Optimal splitting rates for traffic routing and control in simple networks

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This paper considers the problem of traffic routing and control in simple networks. Following [4], we also consider idealised splitting rate models for both routing and signal light control changes. The drivers seek for better routes and the signal control reacts to link flow changes corresponding to the routing decisions. The proposed problem combines the modeling of routing and traffic signal control, and the derived analytical solutions shed some lights to have a better understanding of routing and control interactions. Unlike previous works, e.g. [4], this paper considers *dynamic transient periods*, rather than steady-state periods or equilibrium conditions. This means that the current paper considers the optimal control synthesis for bringing initial queue lengths at an intersection to a predefined steady-state or equilibrium queues, by manipulating traffic routing- and signal-control inputs.

The simple network considered in this paper is shown in Fig. 1. There are two links or routes from point A to point B. Drivers are routed at point A, and a signalized intersection located at point B splits the green light between two links. From point B the traffic flow continues further to the destination point C. This scheme is adopted from the classical scheme presented in [4].

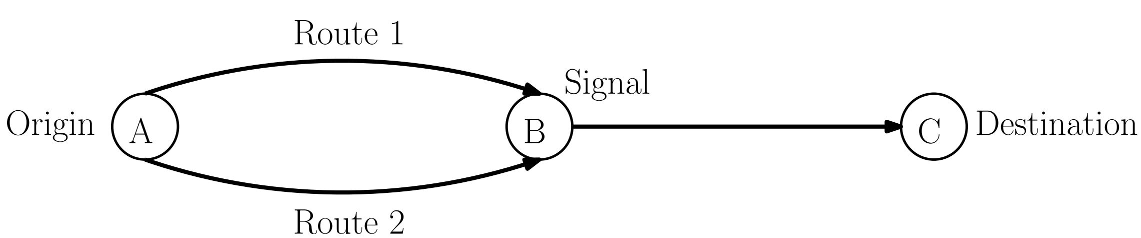

Figure 1: A simple signal-controlled network (based on [4]).

A continuous-time model, which is an extended model based on [2], is utilized to describe the dynamics:

$$\frac{\mathrm{d}q_1(t)}{\mathrm{d}t} = a \cdot v_1(t) - d_1 \cdot \big(u_1(t) - w_1(t)\big)\,, \tag{1}$$

$$\frac{\mathrm{d}q_2(t)}{\mathrm{d}t} = a \cdot v_2(t) - d_2 \cdot \big(u_2(t) - w_2(t)\big)\,, \tag{2}$$

$$u_1(t) + u_2(t) = 1\,, \quad \underline{u} = 1 - \overline{u}\,, \quad \underline{u} \le u_1(t) \le \overline{u}\,, \quad \underline{u} \le u_2(t) \le \overline{u}\,, \tag{3}$$

$$v_1(t) + v_2(t) = 1\,, \quad \underline{v} = 1 - \overline{v}\,, \quad \underline{v} \le v_1(t) \le \overline{v}\,, \quad \underline{v} \le v_2(t) \le \overline{v}\,, \tag{4}$$

$$0 \le w_1 \le \overline{w}_1\,, \quad 0 \le w_2 \le \overline{w}_2\,, \tag{5}$$

$$\overline{w}_1 = \big(a \cdot v_1(t) - d_1 \cdot u_1(t)\big)/d_1\,, \quad \overline{w}_2 = \big(a \cdot v_2(t) - d_2 \cdot u_2(t)\big)/d_2\,, \tag{6}$$

$$q_1(T) = 0\,, \quad q_2(T) = 0\,, \quad 0 \le q_1(t)\,, \quad 0 \le q_2(t)\,. \tag{7}$$

where the state variables $q_1(t)$ and $q_2(t)$ (veh) respectively represent the queue lengths at routes (links) 1 and 2, $w_1(t)$ and $w_2(t)$ $(-)$ are the slack variables that can prevent negative queue lengths, a (veh/s) is the total inflow rate at point A, $v_1(t)$ and $v_2(t)$ $(-)$ are the routing-control inputs that split the flow a between links 1 and 2, and the signal-control inputs $u_1(t)$ and $u_2(t)$ $(-)$ represent the green splits of the signal light at point B for links 1 and 2, respectively. All control inputs, i.e. green duration and routing splits, are considered to be constrained, see (3) and (4). Parameters d_1 and d_2 (veh/s) are the output saturation flows (veh/s) of links 1 and 2 at Point B, respectively.

The objective is to minimize the total delay, [1]. It has the following form

$$J = \int_0^T [q_1(t) + q_2(t)]\mathrm{d}t \to \min\,. \tag{8}$$

where T (s) is the final time, which is not fixed and has to be found from the condition of dissolving both queues, see (7).

According to the optimal control theory (OCT) and Pontryagin maximum principle (PMP), [3], we could form the Hamiltonian $H(q_1, q_2, p_1, p_2, u_1, u_2, v_1, v_2, w_1, w_2)$ to be maximized at each point of time t over the control inputs, and write the differential costate equations for the costate variables $p_1(t)$ and $p_2(t)$ as follows

$$H = p_1(t) \cdot \big[a \cdot v_1(t) - d_1 \cdot \big(u_1(t) - w_1(t)\big)\big] + p_2(t) \cdot \big[a \cdot v_2(t) - d_2 \cdot \big(u_2(t) - w_2(t)\big)\big] - q_1(t) - q_2(t)\,, \tag{9}$$

$$\frac{\mathrm{d}p_1(t)}{\mathrm{d}t} = 1\,, \quad \frac{\mathrm{d}p_2(t)}{\mathrm{d}t} = 1\,. \tag{10}$$

Our preliminary analysis shows that $p_1(t)$ and $q_1(t)$ might become zero only simultaneously

$$p_1(t) = 0\,, \quad q_1(t) = 0\,. \tag{11}$$

This also holds for $p_2(t)$ and $q_2(t)$. From (10) and conditions (11) it follows that both $p_1(0), p_2(0)$ are non-positive. Two switching functions are determined as

$$S_v(t) = p_1(t) - p_2(t)\,, \quad S_u(t) = -d_1 \cdot p_1(t) + d_2 \cdot p_2(t)\,. \tag{12}$$

From maximization of the Hamiltonian (9) over the control inputs, it follows that

$$\forall S_v(t) > 0: \; v_1(t) = \overline{v}\,,\, v_2(t) = \underline{v}\,; \qquad \forall S_v(t) < 0: \; v_1(t) = \underline{v}\,,\, v_2(t) = \overline{v}\,; \tag{13}$$

$$\forall S_u(t) > 0: \; u_1(t) = \overline{u}\,,\, u_2(t) = \underline{u}\,; \qquad \forall S_u(t) < 0: \; u_1(t) = \underline{u}\,,\, u_2(t) = \overline{u}\,. \tag{14}$$

Without lack of generality we assume that $d_1 > d_2$. Given (12) and (10), one gets

$$\frac{\mathrm{d}S_u}{\mathrm{d}t} = -d_1 + d_2 < 0. \tag{15}$$

The switching function $S_u(t)$ decreases, but there is no possibility for singular arcs with control inputs $u_1(t)$ and $u_2(t)$. The results of the analysis show that there exist four different types of the optimal solutions depending on the parameters and initial values of the state variables. All solutions are derived analytically.

References

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