A feedback linearization approach for closed-loop control design: Application to variable speed limits and ramp metering

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Extended abstract

The growing level of freeway traffic congestion comprises an everyday life issue with social, economic, and environmental implications for modern metropolitan areas. Although there is evidence that Variable Speed Limits (VSL) and Ramp Metering (RM) are two effective practical approaches to ameliorate traffic congestion, their real-time field application is deemed cumbersome due to computational complexities. The positive effects that these approaches can have on traffic flow and congestion can be demonstrated with the augmented METANET model, which is one of the most widely used macroscopic models for freeway traffic. From the mathematical and systems theory viewpoint, METANET is a nonlinear, non-affine, Multiple Input Multiple Output (MIMO) system, which is affected by disturbances. Feedback linearization is a useful methodology in the literature to deal with nonlinear MIMO systems, and simplify the control design procedure. In the current work, by applying this method, we manage to represent the closed-loop system with a linear model that under certain conditions be an exact replication of the original system. Furthermore, the existence of a zero dynamic system, the controller stability, and the disturbance decoupling problem are investigated. Afterwards, we present and discuss the closed-loop linear representation of the METANET model by applying an appropriate feedback linearization method. Finally, a pole placement feedback loop is developed to regulate the closed-loop linear model.

METANET model

METANET is a macroscopic second order traffic flow model, mostly used for freeway networks, that represents the dynamics of each freeway segment with length Δ_i and λ_i number of lanes as follows [1]:

$$\dot{\rho}_i(t) = \frac{1}{\Delta_i \lambda_i}\big(q_{i-1}(t) - q_i(t) + r_i(t) - s_i(t)\big), \tag{1}$$

$$s_i(t) = \beta_i(t) q_{i-1}(t), \tag{2}$$

$$\dot{v}_i(t) = \frac{1}{\tau}\Big(V\big(\rho_i(t)\big) - v_i(t)\Big) + \frac{1}{\Delta_i} v_i(t)\big(v_{i-1}(t) - v_i(t)\big) - \frac{\nu}{\tau \Delta_i}\frac{\rho_{i+1}(t) - \rho_i(t)}{\rho_i(t) + k} - \frac{\delta}{\Delta_i \lambda_i}\frac{r_i(t) v_i(t)}{\rho_i(t) + k}, \tag{3}$$

$$V\big(\rho_i(t)\big) = v_{f,i} \exp\left(-\frac{1}{a_i}\left(\frac{\rho_i(t)}{\rho_{cr,i}}\right)^{a_i}\right), \tag{4}$$

$$q_i(t) = \rho_i(t) v_i(t) \lambda_i, \tag{5}$$

where $\rho_i(t)$, $q_i(t)$ and $v_i(t)$ denote the traffic density, flow, and space mean speed in segment i, respectively; τ, ν, k, and δ denote model parameters. Based on (1), known as conservation equation, the difference between total input flows (i.e. $q_{i-1}(t)$ and the on-ramp inflow $r_i(t)$) and total amount of output flows (i.e. $q_i(t)$ and off-ramp outflow $s_i(t)$) in segment i, depicts the change in density $\rho_i(t)$. The exiting rate $\beta_i(t)$ denotes the ratio of $s_i(t)$ to $q_{i-1}(t)$. Equation (3) corresponds to the dynamic speed equation and is composed by four different terms. The first is the so-called relaxation term, which demonstrates the tendency of vehicles to achieve the desired speed (i.e. the stationary speed $V\big(\rho_i(t)\big)$). The second and third terms model the impact of special heterogeneity. More precisely, the second term is the so-called convection term, which expresses the effect of inflow, and the third one the so-called anticipation term that demonstrates the effect of upcoming change in density. The relationship between stationary speed and traffic flow is described by the well-known fundamental diagram and is demonstrated in (4), where $v_{f,i}$ and $\rho_{cr,i}$ denote the free flow speed and critical density, respectively.

VSL impact

In order to model the impact that the application of VSL could have on the traffic conditions (i.e. on the fundamental diagram), [2] proposes the following equation:

$$v_{f,i}(b_i(t)) = \hat{v}_{f,i} b_i(t) \tag{6}$$

$$\rho_{cr,i}(b_i(t)) = \hat{\rho}_{cr,i}(1 + A_i(1 - b_i(t))) \tag{7}$$

$$a_i(b_i(t)) = \hat{a}_i(E_i - (E_i - 1)b_i(t)) \tag{8}$$

where $\hat{v}_{f,i}$, $\hat{\rho}_{cr,i}$, and $\hat{a}_i$ denote the parameters of equation (4) before applying VSL. The VSL rate $b_i(t) \in (0,1]$ is a control variable; when $b_i(t) = 1$, no VSL is applied. Furthermore, we can assume that $b_i(t)$ is roughly equal to the ratio of VSL-induced free flow speed to non-VSL free flow speed. Finally, A_i and E_i are constant parameters equal to 0.67 and 1.82 respectively, that are estimated from real data.

RM impact

The following equations present the impact of applying RM on the on-ramp outflow $r_i(t)$ according to [2].

$$\dot{w}_i(t) = d_i(t) - r_i(t), \tag{9}$$

$$r_i(t) = c_i(\mathrm{t})\min(Q_0, Q_0 \frac{\rho_{\max} - \rho_i(t)}{\rho_{\max} - \rho_{cr,i}}, d_i(t) + \rho_{w_i}(t) v_{f,i}). \tag{10}$$

The dynamics of the queue length $w_i(t)$ (in veh)are presented in (9), where $d_i(t)$ is the on-ramp demand flow (in veh/s). Another control variable is the metering rate $c_i(\mathrm{t}) \in [c_{\min}, 1]$, with $c_{\min}$ denoting the minimum admissible value. According to (10) RM-induced on-ramp outflow is a portion of outflow in absence of RM, which is the minimum of three terms. The first term is the on-ramp flow capacity Q_0, the maximum ramp inflow due to the physical characteristics of the infrastructure (i.e. number of lanes). The second term (also called supply of space), expresses the effect of the mainline congestion on the ramp outflow, where $\rho_{\max}$ denotes the jam density. Finally, the third term (also called demand for space), is the actual demand flow including new arrivals $d_i(t)$ and the vehicles already waiting in the queue; $\rho_{w_i}(t)$ denotes the queueing density at time t.

Feedback linearization

One of the well-known approaches in the literature to deal with nonlinear systems is feedback linearization. With this approach, we can achieve a linearized representation of the (whole or part of the) closed-loop nonlinear system. Afterwards, we can apply the linear control methods to regulate the system. There are some important notions that are used frequently in the feedback linearization approach, such as affine systems, relative degrees, internal dynamics, zero dynamics, and the disturbance decoupling problem. These notions are briefly defined as below (the interested reader is referred to [3] for more details):

- **Affine system:** a nonlinear system is input or disturbance affine if it is linear with respect to the input or disturbance.
- **Relative degree:** input or disturbance relative degree is the number of times you need to take the derivative of the output before the input or disturbance appears. This concept is an analogous to the difference in the number of zeros and poles in linear systems.
- **Internal dynamics:** a part of the system that is unobservable from the external linearized input-output relationship. The dimension of internal dynamics is equal to the difference between the number of system states and the input relative degree.
- **Zero dynamics:** is the internal dynamics of the system when the system output is zero. The zero dynamics is functionally analogous to zero in linear system transfer function.
- **Disturbance decoupling problem:** the process of designing a proper nonlinear feedback controller that simultaneously linearizes the relation between the transformed input and output, while completely eliminates the effects of disturbance on output signals.

The feedback linearization method is applicable for affine nonlinear systems with respect to inputs and disturbances. However, from equations (1)-(5), and by considering $c_i(\mathrm{t})$ and $b_i(\mathrm{t})$ as input signals, METANET is a non-affine system with respect to inputs. To deal with this problem, we can present an extended system that is affine. Assume the previous input signals as the new states of the extended system and their derivatives as the input signals of the new system (see [4]). The extended system of METANET model is presented in the state-space form as follows:

$$\dot{X}(t) = f(X) + g(X)U(t) + p(X)D(t) \tag{11}$$

$$Y(t) = h(X) \tag{12}$$

$$\forall i, 1 \le i \le N \tag{13}$$

$$X(t) = [\rho_i(t)\ v_i(t)\ w_i(t)\ c_i(\mathrm{t})\ b_i(\mathrm{t})]^T \tag{14}$$

$$U(t) = [C_i(t)\ B_i(t)]^T \tag{15}$$

$$D(t) = [d_i(t)\ \beta_i(t)]^T \tag{16}$$

$$Y(t) = [\rho_i(t)\ v_i(t)]^T \tag{17}$$

where, $X(t)$ is the system state vector which consists of the densities, mean speeds, queue lengths, metering rates, and VSL rates of all freeway sections. $U(t)$ is the vector with the input (control) signals, which consist of the first derivatives of the metering rates $C_i(t)$ and VSL rates $B_i(t)$. Note that the on-ramp demands and off-ramp exiting rates are disturbances that are denoted with the vector $D(t)$. Finally, we assume all the densities and mean speeds as the system outputs denoted with $Y(t)$.

The functions $f(X)$, $g(X)$, and $p(X)$ are defined as below.

$$f(X) = \begin{bmatrix} \frac{1}{\Delta_i \lambda_i}[\rho_{i-1}(t)v_{i-1}(t)\lambda_{i-1} + c_i(t)\alpha^* - \rho_i(t)v_i(t)\lambda_i]_{(N\times 1)} \\ \frac{V(\rho_i(t)) - v_i(t)}{\tau} + \frac{v_i(t)}{\Delta_i}\big(v_{i-1}(t) - v_i(t)\big) - \frac{v}{\tau\Delta_i}\frac{\rho_{i+1}(t) - \rho_i(t)}{\rho_i(t)+k} - \frac{\delta}{\Delta_i\lambda_i}\frac{v_i(t)c_i(t)}{\rho_i(t)+k}\alpha^*{}_{(N\times 1)} \\ -c_i(t)\alpha^*{}_{(N\times 1)} \\ 0_{(2N\times 1)} \end{bmatrix} \tag{18}$$

$$\alpha^* = \alpha_{i1}Q_0 + \alpha_{i2}Q_0\frac{\rho_{\max} - \rho_i}{\rho_{\max} - \rho_{\mathrm{cr,i}}} + \alpha_{i3}\rho_{w_i}v_{f,i} \tag{19}$$

$$g(X) = \begin{bmatrix} 0_{(3N\times 2N)} \\ I_{(2N\times 2N)} \end{bmatrix} \tag{20}$$

$$p(X) = \begin{bmatrix} 0_{(1\times i-1)} \quad \alpha_{i3}\frac{c_i(t)}{\Delta_i\lambda_i} \quad 0_{(1\times N-1)} \quad \frac{-1}{\Delta_i\lambda_i}\rho_{i-1}(t)v_{i-1}(t)\lambda_{i-1} \quad 0_{(1\times N-i)}{}_{(N\times 2N)} \\ 0_{(1\times i-1)} \quad \alpha_{i3}\frac{\delta}{\Delta_i\lambda_i}\frac{v_i(t)c_i(t)}{\rho_i(t)+k} \quad 0_{(1\times 2N-i)}{}_{(N\times 2N)} \\ 0_{(1\times i-1)} \quad 1 - \alpha_{i3}c_i(t) \quad 0_{(1\times 2N-i)}{}_{(N\times 2N)} \\ 0_{(2N\times 2N)} \end{bmatrix} \tag{21}$$

For the sake of simplicity, and in order to make it practical to work with the minimum operator in (10), we introduce equation (19), where α_{i1}, α_{i2} and α_{i3} are constant parameters and in every t just one of them is one and the others are zero.

Relative degrees

The following theorems are presented in [5] in order to calculate the relative degrees of a system:

Theorem 1

A nonlinear system has vector input relative degree $IRD = \{ird_1, \cdots, ird_m\}$ at the equilibrium point if:

1. $L_{g_j}L_f^k h_i(X) = 0$ for all $1 \le i,\ j \le m$, for all $k < ird_i - 1$, and for all X in the neighborhood of equilibrium point.

2. The following decoupling matrix is nonsingular at the equilibrium point:

$$A(X) = \begin{bmatrix} L_{g_1}L_f^{ird_1 - 1}h_1(X) & \dots & L_{g_m}L_f^{ird_1 - 1}h_1(X) \\ \vdots & \ddots & \vdots \\ L_{g_1}L_f^{ird_m - 1}h_m(X) & \dots & L_{g_m}L_f^{ird_m - 1}h_m(X) \end{bmatrix}_{m\times m} \tag{22}$$

Theorem 2

The system has disturbance relative degree $DRD = \{drd_1, \cdots, drd_m\}$ at the equilibrium point if:
1. $L_{p_j} L_f^k h_i(X) = 0$ for all $1 \leq i,\ j \leq m$, for all $k < drd_i - 1$, and for all X in the neighborhood of equilibrium point.
2. For each i there exists at least one j where $L_{p_j} L_f^{drd_i - 1} h_i(X_0) \neq 0$.
The functions involved in (11) are summarized as described below in order to avoid cumbersome calculations for this problem:

$$f(X) = \begin{bmatrix} f_{1(N\times1)} \\ f_{2(N\times1)} \\ f_{3(N\times1)} \\ 0_{(2N\times1)} \end{bmatrix},\ g(X) = \begin{bmatrix} 0_{(3N\times2N)} \\ I_{(2N\times2N)} \end{bmatrix},\ p(X) = \begin{bmatrix} p_{1(N\times2N)} \\ p_{2(N\times2N)} \\ p_{3(N\times2N)} \\ 0_{(2N\times2N)} \end{bmatrix} \tag{23}$$

$$\frac{\partial h(X)}{\partial X} = [I_{2N\times2N} \quad 0_{2N\times3N}] \tag{24}$$

$$k = 0 \rightarrow \begin{cases} L_g h(X) = \frac{\partial h(X)}{\partial X} g(X) = \left[0_{(2N\times2N)}\right] \\ L_p h(X) = \frac{\partial h(X)}{\partial X} p(X) = \begin{bmatrix} p_{1(N\times2N)} \\ p_{2(N\times2N)} \end{bmatrix}. \end{cases} \tag{25}$$

From the above calculation, in the studied system when $k = 0$ there is at least one nonzero element in each row of $L_p L_f^k h(X)$, therefore the disturbance relative degree is $DRD = \{1\}_{1\times2N}$. In order to compute the input relative degree we need to take further steps:

$$k = 1 \rightarrow L_g L_f h(X) = \frac{\partial}{\partial X}\left(L_f h(X)\right) g(X) = \frac{\partial}{\partial X}\left(\frac{\partial h(X)}{\partial X} f(X)\right) g(X) = \frac{\partial}{\partial X}\left(\begin{bmatrix} f_{1(N\times1)} \\ f_{2(N\times1)} \end{bmatrix}\right) g(X) =$$

$$\begin{bmatrix} \frac{\partial f_1(X)}{\partial c_i}_{(N\times N)} & \frac{\partial f_1(X)}{\partial b_i}_{(N\times N)} \\ \frac{\partial f_2(X)}{\partial c_i}_{(N\times N)} & \frac{\partial f_2(X)}{\partial b_i}_{(N\times N)} \end{bmatrix} \tag{26}$$

In this step, $L_g L_f^k h(X)$ is a lower triangular matrix with nonzero diagonal elements. Therefore, this matrix is nonsingular and the input relative degree is $IRD = \{2\}_{1\times2N}$.

Internal and zero dynamics

As mentioned earlier, if $\sum ird_i < n$ a part of the nonlinear system with $n - \sum ird_i$ dimension is unobservable from the external input-output relationship; this part is called internal dynamics. Therefore, it is essential to investigate the stability of the internal dynamics. For this reason, we utilize a concept called zero dynamics, which is related to the concept of zeros for linear systems. Zero dynamics is defined as the internal dynamics when the system output is zero. Local asymptotic stability of the zero dynamics is a necessary and sufficient condition for local asymptotic stability of the internal dynamics. In the problem studied here, the sum of the input relative degree elements is $4N$ which is smaller than the number of system states (which is $5N$). Therefore, our analysis has encountered system internal dynamics and its stability should be further investigated.
Essentially, we can transform the nonlinear system into a linear system with a normal form by defining $\xi_k^i = L_f^{k-1} h_i(x)$ for all $1 \leq k \leq ird_i,\ 1 \leq i \leq 2N$. The following equations represent the normal form, where ξ denotes the states of the linearized model and η the vector of unobservable states, the so-called internal dynamics.

$$\dot{\xi}_1^i = \xi_2^i + c_i(\xi, \eta) D \tag{27}$$

$$\dot{\xi}_2^i = b_i(\xi, \eta) + a_i(\xi, \eta) U + s_i(\xi, \eta) D \tag{28}$$

$$\dot{\eta}_j = q_j(\xi, \eta) + t_j(\xi, \eta) D \tag{29}$$

$$y_i = \xi_1^i \tag{30}$$

Suppose that g_i denotes the vectors of $g(X)$; if $G = span\{g_1, \dots g_m\}$ is involutive it is always possible to choose the states of the internal dynamics in such a way that $L_{g_i}\eta_j = 0$ for all $1 \leq i \leq m, 1 \leq j \leq N$. From (20), G is obviously involutive, therefore

$$L_{g_i}\eta_j = \frac{\partial \eta_j}{\partial X} g_i = 0 \Rightarrow \left[\frac{\partial \eta_j}{\partial C_k}, \frac{\partial \eta_j}{\partial B_k}\right] = 0 \text{ for all } 1 \leq k \leq N \tag{31}$$

On the other hand, $\Phi(X) = \mathrm{col}(\xi_1^1, \dots, \xi_1^{2N}, \xi_2^1, \dots, \xi_2^{2N}, \eta_1, \dots \eta_N)$ has to be nonsingular at the equilibrium point. As a result, we determine the internal dynamics as $\eta = w$.

Disturbance decoupling problem

As discussed before, in this problem the disturbance relative degree is less than the input relative degree. One can interpret that this as follows: in METANET model the disturbances affect the outputs more directly than the input (control) signals. We can rewrite equations (27)-(29) as follows:

$$\dot{\xi}_1^i = \xi_2^i + \textstyle\sum_{j=1}^{2N} L_{p_j} h_i(\xi) D_j, \tag{32}$$

$$\dot{\xi}_2^i = L_f^2 h_i(\xi) + \textstyle\sum_{j=1}^{m} L_{g_j} L_f h_i(\xi)\, U_j + \sum_{j=1}^{2N} L_{p_j} L_f h_i(\xi) D_j, \tag{33}$$

$$\dot{\eta}_j = L_f \eta_j + \textstyle\sum_{j=1}^{2N} L_{p_j} \eta_i D_j, \tag{34}$$

where D_j and U_j are the j^{th} column of D and U, respectively. In case of measurable disturbances, [6] proposes a solution for the disturbance decoupling problem, under the same conditions explained above. As it is shown below, the proposed control law contains an anticipatory factor for the disturbances:

$$U = (L_g L_f h(x))^{-1}(V - \textstyle\sum_{j=0}^{2} L_f^j h(X) - \sum_{l=0}^{1} \frac{d^l\left(L_p L_f^{1-l} h(x) D\right)}{dt^l}) \tag{35}$$

Applying the equation above results in a linear representation of METANET model, where V is the control signal for the linear closed-loop representation of the system. Note that V can be derived from any of the available control methods for linear systems (e.g. pole placement).

Conclusion and future work

In this work, we have investigated the characteristics of METANET model as a nonlinear system. METANET belongs to the group of non-affine systems with respect to the input signals. A useful method to deal with this type of nonlinear systems is by adding an integral block before the input signals, which results in an affine representation of the system. The study of relative degrees has revealed the existence of internal dynamics for this problem. Moreover, this study indicates that the disturbances affect the system outputs more directly than the input signals. Consequently, in order to solve the disturbance decoupling problem, we need to add an anticipatory component to the control law, that involves factors of disturbance anticipation in terms of derivatives.

For the future work, we aim at designing a pole placement to control the closed-loop linear representation of the original system. As already mentioned, due to the existence of internal dynamics, applying a proper controller is not sufficient for the stability of the system. Since the states of internal dynamics are unobservable from the external input-output relationship, we need to investigate the stability of the internal dynamics separately. After completing these steps, our methodology would be ready to be tested with real world freeway traffic datasets.

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