Extended abstract for hEART2013

A dynamic link performance model and convergence in day to day traffic assignment

Mike Smith and Richard Mounce (Department of Mathematics, University of York, UK)

1. INTRODUCTION

In order to estimate correctly the impact of a proposed traffic strategy for an urban area, it is important to estimate the network-wide consequences of all bottlenecks; and if controls such as gating strategies are to be tested then detailed models are a great help so that individual lanes of traffic are represented. Furthermore an attempt must be made to use such models to study the evolution from day to day (say) of city traffic. Here we outline a link model with spatial queues. This link model permits us to prove that certain day to day dynamical systems converge to the set of Wardrop equilibria as days pass. Balijapelli et al (2013) put forward a more general two-regime link model. There are also connections with ideas expressed in Yperman (2007).

2. A SPATIAL QUEUEING LINK PERFORMANCE MODEL

2.1. A link model with a dynamic spatial representation of traffic queues

As is usual in traffic modelling, each real-life traffic lane is here represented by

1. a node which represents the entry point of the lane,
2. a node which represents the exit point or the stop line of the lane, and
3. a directed link joining these two nodes which represents the stretch of lane between the entry and the exit of the lane.

The words lane and link will be used interchangeably.

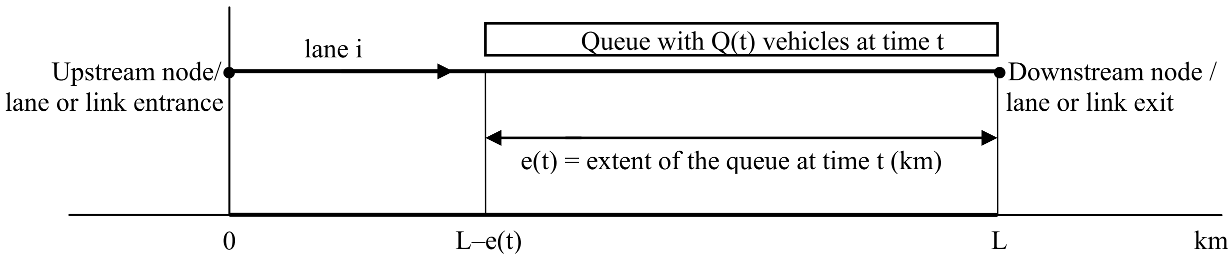

Figure 1. The closed interval [0, L] here represents real life traffic lane i at time t. 0 corresponds to the upstream node, L marks the stop line at the link exit (which is also the head of any queue). The extent of the queue is e(t) km. and L-e(t) marks the tail of the queue.

A representation of link i is shown in figure 1 as the closed interval [0, L] comprising all real numbers x such that $0 \leq x \leq L$. Two links in the network model are connected by short links when traffic may, in reality, pass from one link to the other and are otherwise unconnected. These additional short links represent all possible movements at junctions and junctions are thus represented in an "expanded" form.

At each time the link is divided into two parts; an uncongested or freeflowing part where the speed of all vehicles is u km/hr and a queueing part. A position on the link will be determined by a real number x where $0 \leq x \leq L$. This position on the network link is x km downstream of the upstream node and so the x-axis is labelled "km". The flow rate v(t) (vehicles per hour) is the rate at which vehicles enter the link at time t and, given any v(.). Let the time $t > 0$ and location x in [0, L] be such that the link is uncongested at x at time t. For such uncongested pairs (t, x), the flow rate v(x, t) at location x at time t is to be determined by

$$v(x, t) = v(t-x/u). \qquad (1)$$

For any time $t > 0$, the density of vehicles in the queue at the link exit is ρ_0 (vehs/km). The number of vehicles in the queue is denoted by $Q_i(t)$ (vehicles); for each link i the maximum possible value of Q_i is given and is M_i (or

$MAXQ_i$). This is the greatest number of vehicles that the link can contain. The saturation flow at the exit of link i is to be fixed at s_i vehicles per minute and is constant. Also the time to traverse the entire length of link i (when the queue $Q_i(t) = 0$ for all t) is c_i and is constant.

Consider a link i with a queue volume of $Q_i(t)$ at time t. To calculate the queueing delay function $b_i(t)$ (hours) on link i it has often been proposed in the steady state that:

$$b_i(t) = Q_i(t) / s_i. \tag{2}$$

Then the whole time of travel along link i has often been written

$$c_i + b_i(t) \tag{3}$$

See for example Thompson and Payne (1975), Smith (1987) and Smith and Ghali (1990) (dynamic case).

This formulation of a link performance function involving queueing is called "vertical queueing"; the space taken up by the physical vehicles in the queue is not represented. Further, blocking back effects are not represented. The model may be thought of as representing traffic comprising very short vehicles. In this paper we show how this vertical or point queueing model may be amended to take account of the space occupied by queues and of blocking back. The revised model presented here may be called a *spatial queueing model*, as opposed to the point queueing model. This natural spatial queueing model was first designed in the steady state in Smith (2011).

2.2. All notations in the basic spatial queueing model of a single link.

In this spatial queueing model we allow for the physical representation of the length or the physical extent of queues; the notation is as follows, and refers to the single link above.

$v(t)$ = rate at which vehicles enter link i (vehs./hr.);
$e(t)$ = the length or extent of the queue at time t (km);
u = uncongested speed on the unqueued part of the link (km/h; supposed constant);
$v(x, t) = v(t-x/u)$ = rate at which vehicles arrive at location x in $[0, L – e(t)]$ at time t (veh/hr);
$Q(t)$ = number of vehicles at time t in the queue seeking to exit the link (veh);
M = MAXQ = the maximum possible queue on the whole link (veh);
L = length of the link (km);
s = maximum possible outflow from the link (veh/h);
$\rho(x, t) = v(x, t)/u$ = the traffic density at x at time t, for all x in $[0, L – e(t)]$;
$\rho_0 = M/L$ = the traffic density in the queue (jam density); and
$c = L/u$ = uncongested traversal time (assuming that there is no queue.

The spatial queueing model is specified by specifying $v(t)$ for each $t > 0$. Then for each $t > 0$, each $x \in [0, L]$ and each $e(t)$ as follows:

$$\begin{aligned}\rho(x, t) &= v(x, t)/u \quad \text{if} \quad 0 \le x \le L - e(t)\\ &= M/L \quad \text{if} \quad L - e(t) < x \le L.\end{aligned}$$

We are here defining $\rho(x, t)$ at $L – e(t)$ to be $v(L – e(t), t)/u$; so we are really saying that $\rho(x, t)$ is here the left density at each x in $[0, L]$. This spatial queuing model is to have two distinct regimes; for each time $t > 0$ these correspond to two distinct parts of the link representation $[0, 1]$:

UNCONGESTED REGIME: for any $t > 0$ this corresponds to the interval $[0, L – e(t)]$; and
CONGESTED REGIME: for any $t > 0$ this regime corresponds to the interval $(L-e(t), L]$.

In the model described here, for any given inflow rate $v(.)$, the queue extent $e(t)$ and the queue volume $Q(t)$ evolve as specified in the next section.

2.3. Rate of change of e(t) and Q(t) at t when Q(t) > 0 in the spatial queueuing model

Using standard techniques and putting $h = L/M$ (the "length" of a single vehicle), and tau = the time the link is entered;

$$de(t)/dt = h[v(\tau (t)) - s]/\{1 – hv(\tau (t))/u\}$$

and

$$dQ(t)/dt = [v(\tau (t)) - s]/[1 – hv(\tau (t))/u].$$

2.4. Rate of change of e(t) and Q(t) at t when Q(t) > 0 in the vertical queueing model

We now consider vertical queueing; this corresponds to the limit as $h = L/M \rightarrow 0$ in the above equations. Let $e_h(t)$, $Q_h(t)$ be the queue extent and queue volume in the above spatial queueing model when the vehicle length = h km. where h > 0. We obtain:

$$de_0(t)/dt = \lim_{k \text{ tends to } 0} [v(L-e_k(t), t) - s]/[1/h - v(L-e_k(t), t)/u] = 0 < de_h(t)/dt$$

and

$$dQ_0(t)/dt = \lim_{k \text{ tends to } 0} [v(L-e_k(t), t) - s]/[1 - kv(L-e_h(t), t)/u] = [v(L-e_0(t), t) - s] < dQ_h(t)/dt.$$

2.5. Link traversal times are the same in the two cases

Consider an entrance time τ and suppose that, for vehicle length k (km), traffic entering at this time τ reaches the queue when the extent of the queue is $ext_k(\tau)$. Then, by the results above,

$$ext_0(\tau) < ext_h(\tau) \text{ and } Q_0(\tau) < Q_h(\tau).$$

The non-queueing time is now (with vertical queueing) c instead of $c - ext_h(\tau)/u$ and so is increased, over the spatial queueing case, by $ext_h(\tau)/u$. The queue is now (with vertical queueing) smaller when the queue is reached as there has been $ext_h(\tau)/u$ extra minutes of outflow from the queue. Hence when reached the vertical queue volume is now $[ext_h(\tau)/u]s$ smaller than the spatial queue.

So the queue experienced if the link is entered at τ is

$$QUE_0(\tau) = QUE_h(\tau) - [ext_h(\tau)/u]s$$

and the queueing delay if the link is entered at time τ(t) is now, with vertical queueing,

$$QUE_0(\tau)/s = QUE_h(\tau)/s - [ext_h(\tau)/u]s/s = QUE_h(\tau)/s - ext_h(\tau)/u$$

and so is reduced by $ext_h(\tau)/u$.

Thus for any given τ, compared to the spatial queue, the freeflowing link travel time with the point queue is increased by $ext_h(\tau)/u$ and the queueing time is decreased by $ext_h(\tau)/u$.

Thus the traversal time is unchanged by moving to the vertical queue.

3. MONOTONICITY OF THE SPATIAL QUEUEING LINK MODEL AND CONVERGENCE

Smith and Ghali (1990) show that with *vertical* queueing the delay on entering a link is a monotone function of the time-varying inflow to a link. Hence by the result above the same is true for the delay generated in the spatial queueing model.

Further Smith and Wisten (1994) show that, in a day to day model, if at each instant of time and for each pair of routes joining each *OD* pair, the proportion of route inflow moved to the less expensive route is proportional to

the difference in cost between the two routes× the flow entering the more costly route

then natural day to day dynamical route choice systems are stable; and the main condition is monotonicity of the route cost functions.

It follows that if each route has one bottleneck and the above link model is used then the day to day system converges to the set of dynamic equilibria. For many networks monotonicity of route costs (travel times) follows from monotonicity of link delays. For these also we thus show that the day to day evolution converges to the set of dynamic equiibria.

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