = \int_{\mathcal{T}_a} dt_a\, f(t_a,x,t) & (7.3) \end{array} \right. \tag{7}$$

(7) can be deduced from (5) by setting $x = 0$ in the latter and integrating it over t_a. It can be shown, adapting the classical Picard argument (refer to (Ameli, Faradonbeh, et al., 2021)) that (7) admits a unique solution in z and H which depends continuously on the data and initial conditions. The reason for which it is possible to state an equation for $H(t)$ is that the speed of traffic is considered uniform following (4). The solution of (7) is denoted

$$(7) \iff \left| \begin{array}{l} H = \mathscr{H}(f,h) \\ z = \mathscr{Z}(f,h) \end{array} \right. \tag{8}$$

In the space of continuous functions, $\mathscr{H}$ and $\mathscr{Z}$ are Lipschitz continuous wrt h and weak- continuous wrt f (recall that f is a bounded positive measure). (refer to to (Ameli, Faradonbeh, et al., 2021)).

Once (7) has been solved, it is straightforward to calculate K by evaluating the integrals (5.2) and (5.3).

Stochastic dynamic user equilibrium.

The time at which a traveller entering the system at time t with trip length x can exit the system is denoted as $TA(x,t)$ (arrival time) and is given by:

$$TA(x,t) = z^{-1}(x+z(t)) \tag{9}$$

Indeed $z(TA(t)) - z(t) = x$. Thus the travel time is given by $TA(x,t) - t$. Given that $v(t) \geq V_{min} > 0$, it follows that z^{-1} is Lipschitz continuous and thus TA is also Lipschitz continuous.

The travel cost perceived by a traveller entering the system at time t, with trip length x and desired arrival time t_a, is given by:

$$J(t_a,x,t) = \alpha\,(TA(x,t)-t) + \phi\,(TA(x,t)-t_a) \tag{10}$$

where ϕ denotes the penalty. It is a convex function of the difference between the actual and the desired arrival time. A frequently used expression for the penalty ϕ is the following:

$$\phi(\vartheta) = \beta(-\vartheta)_+ + \gamma(\vartheta)_+ \tag{11}$$

with $\alpha > \beta$. Note that $(y)_+ \stackrel{def}{=} \max\{y,0\}$. J is a function of f and h via (10), (9) and (8):

$$J = \mathscr{J}(f,h) \tag{12}$$

The Logit stochastic user equilibrium specifies that the demand $m(t_a,x)$ is disaggregated wrt t proportionally to $\exp[-\varsigma J(t_a,x,t)]$:

$$f(t_a,x,t) = m(t_a,x)\frac{\exp[-\varsigma J(t_a,x,t)]}{\int_{\mathscr{T}} ds\, \exp[-\varsigma J(t_a,x,s)]} \tag{13}$$

ς denotes the sensitivity coefficient of the model. For numerical evaluation (13) is better reformulated as:

$$f(t_a,x,t) = \frac{m(t_a,x)}{\int_{\mathscr{T}} ds\, \exp - [\varsigma\, (J(t_a,x,s) - J(t_a,x,t))]} \tag{14}$$

(13) or (14) express f as a function of J:

$$f = \mathscr{F}(J) \tag{15}$$

Note that (13), (14) or (15) implies that f satisfies the demand constraints $(\mathscr{K})$.

The SDUE problem can be formulated as the following fixed point problem:

$$\left| \begin{array}{l} f = \mathscr{F}(J) \\ J = \mathscr{J}(f,h) \end{array} \right. \tag{16}$$

If m is a bounded integrable function (i.e. $m \in L^\infty(\mathscr{T}_a \times \mathscr{X})$ then it can be shown that (16) admits a fixed point in $L^\infty(\mathscr{T}_a \times \mathscr{X} \times \mathscr{T})$. The idea is to show the result first in L^2 by a Brouwer-type argument based on the fact that the fixed point problem is set in $(\mathscr{K})$ which is convex and closed. A similar result can be shown if m is distributed over a finite number of values of t_a. Then (16) admits a fixed point which is in $L^\infty(\mathscr{X} \times \mathscr{T})$ for all possible values of t_a.

A final point: care must be taken with boundary effects. To be specific, let us consider that the fixed point problem (16) is set in a departure time interval $[0,T]$. Then the travel cost of travellers must be evaluated in the time interval $[0, T + \max_{x\in\mathscr{X}}(TA(x,T))]$. Since the travel time of a traveller leaving at time T depends on the velocity of travellers during the time-interval $[T, T + \max_{x\in\mathscr{X}}(TA(x,T))]$ which in turn depends on the inflow into the system during that interval. This inflow must be specified independently of f. Typically the corresponding demand will be assumed to be 0.

Discretization

The discretization of the problem addresses the following tasks:

- to calculate H given f and the initial condition h i.e. to provide a numerical approximation of (8);
- to calculate J given H i.e. provide a numerical approximation of (9) and (12);
- to approximate numerically the Logit assignment formula (14) (i.e. the expression of f as a function of J (15)).

Three possible methods can be considered.

1. Discretize (6) directly. Indeed (6.1) can be recast as an advection type equation the discretization of which is simple because the wave speed is given by $v(t)$ in (6.2). The size of the system can be reduced by integrating (6.1) over $t_a \in \mathscr{T}_a$, thus solving with respect to $KR(x,t) \overset{def}{=} \int_{\mathscr{T}_a} dt_a K(t_a,x,t)$ which satisfies the same equation (6.1) as K but with one order lower dimensionality, since KR is only a function of x and t. Actually since it is possible to solve the dynamics wrt H only as in (7) we chose instead to solve (8) because H depends on t only.

2. Cell-wise discretization of (7). The idea is to discretize $\mathscr{T}$ into cells $[t_{n-1},t_n]$, $n \in \mathscr{N}$, $\mathscr{X}$ into cells $[x_{\ell-1},x_\ell]$, $\ell \in \mathscr{L}$, and for simplicity's sake we assume the set $\mathscr{T}_a$ discrete. Then z and H are approximated by piecewise linear functions defined by the values $z_n = z(t_n)$ and $H_n = H(t_n)$. z^{-1} need not be explicitly calculated as we need essentially TA. TA is approximated by a piecewise linear function defined by its values $TA_{\ell n} = TA(x_\ell,t_n)$. The disaggregated demand f is approximated by a piecewise constant function which takes the values $f_{t_a \ell n} = \int_{x\ell-1}^{x\ell} d\xi \int_{tn-1}^{tn} d\tau f(t_a,\xi,\tau)$. The demand m is also approximated by a piecewise constant function. Thus (7) boils down to an explicit Euler scheme for the integration of z i.e. (7.1), whereas (7.2) can be solved by simply evaluating the right-hand-side integral. $\bar{F}$ is evaluated as a piecewise constant function by calculating the integral in (7.3). TA results from (9) and then J is deduced as a piecewise linear function with values $J_{t_a \ell n} = TA_{\ell n} - t_n + \phi(TA_{\ell n} - t_a)$. Finally f follows from J by (14).

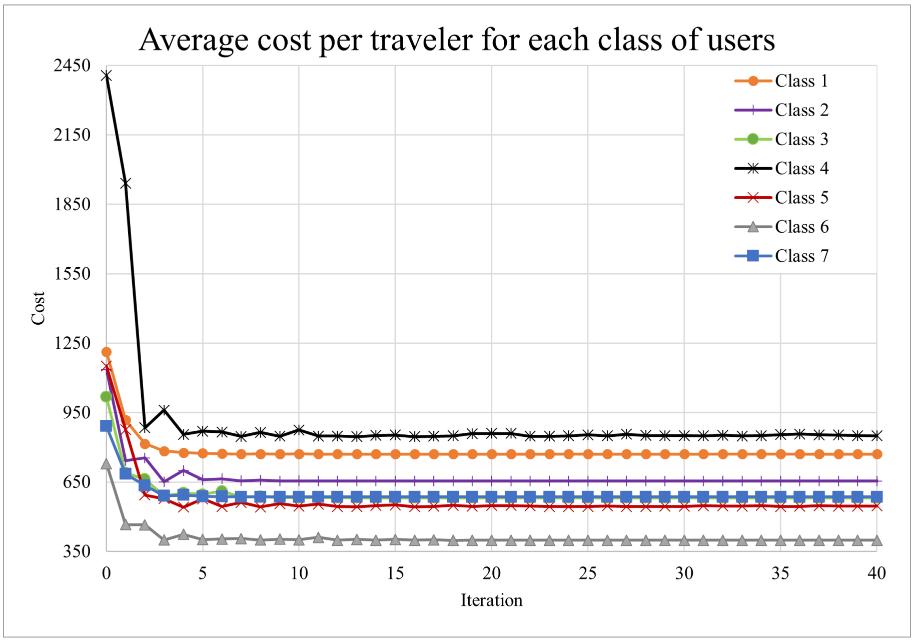
3. Particle discretization of (7). The main difference with the second approach is that the disaggregated demand f is described also by particles $p \in \mathscr{P}$. The particle discretization is easily put in correspondence with a micro-simulation. Each particle is endowed with a departure time TD_p (which results from f), an arrival time TA_p, a remaining trip length x_p, and a desired arrival time $t_{a,p}$. The treatment of the dynamics of the system (7) are now different. The particle p enters the system at time TD_p which is a data, x_p decreases at a rate $v_n = V(H_n)$, and the particle p exits the system when $x_p = 0$ which defines TA_p. The number of particles present in the system yields H at any time, thus yield v and z. The travel cost for particle p is given by $J_p = \alpha(TA_p - TD_p) + \phi(TA_p - t_{a,p})$. Finally the $f_{t_a \ell n}$ values follow from J by averaging (14) for all particles in a cell.

3. RESULTS AND DISCUSSION

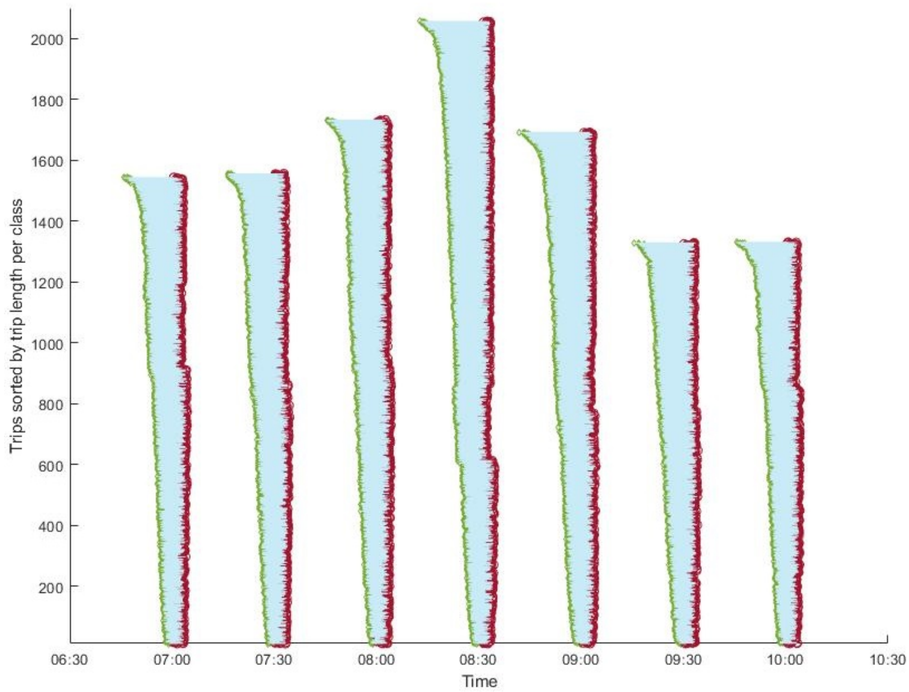
We implemented and applied our SUE model to the Lyon North network, which includes 1,883 nodes and 3,383 links. The network characteristics are presented in (Mariotte, Leclercq, Batista, Krug, & Paipuri, 2020). The demand profile includes 62,450 trips trips during the morning peak hours (6:30 AM to 10:40 AM). The data set of the demand profile is published in (Ameli, Alisoltani, & Leclercq, 2021) and its characteristics considered in this study are detailed in (Ameli, Faradonbeh, et al., 2021). The parameters of the scheduling preferences are set based on the study of (Lamotte, 2018) and $\varsigma = 0.75\ \mathrm{s}^{-1}$ based on (Long, Szeto, Shi, Gao, & Huang, 2015).

To solve the SUE problem, we followed most of the studies in the literature and used a heuristic method based on the method of successive average. The MSA ranking, introduced by (Sbayti, Lu, & Mahmassani, 2007), is employed to calculate the equilibrium. The main advantage of this method is that it guarantees convergence as it relies on predetermined step size. The convergence results for all user classes of trips are presented in Figure 1(a). The convergence pattern of the algorithm shows that it is stabilized after the fifth iteration. The few variations of different classes are caused by the heuristic nature of the algorithm that explores the solution space. Note that the exploration rate of heuristic methods also depends on the step size (Ameli, Lebacque, & Leclercq, 2020).

Figure 1(b) illustrates the final equilibrium departure time and arrival time distributions. Each green diamond denotes the departure time, and each red circle represents the arrival time of a trip, with the duration denoted by a horizontal blue line between the departure and the arrival time. In each class, the trips are sorted based on their trip length. In Figure 1(b), the deformations of the distributions for all classes show that non-regular sorting pattern which is expected for the stochastic UE. The final solution validates the good performance of the solution method.



(a) The convergence of the travel cost for each class of users.



(b) Departure and arrival time distributions of the final solution.

Figure 1: Convergence and the solution characteristics.

4. CONCLUSIONS

This study proposed a new formulation for departure time stochastic dynamic user equilibrium problem based on the generalized bathtub model. Based on the analytical analysis, the continuous formulation of the model can consider any distribution for trip length and desired arrival time of the users. In addition, the particle discretization approach is used to extend the model to trip-based formulation in order to address realistic test cases. The application of the proposed framework to the large-scale test case showed that the model and the solution method are capable of representing and solving the stochastic dynamic user equilibrium problem with multiple desired arrival times and heterogeneous trip length for a large number of trips that was little addressed before. The authors are currently preparing an analytical test case to investigate the features of the continuous model and provide some analytical proofs. The second direction is to benchmark the model with other SUE models for macroscopic and microscopic models.

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