A demand-based optimization approach to study oligopolistic markets

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1 Introduction

Oligopolistic competition occurs when a small number of operators compete for the same pool of customers. This is often the case in transportation, due to reasons such as external regulations, limited capacity of the infrastructure and difficulty in entering a well-established market.

In our work, we propose a demand-based optimization approach to study oligopolistic markets. The framework takes into account interactions between demand and supply as well as competition among suppliers. In particular, the preferences of the customers are modelled at a disaggregate level according to random utility theory, while competition is modelled explicitly as a multi-leader-follower game in which all suppliers (leaders) simultaneously optimize their decisions based on their knowledge of the customers (followers).

A fixed-point optimization model to find equilibrium solutions of such games is presented which can incorporate both nonlinear and linearized customer choices probabilities. Due to its complexity, the model can currently tackle small-size instances with restricted strategy sets. The model has been conceived as a final block of a heuristic framework in which efficient strategy generation techniques are employed to select for all competitors a subset of feasible strategies that could produce equilibrium solutions. This is fundamental in real-life problems, where the enumeration of all strategies is not possible for computational reasons. A parallel research stream is currently investigating algorithmic approaches to generate candidate equilibrium strategies.

The rest of the paper is structured as follows. Section 2 reviews two formulations of the demand-based optimization problem. Section 3 proposes a mixed integer program that solves the fixed-point problem to find equilibrium solutions of multi-leader-follower games. Section 4 presents the results of some numerical experiments. Section 5 illustrates the future steps of this research.

2 Demand-based optimization

Demand-based (or choice-based) optimization models have attracted more and more attentions in the recent years. The goal of such models is to study customer behavior at a disaggregate level in order to include it into the optimization problem of the suppliers. By incorporating customer behavior inside their optimization problem, suppliers can improve many of their strategic decisions. Generally, demand-based optimization problems can be modelled as Stackelberg games [1]. Equivalent Stackelberg problems are frequent in transportation when a supplier or regulator knows the utility functions of its potential customers, who collectively play the follower role. From a modelling perspective, the result is an optimization problem having optimization problems in the constraints, also known as bilevel program [2].

Applications of demand-based optimization models include revenue management [3, 4] and road tolling [5], among others. The majority of the papers propose nonlinear formulations and estimate choice probabilities with the multinomial logit model (MNL), whose advantage is the existence of a closed-form expression. To overcome MNL limitations on random taste variation or correlation between alternatives, more complex discrete choice models such as the nested logit and the mixed multinomial logit have also been used.

A framework that can integrate any discrete choice model in a MILP is presented in [6]. More specifically, choice probabilities can be linearized by using simulation to draw from the utility function's known error term distribution. For all customers and alternatives, a number of draws are extracted, corresponding to different behavioral scenarios. In each scenario customers deterministically choose the utility-maximizing alternative. Over multiple scenarios, the choice probability of an alternative is equal to the number of times the alternative is chosen over the number of draws.

Nonlinear and linear demand-based optimization models

Consider a set N of customers, a set I of alternatives and a set $I_k \subseteq I$ of alternatives managed by the supplier. Let V_{in} and P_{in} be the utility associated by customer $n \in N$ to alternative $i \in I$ and the corresponding choice probability. For the sake of simplicity, we assume that prices p_{in} are the only upper-level decision variables, that the supplier has no operational costs and that choice probabilities are estimated by using a MNL. These assumptions could be removed by considering a generic vector of decision variables which affect both profits and costs and by modelling customer choices with any desired discrete choice model. The optimization problem is then

$$\max \quad \sum_{i \in I_k} \sum_{n \in N} p_{in} P_{in} \tag{1}$$

$$s.t. \quad P_{in} = \frac{\exp(V_{in})}{\sum_{j \in I} \exp(V_{jn})} \qquad \forall i \in I, \forall n \in N \tag{2}$$

$$V_{in} = \beta_{in} p_{in} + q_{in} \qquad \forall i \in I, \forall n \in N. \tag{3}$$

The objective function (1) maximizes the supplier's revenue. Constraints (2) derive the choice probabilities. Constraints (3) define the deterministic utility functions, composed of an exogenous term q_{in} and an endogenous term that depends on the price, which is the variable linking the upper-level problem with the lower-level problem.

For the linearized version of the model, let R be the set of behavioral scenarios. For each $r \in R$, an error term parameter ξ_{inr} is drawn from the known distribution. The variables $U_{nr} = \max_i U_{inr}$ capture the value of the highest utility for customer n in scenario r, while the binary decision variables w_{inr} identify the alternative i chosen by each customer n in each scenario r. Constraints (2-3) can be now written as

$$s.t. \quad P_{in} = \frac{\sum_{r \in R} w_{inr}}{R} \qquad \forall i \in I, \forall n \in N \tag{4}$$

$$U_{inr} = \beta_{in} p_{in} + q_{in} + \xi_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{5}$$

$$U_{inr} \leq U_{nr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{6}$$

$$U_{nr} \leq U_{inr} + M_{U_{nr}}(1 - w_{inr}) \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{7}$$

$$\sum_{i \in I} w_{inr} = 1 \qquad \forall n \in N, \forall r \in R. \tag{8}$$

The utility functions (5) now includes a drawn error term. Constraints (6-7) ensure that in each behavioral scenario customers deterministically choose the alternative yielding the highest utility.

3 A MIP for the fixed-point problem

In oligopolistic markets there are multiple players that simultaneously solve a demand-based optimization problem. The result is a multi-leader-follower game in which the payoffs are a function of both the decisions of the customers and the strategies of the competitors. Given the complexity of the demand-based optimization framework, well-known results on the existence or uniqueness of pure or mixed strategy Nash equilibria cannot be exploited and alternative approaches are needed.

Several works dealing with Nash equilibria in transportation adopt an algorithmic approach based on the fixed-point iteration method (see for example [7] and [8]). Starting from an initial feasible solution to the problem, operators take turns to play their best response pure strategy to the last strategy played by the competitors. Such sequential game terminates when a solution is repeated, as it induces the same sequence of best responses as before. Such solution is either a Nash equilibrium for the game or a set of n strategies for each player, with $n > 1$, which would continue to be played cyclically.

Solving the multi-leader-follower game as a sequential game is attractive from a computational perspective. The sequential game is also easily interpretable, since it reproduces the behavior of two or more players that do not know the competitors' objective function. However, the convergence proof of such algorithm depends on conditions such as having a convex payoff function [9], which are not verified in the multi-leader-follower games we want to solve. Consequently, by solving the problem as a sequential game there is no guarantee of existence or uniqueness of a pure strategy Nash equilibrium. Finally, different initial solutions could lead to different equilibria.

We propose a new mathematical model to find equilibria in multi-leader-follower games. It models the sequential game as a one-step approach by considering only two iterations of the fixed-point problem. We define as *distance* between two solutions a non-negative value measuring the difference in operators' decisions, in customers' decisions, or a combination. If we start from an equilibrium point, the distance between the initial solution and the next iteration solution is equal to 0. Else, the distance is greater than 0, since at least one of the players changes its strategy.

The notation is now introduced for the linear model. Let K be the set of the operators and let S_k be the given finite set of strategies that can be played by operator $k \in K$. The parameters p_{ins} indicate the price at which alternative i is offered to customer n by operator k if playing strategy $s \in S_k$. The superscripts $'$ and $''$ refer to the variables of the initial configuration and of the best response configuration, respectively. The variables V_s store the value of the payoff for operator k if responding with strategy $s \in S_k$, while the variables V_k^{max} store the highest of these values for each operator. The binary variables x_s are equal to 1 if strategy $s \in S_k$ is the best response of operator k to the initial configuration. Then, the mathematical model can be written as

$$\min \sum_{i \in I} \sum_{n \in N} |p''_{in} - p'_{in}| \tag{9}$$

$s.t.:$

Initial configuration:

$$\sum_{i \in I} w'_{inr} = 1 \qquad \forall n \in N, \forall r \in R \tag{10}$$

$$U'_{inr} = \beta_{in} p'_{in} + q_{in} + \xi_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{11}$$

$$U'_{inr} \leq U'_{nr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{12}$$

$$U'_{nr} \leq z'_{inr} + M(1 - w'_{inr}) \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{13}$$

Final configuration:

$$\sum_{i\in I} w''_{inrs} = 1 \qquad \forall n \in N, \forall r \in R, \forall s \in S_k, \forall k \in K \tag{14}$$

$$U''_{inrs} = \beta_{in} p_{ins} + q_{in} + \xi_{inr} \qquad \forall i \in I_k, \forall n \in N, \forall r \in R, \forall s \in S_k, \forall k \in K \tag{15}$$

$$U''_{inrs} = U'_{inr} \qquad \forall i \in I \setminus I_k, \forall n \in N, \forall r \in R, \forall s \in S_k, \forall k \in K \tag{16}$$

$$U''_{inrs} \leq U''_{nrs} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S_k, \forall k \in K \tag{17}$$

$$U''_{nrs} \leq z''_{inrs} + M(1 - w''_{inrs}) \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S_k, \forall k \in K \tag{18}$$

Best response constraints:

$$p''_{in} = \sum_{s\in S_k} p_{ins} x_s \qquad \forall i \in I_k, \forall n \in N, \forall k \in K \tag{19}$$

$$V_s = \frac{1}{R} \sum_{i\in I_k} \sum_{n\in N} \sum_{r\in R} p_{ins} w''_{inrs} \qquad \forall s \in S_k, \forall k \in K \tag{20}$$

$$V_s \leq V_k^{max} \qquad \forall s \in S_k, \forall k \in K \tag{21}$$

$$V_k^{max} \leq V_s + M(1 - x_s) \qquad \forall s \in S_k, \forall k \in K \tag{22}$$

$$\sum_{s\in S_k} x_s = 1 \qquad \forall k \in K \tag{23}$$

$$w'_{inrs}, w''_{inrs} \in \{0, 1\} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S_k, \forall k \in K \tag{24}$$

$$x_s \in \{0, 1\} \qquad \forall s \in S_k, \forall k \in K. \tag{25}$$

The objective function (9) minimizes the distance between the two solutions in terms of operators' strategies. The absolute value can be linearized by expressing the argument as the difference of two non-negative variables and by minimizing the sum of these variables in the objective function. Constraints (10-13) define the utilities and force customers to always choose the alternative with the highest utility in the initial configuration. Notice that the price variables p'_{in} are modeled as free continuous variables. Constraints (14-18) impose the utility maximization principle in the best response configurations. In each strategic scenario, the decisions of the optimizing operator only affect the utility of its alternatives (15), while the utility of the competitors' alternatives remain unchanged (16). Finally, constraints (19-23) state that operators always select the best response strategy to the initial configuration.

Compared to the sequential game, this model enables discrimination between different equilibrium solutions by modifying the objective function, and it can also find near-equilibrium solutions, if no Nash equilibrium exists. It can be also applied to a nonlinear model having probabilistic customer choices. The description of the nonlinear case is omitted here.

4 Numerical experiments

The case study used for the tests is derived from [10], where the choice of customers among three different parking alternatives is modelled with a mixed logit model. In our experiments, the nonlinear and the linear optimization models for both the Stackelberg game and the multi-

Instance			MILP						NLP					
DCM	N	R	Time (s)	Obj	p_1	p_2	d_1	d_2	Time (s)	Obj	p_1	p_2	d_1	d_2
Logit	10	100	921	6.44	0.67	0.72	0.92	8.07	0.02	6.36	0.83	0.71	0	8.92
Logit	10	200	7027	6.43	0.66	0.72	0.99	8.05	0.02	6.36	0.83	0.71	0	8.92
Logit	50	50	7105	32.09	0.68	0.71	1.42	43.88	0.06	31.93	0.71	0.72	0.43	43.86
Logit	50	100	55020	32.19	0.68	0.73	2.80	41.66	0.06	31.93	0.71	0.72	0.43	43.86
Mixed	10	100	2378	5.38	0.55	0.63	2.72	6.18	0.05	5.31	0.55	0.63	2.79	6.03
Mixed	10	200	3942	5.21	0.54	0.61	2.94	5.95	0.29	5.22	0.56	0.64	2.96	5.60
Mixed	50	50	13285	27,33	0,58	0.67	13.80	29.08	0.45	27.20	0.58	0.66	13.64	29.06
Mixed	50	100	72000*	27,00*	0,56*	0.65*	13.92*	29.58*	0.70	26.92	0.56	0.66	14.79	28.39

Table 1: Numerical experiments on the single operator optimization problem (uncapacitated case)

Instance			MILP				MINLP			
DCM	N	R	Time (s)	Obj	p_1	p_2	Time (s)	Obj	p_1	p_2
Mixed	5	100	518	2.28	0.58	0.74	3	1.96	0.70	0.84
Mixed	5	200	4428	2.30	0.55	0.70	35	2.04	0.70	0.85
Mixed	10	100	4564	4.85	0.58	0.73	12	4.84	0.64	0.78
Mixed	10	200	72000*	4.70*	0.58*	0.68*	26	4.75	0.63	0.77
Mixed	50	50	72000*	26.09*	0.61*	0.77*	163	26.51	0.60	0.76
Mixed	50	100	72000*	25.71*	0.60*	0.74*	661	26.19	0.59	0.75

Table 2: Numerical experiments on the single operator optimization problem (capacitated case)

leader-follower game were tested on two discrete choice specifications, namely the multinomial logit model and the mixed logit model. All the proposed model are solved through the NEOS Server [11]. MILP models are solved using CPLEX 12.7.0, while NLP and MINLP models are solved using Artelys Knitro 10.3.0.

For the Stackelberg game, Tables 1 and 2 show the results of the uncapacitated and of the capacitated instances, respectively. The latter ones include capacity constraints on the two operated alternatives, which require the use of binary variables that express whether an alternative is or is not available to a customer due to capacity limits. The experiments show that the nonlinear model converges to optimality much faster than the MILP model in all cases, and that computational times for the capacitated case are always higher than for the uncapacitated case. The performance of the MILP model is primarily related to its combinatorial nature and to the weak formulation of the linear relaxation, which could be improved by adding valid inequalities.

Table 3 shows that in the case of a logit formulation the nonlinear model converges faster to optimality, as there is no need for simulation. On the other hand, when using a mixed logit formulation, the linear model generally outperforms the nonlinear model, which fails to converge on larger instances. Compared to the results of the Stackelberg game, the substantial worsening of the computational performance of the nonlinear model can be imputed to the discretized price parameters and to the binary decision variables of the upper-level problems, while the relatively good performance of the MILP model can be explained by the reduction of the solution space due to the limited set of response strategies. In particular, the linear model, which is structured around a simulation framework, has similar computational performances on the logit and the

			MILP					MINLP					
DCM	N	R	Time (s)	Obj	p_1	p_2	d_1	d_2	Time (s)	Obj	p_1	p_2	d_1	d_2
Logit	5	50	68	0	0,05	0,15	1,28	3,72	78	0	0,05	0,15	1,54	3,46
Logit	5	100	203	0	0,05	0,15	1,65	3,35	78	0	0,05	0,15	1,54	3,46
Logit	5	200	818	0	0,05	0,15	1,55	3,45	78	0	0,05	0,15	1,54	3,46
Logit	10	50	208	0	0,05	0,15	2,74	7,26	94	0	0,05	0,15	2,85	7,15
Logit	10	100	3679	0	0,05	0,15	2,86	7,14	94	0	0,05	0,15	2,85	7,15
Logit	10	200	5595	0	0,05	0,15	2,84	7,16	94	0	0,05	0,15	2,85	7,15
Logit	50	25	6894	0	0,05	0,15	11,20	38,80	1151	0	0,05	0,15	10,72	39,29
Logit	50	50	16400	0	0,05	0,15	10,60	39,40	1151	0	0,05	0,15	10,72	39,29
Logit	50	100	6124	0	0,05	0,15	10,81	39,19	1151	0	0,05	0,15	10,72	39,29
Mixed	5	50	70	0	0,10	0,20	1,96	3,04	849	0	0,10	0,20	2,05	2,95
Mixed	5	100	170	0	0,15	0,20	1,52	3,48	747	0	0,10	0,20	2,22	2,78
Mixed	5	200	1013	0	0,10	0,20	2,13	2,87	2962	0	0,10	0,20	2,07	2,93
Mixed	10	50	291	0	0,15	0,25	4,16	5,84	2019*	0,09*	0,30*	0,39*	3,95*	6,05*
Mixed	10	100	2204	0	0,15	0,25	3,84	6,16	3499	0	0,10	0,20	3,92	6,08
Mixed	10	200	3589	0	0,10	0,20	4,17	5,83	4413	0	0,10	0,20	4,18	5,82
Mixed	50	25	985	0	0,10	0,20	17,24	32,76	7035	0	0,10	0,20	17,09	32,91
Mixed	50	50	13923	0	0,15	0,25	18,28	31,72	16242*	0,19*	0,13*	0,32*	31,42*	18,58*
Mixed	50	100	28682	0	0,15	0,25	18,31	31,69	36000*	-	-	-	-	-

Table 3: Numerical experiments on the fixed-point MIP model

				MILP				MINLP			
DCM	N	R	$\lvert S_k \rvert$	Time (s)	Obj	p_1	p_2	Time (s)	Obj	p_1	p_2
Logit	10	100	11	3679	0	0,05	0,15	94	0	0,05	0,15
Logit	10	100	21	16524	0	0,02	0,10	194	0	0,02	0,10
Logit	10	100	31	59096	0	0,02	0,11	719	0	0,02	0,11
Mixed	10	100	11	2204	0	0,15	0,25	3499	0	0,10	0,20
Mixed	10	100	21	4023	0	0,12	0,22	11006*	0,05*	0,19*	0,26*
Mixed	10	100	31	5017	0	0,12	0,21	19401*	0,02*	0,11*	0,19*

Table 4: Numerical experiments to test the effect of the strategy set size

mixed logit model. The latter finding is particularly encouraging, because it indicates that the MILP formulation for the demand-based optimization model could potentially embed even the most complex and accurate discrete choice models. Finally, Table 4 shows that, as expected, computational times are influenced by the size of the players' strategy set.

5 Conclusions

We introduced a demand-based optimization framework that allows to study oligopolistic markets by explicitly modelling demand-supply and supply-supply interactions. A fixed-point optimization model to find equilibrium solutions which can incorporate both nonlinear and linearized customer choices probabilities has been proposed and tested on two different discrete choice specifications. The numerical experiments performed so far indicate that different formulations could be more or less effective depending on the type of decision variables and on the chosen discrete choice model. The nonlinear formulation is non-convex and becomes intractable when many discrete variables are introduced, while the linear formulation is convex but combinatorial due to the nature of the simulation framework.

In the next phases of this research, we plan to (i) propose an algorithmic framework in which candidate equilibrium solutions are found by means of ad-hoc algorithms and used as input strategies in the fixed-point MIP model, and (ii) apply the methodology to a realistic transport application, such as a competitive high-speed rail or airline market.

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