Targeted Stiff – Optimal Face Identification through Higher Order Conjugation in the Frank-Wolfe Method for Traffic Assignment

P O Lindberg, polin@kth.se

Centre for Transport Studies, and Divs. of Transport and Location Analysis, and Numerical Analysis, KTH Royal institute of Technology, Stockholm

In spite of the fact that most traffic phenomena are temporal, the network route choice models most commonly used in transportation planning are steady state models (e.g. Florian & Morosan, 2014), in particular the classical Traffic Assignment Problem (TAP). For TAP the Frank-Wolfe (FW) Method has been the dominating method until recently. For TAP it shows fast initial convergence. But asymptotically the convergence is so dishearteningly slow (e.g. Patriksson, 1994, Ch. 4.1.5), that methods oriented researchers have considered it dead.

The FW method determines search directions in the link flow space by linearizing the problem, resulting in extreme point solutions, towards which line searches are done. This implies that the current solution will be a convex combination of previously generated extreme points. One explanation for the slow convergence, is that the generated directions become successively worse, resulting in successively shorter steps. Another explanation is that the weights of the early generated extreme points die out very slowly, due to the short steps.

But as shown in recent papers, the stiff has begun to move: Through Conjugation of search directions, the number of iterations is reduced (Mitradjieva and Lindberg, 2013). Moreover, the resulting method is apt for parallelization, resulting in that CitiLabs (Zhou et al, 2010), and Caliper (Caliper Corporation, 2010), has it as winner on 8 core computers, running against Bush-based and gradient projection methods. Further, the iterations may be speeded up through sub-problem updating (Holmgren and Lindberg, 2014), resulting in non-parallelized speedups of 25-50% on top of those from conjugation.

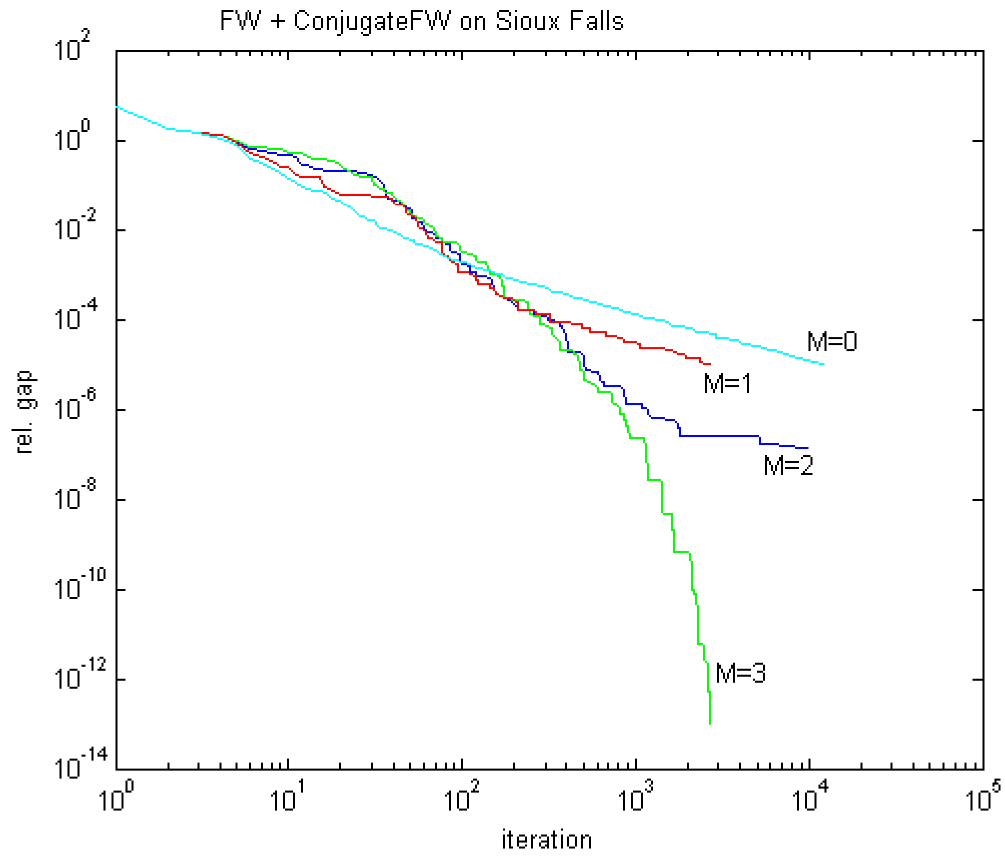
Another benefit of the FW method is that it, at least approximately, achieves route flow proportionality, a proxy for maximum entropy (Florian & Morosan, 2014). This property is important in analyses of individual route flows, e.g. through select link analysis.

Still, conjugate FW methods display the slow $1/n$ asymptotic convergence, although it sets in later. However, if FW is started at a point in the optimal face, it has linear convergence, as shown already by Wolfe (1970). Thus, if the method could be made to enter the optimal face, we would get linear convergence from there on.

The Conjugate FW methods in (Mitradjieva and Lindberg, 2013) are on the way of achieving this. There the current search direction is made conjugate to the previous (CFW), or the two previous (BFW) search directions. This results in better search directions, and hence longer steps. Indeed, every now and then the methods take steps of length 1, i.e. all the way towards the aiming point, which is a convex combination of previous extreme points (more often so in BFW than in CFW). Taking steps of length 1 moreover implies that the weight of some earlier generated extreme point is set to zero. So, the question naturally arises, whether one by higher order conjugation can achieve that the weights of all non-optimal extreme points (i.e. not in the optimal face) are zeroed out.

We show that this is indeed the case. With higher order conjugation (3rd order and higher), conjugate FW will typically arrive at the optimal face in a finite number of iterations, achieving relative gaps down to 10^{-17}.

A typical test case is shown is shown in the following graph, where FW with no conjugation (M=0), first order (M=1), second order (M=2), or third order (M=3), is applied to the classical Sioux Falls test case. The graph shows the relative gap vs. the iteration number (which corresponds to time, since the conjugation overhead is negligible) in log-log scale for all three methods. (This scale gives a linear appearance with slope -1 for a $1/n$ convergence.)



As can be seen, the conjugate methods lag behind the pure FW in the beginning (but this is not generally the case). Further, for lower order conjugation, the methods eventually experience the 1/*n* convergence rate; around *it*(eration) 200 for M=1, and around *it* 2000 for M=2, and not all for M=3.

The results imply that conjugate FW methods can be used to achieve highly converged solutions for the TAP. Such solutions are necessary e.g. for sensitivity analyses.

In the talk we will further discuss experiments with different version of conjugate FW, aiming at arriving early at the optimal face.

This talk represents the third step in a series of research efforts to re-establish the FW method as the leading method for TAP.

References

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