A mathematical approach to the modeling of taxi services

Josep Maria Salanova Grau*

Hellenic Institute of Transport,

Centre for Research and Technology Hellas (CERTH), Hellenic Institute of Transport (HIT), Thessaloniki, Greece

Miquel Estrada Romeu

Center for Innovation in Transport (CENIT), Barcelona Tech, Technical University of Catalonia (UPC), Barcelona, Spain

***** Email: jose@certh.gr

Abstract

This paper presents a mathematical model for the estimation of the key performance indicators of the taxi markets in urban areas. These key parameters are basically the demand and supply for taxi services, the waiting time of users and the cost of the different involved actors. The identified actors are the drivers, the taxi users and the city, represented by the other drivers and the inhabitants. Optimum values for the taxi supply are obtained from the mathematical formulations depending on the demand level and the size of the city, presenting the users waiting time and the unitary system costs related to this optimum fleet size. The model is applied to two modes of operations, hailing and dispatching market. Conclusions are drawn for the best type of market for each demand level and city size.

1. Introduction

Many modern cities are oversaturated, on one hand there is a population concentration in urban agglomerations, with more than 80% of the total population living in urban areas in 2030 (UNFPA 2007), on the other hand mobility needs due to modern way of life are increasing continuously. Last years, mobility management is focusing the attention of different stakeholders, research centers, universities, industry, policy makers and the users themselves, which are cooperating in this intend to optimize the use of existing infrastructures through a better use of Information and Communication Technologies. The application of these technologies to the Transport Sector provides society with Intelligent Transport Solutions (ITS) schemes where the user is fully informed about everything related to their trip. At the same time the user can provide information to the Transportation System Managers, improving importantly the data quantity and quality.

The most effective measure on mobility management is the shift of personal trips from private cars to public mass transportation systems, increasing the utilization factor of vehicles and reducing the stopped or parked vehicles along the streets. In order to attract citizens to this kind of transportation, well planned, efficiently operated, and cost-effective transportation system management (TSM) strategies are applied in order to manage this demand for public transportation services in an optimum way. ITS technologies are applied for providing real time information to users, increasing importantly the offered Level of Service while collecting a great quantity of data. This data is used in both a dynamic scheme for short term services and a static scheme for long terms policy applications.

The taxi service is one of the public transportation services commented above, combining the benefits of the private transportation (good coverage, door to door and comfort) with the benefits of the public transportation systems (no parking, no stress). Taxi markets have been traditionally regulated by the cities, controlling the number of issued licenses and the prices of the offered services trying to protect both the users and the taxi drivers. This regulation assured a minimum income to taxi drivers while protecting users from abusive tariffs, but created a market for taxi licenses, where prices were controlled by the free market, and not by policy makers. In order to evaluate the system in terms of waiting time of users and income of taxi drivers, different models have been developed, providing policy makers with methodologies for estimating the optimum number of licenses for each demand level and city (in terms of size, geometry and congestion levels).

A new taxi model taking into account environmental issues has been developed and is presented in this paper. The work presented in the literature by the different authors has been taken into account for the development of the proposed formulations, testing it for the two taxi operation modes (hailing and dispatching). The model analyzes the costs of all the stakeholders and presents the minimum and optimum number of licenses for each combination of demand level and city geographic characteristics.

This paper is organized as follows: the second chapter reviews the different models presented in the literature; the third chapter presents the proposed model formulation and reviews the formulations presented in the literature for estimating the variables of the model; the fourth chapter is dedicated to the application of the proposed methodology to the dispatching and the hailing modes of taxi services, concluding in the comparison of the performance of both modes of operation; a very brief description of the agent-based model under development is presented in chapter five; finally, chapter six presents the conclusions of the above analyses. An extra chapter is added after the conclusions with the proofs of the presented formulations.

2. Literature Review

Since the early 70's many studies have been published in relation to the taxi sector. While first studies (1970-1990) focused in the profitability of the sector and the necessity for regulation using aggregated models, later studies (1990-2010) implemented more realistic models in the taxi sector: from the most simple model of Yang et al. (1998) developed in 1998 for a small taxi fleet until the most sophisticated models of Yang and Wong (1998-2010) that are able to simulate congestion, elasticity of demand, different user classes, external congestion and non linear costs, while taking into account different market configurations. Douglas (1972) developed the first taxi model in an aggregated way, using economic relationships from other sectors (goods and services). Many authors (de Vany (1975), Beesley (1973), Beesley and Glaster (1983) and Schroeter (1983) used the model proposed by Douglas for developing their own models and tested them in different market configurations. Manski and Wright (1976), Arnott (1985) and Cairns and Liston-Heyes (1996) developed structural models, obtaining more realistic results. Yang and Wong (1998-2005) developed accurate and detailed models, taking into account the spatial distribution of demand and supply in the city and using traffic equilibrium models. Latter models proposed by Wong et al. (2005a) and Yang et al. (2010b) assumed a bidirectional function taking into account the willingness to pay of customers, making it much more realistic.

New technologies applied to the taxi market such as GPS, GIS and GPRS were also simulated in the different models, proving their benefits and justifying their use. Many of the models developed have been tested in various cities around the world using data from different sources. Beesley (1973) and Beesley and Glaster (1983) studied the data obtained from questionnaires in different cities in the UK, especially from London. Schroeter (1983) is the first to use data from taximeters in his model, using the data from a taxi company in Minneapolis (EEUU). Schaller (2007) used interviews and questionnaires from taxi agents and customers in different cities of the EEUU. Recently, Kattan et al. (2010) developed regression models for work trips made by taxi in 25 Canadian cities.

A detailed review of the aggregated and equilibrium models named above can be found in Salanova et al. (2011), where the different assumptions of the models are presented and discussed.

3. Description of the proposed model

The proposed model uses the different mathematical formulations presented in the literature (Corominas (1985), Chang et al. (2003, 2009 and 2010), Zamora i Conrado (1996) and Meyer (1961)) for obtaining the optimum size fleet related to each operational mode, city size and demand level. The correspondent generalized cost and Level of Service (waiting time of users) are also obtained, comparing for the same city the characteristics of two different operational modes for taxi services (optimum fleet and the related unitary cost and waiting time of users for the dispatching and hailing taxi markets).

In order to define the optimum fleet size, all the costs (monetary costs and time costs) of the involved stakeholders are added in a unique function. Different quality constraints are defined and the objective function is solved, presenting the results in terms of minimum and optimum fleet size for each operation mode, demand level and city size.

The objective function is composed by the "costs" (in terms of time) of the involved actors and the cost of the taxi services infrastructure. In the case of the users the cost is composed by the total travel time (access time, waiting time and in-vehicle time) and the trip monetary cost. In the case of the city, the costs are composed by the increase of travel time and emissions caused to other users by the taxi drivers. Finally, the taxi drivers cost is the difference between the cost of offering the taxi service and the income (in this case the cost is expected to be negative). The cost of the infrastructure varies depending on the operational mode: zero cost in the hailing market; stand construction and space opportunity cost for the stand market; communications, office and personnel costs in the dispatching market case. For the external cost calculation, Geroliminis and Daganzo (2008) have recently presented the existence and the methodology for the estimation of the accurate and not scattered MFD diagram, that can be used for estimating the average speed increase/reduction of the whole network due to a reduction/increase in the number of taxis (vehicle-kilometers, density) in the network as it is done by Estrada et al (2011) when studying the impact of new bus lanes in the city of Barcelona.

3.1. Model formulation

The proposed objective function is the following:

$$Min\ Z_d + Z_u + Z_c + G \tag{1}$$

$$Z_u = \lambda_u \cdot A \cdot \left[\alpha_A \cdot T_A + \alpha_W \cdot T_W + \alpha_{IV} \cdot T_{IV} + \frac{\bar{c}^1}{VoT_u} \right] \tag{2}$$

$$Z_d = \frac{\lambda_d \cdot A}{VoT_d} \left[\bar{n} \cdot \bar{c} + \left(\bar{n} \cdot \bar{d} \cdot C_{km} + C_h \right) \right] \tag{3}$$

$$Z_c = \lambda_v \cdot A \cdot \Delta T_v + \frac{\lambda_d \cdot A \cdot C_E \cdot E_d}{VoT_d} + \frac{\lambda_v \cdot A \cdot C_E \cdot \Delta T_v \cdot E_d}{VoT_v} \tag{4}$$

Where,

Model outputs:

Z_d is the cost of the drivers (min) - z_d is the unitary cost of the drivers (min/trip)

Z_u is the cost of the users (min) - z_u is the unitary cost of the users (min/trip)

Z_c is the additional cost for the city (min) - z_c is the unitary cost for the city (min/trip)

G is the cost of the infrastructure (min) - g is the unitary infrastructure cost (min/trip)

T_W is the waiting time of users (min)

T_A is the access time of users (min)

$\bar{c}$ is the average trip cost[2] (€)

$\bar{n}$ is the average number of trips per hour and driver (trips)

ΔT_v is the increase in the travel time of the other drivers caused by taxis (min)

Decision variables:

λ_d is the taxi hourly supply (vehicles per hour and area of service)

D is the flag-drop charge (€)

τ_{km} is the taxi fee per unit of distance (€/km)

τ_{sec} is the taxi fee per unit of time (€/min)

Model inputs (variables):

λ_u is the hourly demand for taxi trips (trips per hour and area of service)

A is the area of the region (km^2)

λ_v is the hourly circulating vehicles (vehicles per hour and area of service)

T_{IV} is the in-vehicle time of users (min)

$\bar{d}$ is the average distance of the trip (km)

$\bar{v}$ is the average speed of the trip (km/h)

E_d are the hourly vehicle emissions (kg of CO_2)

Model inputs (parameters):

VoT_u is the value of time of the taxi users (€/min)

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[1] $\bar{c} = D + \bar{d} \cdot \tau_{km} \widehat{\mp} \overline{T_{IV}} \cdot \tau_{sec}$, where $\widehat{\mp}$ symbolizes the special addition of taximeters, depending on the instantaneous speed. For simplifying the model the average trip cost is estimated only with the distance, without taking into account the temporal fare.

[2] Note that the term $\bar{c}$ does not affect the global objective function of the stakeholders since it will appear in both the users' cost and the drivers' cost with opposite signs. However it is an important factor when we analyze the profitability of each particular stakeholder.

VoT_d is the value of time of the taxi drivers (€/min)
VoT_v is the value of time of the other drivers (€/min)
α_A is the user perception factor of the access time
α_W is the user perception factor of the waiting time
α_{IV} is the user perception factor of the in-vehicle time
C_{km} is the operational cost per unit of distance of taxis (€/km)
C_h is the hourly operational cost of the moving taxis (€/min)
C_E is the emission unitary cost for all vehicles (€/kg of CO_2)
r is the area and network parameter (depending on the geometry as proposed in Holroyd (1965) and Smeed (1975))

The above metrics are expressed in terms of output per hour, analyzing the characteristics and providing the results for the typical peak hour of the market. Longer periods can be also selected if there is homogeneity in their characteristics along time. The constraints presented below must be taken into account when modeling taxi services in order to reflect physical or time restrictions of the real world:

- Access and waiting time of users lower than maximum values.
- Benefit of taxi drivers higher than minimum value.
- Emissions level lower than maximum value.
- Congestion level lower than the maximum value.
- Infrastructure cost lower than maximum value.
- Number of licenses between minimum and maximum values.

The problem is to minimize the objective function while respecting the above constraints.

3.2. Review of the formulations presented in the literature and used in the proposed model

3.2.1. In-vehicle travel time

The in-vehicle travel time is the same for the two operational modes. It can be expressed by using the average distance between two interior points within the zone and the average speed, as shown in Zamora and Cornadó (1996):

$$T_{IV} = \frac{\mathrm{r}\mathrm{A}^{1/2}}{2\bar{v}} \tag{5}$$

Smeed (1975) and Holroyd (1965) calculated different r values for different network configurations, presented in Table 1.

Table 1 Network parameters proposed by Smeed and Holroyd. Source: Zamora and Cornadó (1996)

Network	Network parameter		
	Smeed	Holroyd	r
Direct distance		0.905	1.00
Radial		1.333	1.47
External ring		2.237	2.47
Internal ring		1.445	1.59
Radial arc		1.104	1.21
Rectangular	0.78 – 0.97	1.153	1.27
Triangular		0.998	1.1
Hexagonal		1.153	1.27
Irregular	0.80 – 1.06		

3.2.2. Access and waiting time

In the hailing and dispatching markets the access time is either 0 or very small.

The models of Yang (2005) and Chang (2009) use the number of available taxis for obtaining the customers waiting time. In the dispatching market, the average waiting time can be expressed in terms of reaction time (negligible) and access time. The users' waiting time is the average travel time between the customer and the nearest vehicle, related to the density of free taxis in the area. The proposed formulation in Zamora and Cornadó (1996) is:

$$d(\Delta) = \frac{0{,}4\mathrm{r}}{\sqrt{\Delta}} \tag{6}$$

Where Δ is the density of free taxis. They conclude in the following waiting time:

$$\mathrm{T}_w = \frac{0{,}4\mathrm{A}^{1/2}\mathrm{r}}{\bar{v}\sqrt{A\lambda_d - A\lambda_u T_{IV}}} = \frac{0{,}4\mathrm{r}}{\bar{v}\sqrt{\lambda_d - \lambda_u T_{IV}}} \tag{7}$$

Fernandez et al. (2006) present the following formulation for the calculation of the waiting time in the hailing market:

$$\mathrm{T}_w = \frac{\mathrm{K}}{\mathrm{A}\lambda_d - A\lambda_u T_{IV}} \tag{8}$$

In accordance to Douglas (1972), they use the following approximation for the parameter K:

$$\mathrm{K} = \frac{\mathrm{A}}{\bar{v}} \tag{9}$$

3.2.3. Trip monetary cost

The characteristics of the trip (average distance, average duration and average speed) depend on the size and topology of the city. The average travel time has been presented in 0; the average distance depends on the average travel time and the average speed; the average speed depends on the policy applied to the taxi sector (if they can use the bus lanes) and the congestion level of the city for each time interval and zone (obtained using the speed reduction/increase obtained from the Macro Fundamental Diagram[3] (MFD) of the city). Other characteristics, such as unitary fees or emissions pricing depend on the policy applied by the city to the transport sector.

The value of time of both taxi user and other drivers is estimated in order to quantify the economical costs of the total travel time in the case of the users, and the travel time increase in the case of the other drivers. Many studies have obtained specific values for the VoT of the citizens by trip purpose, trip length, income and others. For the value of time, Small (1992) proposes the 50% of the average hourly salary, while Daganzo (2010) assumes it to be 20$/hour.

Three weighting parameters (α_A, α_W and α_{IV}) are proposed in order to use a unique VoT for the taxi users. The parameters weight access time, waiting time and in-vehicle travel time, taking into account the perception of the time by the users in each case. There is the need for calibration of the parameters in each city and society. The proposed parameters in Zamora and Cornadó (1996) are the following:

- Hourly cost of waiting time/hourly cost of in-vehicle time=2-3
- Hourly cost of access time/hourly cost of in-vehicle time=3

3.2.4. Externalities

In the last years there have been many attempts to internalize the externalities of the road transport in terms of congestion and pollution. The users that are paying these costs nowadays are the other road users (congestion) and the citizens (emissions). Both values are related to the average speed of the network, which can be calculated if a neat MFD for the city/area is known. If a neat relation density-speed is available, it is possible to estimate the new average speed of the network when adding or eliminating vehicle-kilometers produced by taxis.

3.2.4.1. Other road users

Taxis are circulating in the road network also used by other types of vehicles, such as buses or private cars. The number of circulating taxis has influence in the travel time of the rest of the users, especially when the percentage of taxis in the daily volume is high[4]. A variation in the number of circulating taxis will affect the travel time of all road users. Estrada et al (2011) have shown how to quantify the impact of this variation in terms of average speed reduction for a whole zone within the city. Using the neat MFD obtained by Geroliminis and

[3] Geroliminis and Daganzo (2008) have proved the existence of a unique and no scattered MFD.
[4] Taxi volume in Hong Kong represents the 60% of the total volume in the peak hours (Yang et al. (2000)).

Daganzo (2008) it is possible to estimate the average speed increase/reduction of the whole network due to a reduction/increase in the number of taxis in the network.

3.2.4.2. Emissions and fuel consumption

The environmental issues are gaining importance when developing policies and planning transport systems. In order to quantify the impact of the taxi services on emissions, an emission unitary cost is applied to the taxi emissions and to the additional emissions from other vehicles caused by the extra time due to the presence of taxis in the network. The fuel consumption and the emission levels are estimated using the travelled distance and the average speed using the formulations[5] proposed in the different environmental models. The impact of the taxi fleet on the average speed of the network can be approximated by the MFD: more taxis (vehicle-kilometers) will increase the density of the network and therefore reduce speed; oppositely, a reduction in the number of vehicles-kilometers produced by taxis will increase the average speed of the network, reducing fuel consumption and emissions.

4. Application of the proposed methodology

For the purpose of this paper two models are developed and presented, the dispatching and the hailing market. In the case of the dispatching market the vehicles wait at taxi stands of a homogeneous network of taxi stops distributed along the city. The externalities costs and the infrastructure costs are not taken into account. The demand is supposed known, and the models present the formulation for the optimum fleet, the waiting time related to this optimum fleet and the system costs. The demand calculation can be introduced into the model by using a bi-level formulation, where the demand is calculated at the upper level and the supply at the lower level. In each iteration the waiting time related to the optimum fleet is used for estimating the demand, and this demand is used for obtaining the new optimum fleet and related waiting time, stopping when the values don’t change significantly between interactions.

4.1. The dispatching market application

Taking into account the defined objective function and the approximations for the different variables presented above, the user and driver costs can be rewritten in terms of demand, supply and area of service. The optimum supply is presented below:

[5] Most of the models propose a second grade function for estimating the kilometric fuel consumption and the emissions using the average speed.

$$\lambda_d{}^{*}\mathrm{A} = \lambda_u \mathrm{A}\frac{\mathrm{rA}^{1/2}}{2\bar{v}}\varepsilon + \lambda_u \mathrm{A}\left(\frac{\alpha_W VoT_u 0{,}4\mathrm{r}}{2\bar{v}C_h}\right)^{2/3} \tag{10}$$

where the first term is the minimum fleet size for serving all trips and the second term is the extra fleet needed for providing a better LoS to customers while maintaining a satisfactory profit to taxi drivers. The constant value of this extra fleet is directly proportional to the VoT and inversely proportional to $\mathrm{C_h}$ and $\bar{\mathrm{v}}$, which means:

- High VoT of taxi users implies a higher extra fleet in order to reduce waiting time.
- High r (longer trips due to the complexity of the network) values implies more taxis
- High hourly operating cost implies fewer taxis for reducing the empty kilometers of taxis.
- Higher speed implies smaller taxi fleet due to the higher performance of the vehicles.

The relation between these four values (r, VoT, operation costs and network speed) defines the extra fleet needed for serving the same demand level and city.

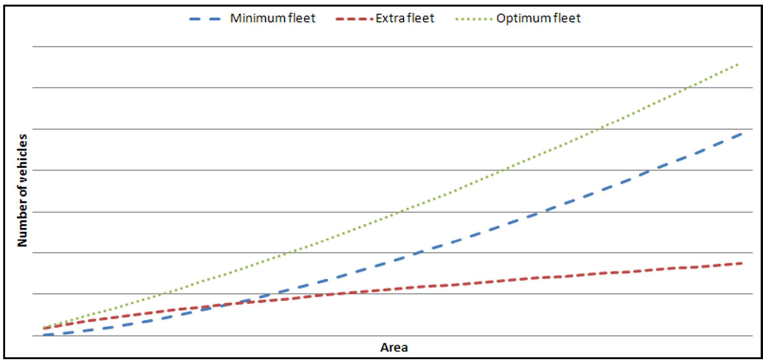

Figure 1 Minimum, extra and optimum fleet in relation to the city area

Figure 1 shows the minimum and the optimum fleet sizes obtained by the formulation presented above in relation to the area size for a generic case. It shows that small cities need a very small minimum fleet due to the small distance of the trips, but the extra fleet is much larger. Larger cities need larger minimum fleets (longer trips), but the extra fleet is smaller in relation to the minimum fleet compared with smaller cities.

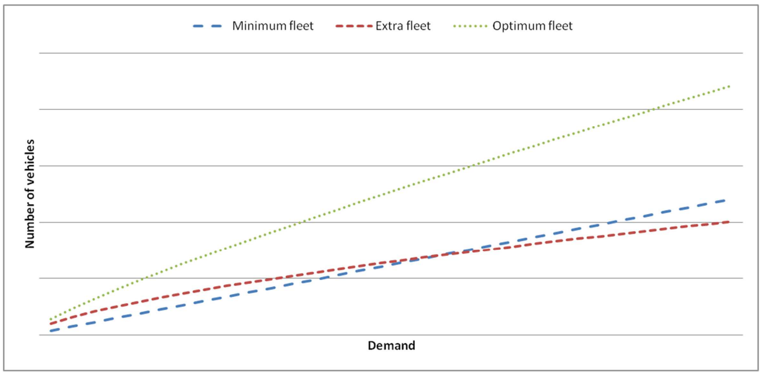

Figure 2 Minimum, extra and optimum fleet in relation to the demand for taxi services

Figure above shows the optimum fleet and its composition (minimum fleet and extra fleet) in relation to the demand. The relation between the minimum fleet and the extra fleet decreases as the demand level increases. Again low demand levels require larger extra fleet in relation to the minimum fleet.

The waiting time related to this optimum fleet size is presented below.

$$T_W{}^* = \left(\frac{\alpha_W VoT_u}{0{,}32 \cdot C_h \mathrm{r}^2} \lambda_u \bar{v}^2\right)^{-1/3} \tag{11}$$

The waiting time is multiplied by a similar factor to the one commented above for the extra fleet (directly proportional to r and C_h and inversely proportional to VoT). Similar conclusions can be obtained:

- Higher VoT or network speed implies less waiting time.
- Higher r or C_h implies higher waiting times.

Finally, by introducing the above results into the generalized cost function (drivers and users), we obtain:

$$Z_u + Z_d = A\left[\frac{\lambda_d{}^* C_h}{VoT_d} + \lambda_u \frac{\mathrm{r}A^{1/2}}{2}\left(\frac{C_{km}}{VoT_d} + \alpha_{IV}\right) + \lambda_u \alpha_W T_W{}^*\right] \tag{12}$$

where the first term corresponds to the fixed hourly cost of the taxi fleet, the second term represents the user and driver variable cost due to the trips and the final term is the waiting time cost of the users. Rearranging the terms and introducing the formulations of the optimum fleet and the related waiting time we obtain the following system optimum unitary cost:

$$\frac{Z_u + Z_d}{\lambda_u \cdot A} = \frac{C_h}{VoT_d}\frac{\mathrm{r}A^{1/2}}{2\bar{v}}\varepsilon + \left(28{,}125\frac{\bar{v}^2 VoT_d{}^3 \lambda_u}{(\alpha_W \mathrm{r})^2 C_h}\right)^{-1/3} + \frac{\mathrm{r}A^{1/2}}{2\bar{v}} \cdot \left(\frac{C_{km}}{VoT_d} + \frac{\alpha_{IV}}{\bar{v}}\right) \tag{13}$$

By representing the unitary costs we obtain the results shown in Figure 3 below, where the minimum and optimum fleet are represented using the formulations presented above (equation (10)). The figure represents the total costs (axis z – contour lines) of the system depending on the demand (axis x) and the supply (axis y) for a fixed area size.

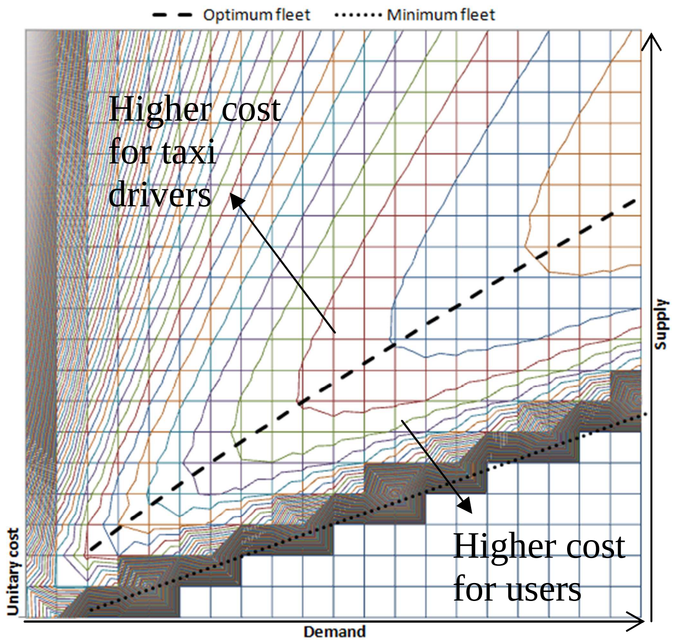

Figure 3 System unitary cost of each demand and supply configurations for the dispatching-stand market

Calculating the unitary costs the results shown in Figure 4 are obtained, where a minimum system unitary cost can be observed. Each demand level has a supply level (or range M) for which the total unitary cost is minimum.

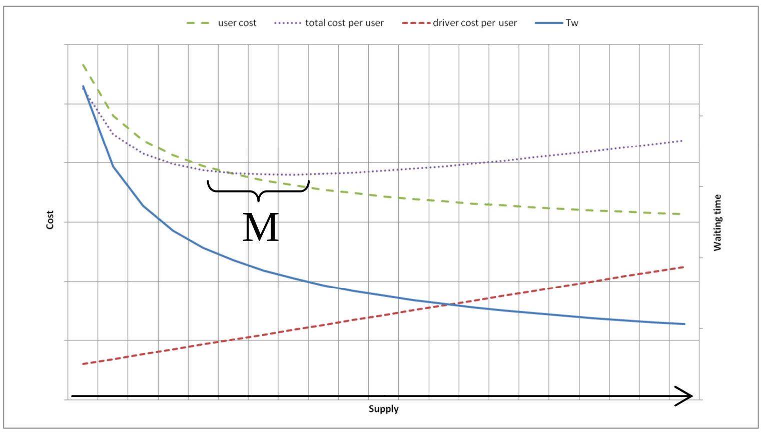

Figure 4 Waiting time, user, driver and system unitary costs of different demand levels for the dispatching-stand market

4.2. The cruising market application

Taking into account the defined objective function and the approximations for the different variables presented above, the user and driver costs can be rewritten in terms of demand, supply and area of service for the hailing market. The optimum supply is presented below:

$$\lambda_d{}^{*}A = \lambda_u A\frac{rA^{1/2}}{2\bar{v}} + A\left(\frac{\lambda_u \alpha_W VoT_d}{\bar{v}(\bar{v}\cdot C_{km} + C_h)}\right)^{1/2} \tag{14}$$

where the first term is the minimum fleet size for serving all trips and the second term is the extra fleet needed for providing a better LoS to customers while maintaining a satisfactory profit to taxi drivers. The constant value of this extra fleet is directly proportional to the VoT and inversely proportional to C_h, C_{km} and $\bar{v}$, presenting similar behavior as the one observed for the dispatching market.

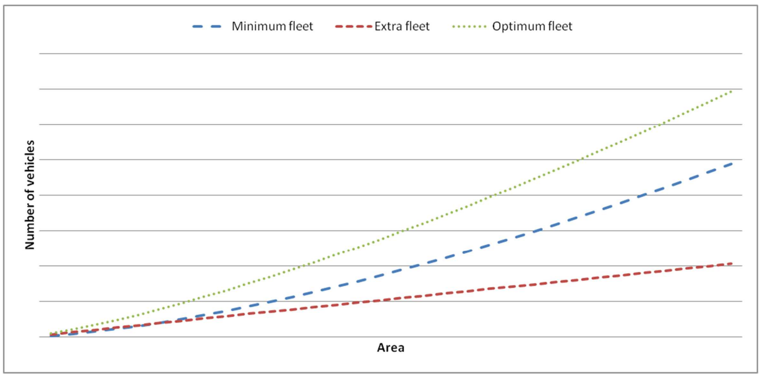

Figure 5 Minimum, extra and optimum fleet in relation to the city area

Figure 1 shows the minimum and the optimum fleet sizes obtained by the formulation presented above in relation to the area size for a generic case. It shows that the extra fleet grows linearly in relation to the city size while the minimum fleet grows with a higher exponent. The number of extra taxis in relation to the extra fleet decreases as the area grows.

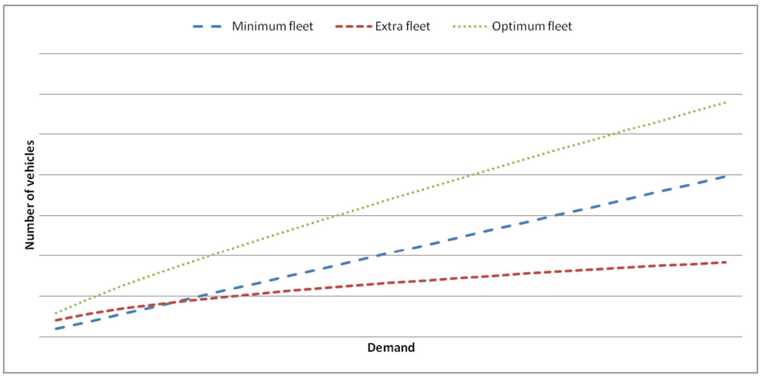

Figure 6 Minimum, extra and optimum fleet in relation to the demand for taxi services

Figure above shows the optimum fleet and its composition (minimum fleet and extra fleet) in relation to the demand level. The relation between the minimum fleet and the extra fleet decreases as the demand level increases.

The waiting time related to this optimum fleet size is presented below.

$$T_W{}^{*} = \left(\frac{\alpha_W VoT_d}{\bar{v}\cdot C_{km} + C_h}\lambda_u \bar{v}\right)^{-1/2} \tag{15}$$

The waiting time is multiplied by the same factor commented above (directly proportional to the VoT and inversely proportional to C_h, C_{km} and $\bar{v}$). Finally, by introducing the above results into the generalized cost function (drivers and users), we obtain:

$$Z_u + Z_d = A\left[\frac{\lambda_d{}^*(C_h + \bar{v}C_{km})}{VoT_d} + \lambda_u \alpha_{IV} \frac{rA^{1/2}}{2} + \lambda_u \alpha_W T_W{}^*\right] \tag{16}$$

where the first term corresponds to the fixed hourly cost of the taxi fleet, the second term represents the user variable cost due to the trips and the final term is the waiting time cost of the users. Rearranging the terms and introducing the formulations of the optimum fleet and minimum waiting time we obtain the following system optimum unitary cost:

$$\frac{Z_u + Z_d}{\lambda_u \cdot A} = 2\left(\frac{\alpha_W(\bar{v} \cdot C_{km} + C_h)}{VoT_d \bar{v} \lambda_u}\right)^{1/2} + \frac{rA^{1/2}}{2\bar{v}} \cdot \left(\frac{\bar{v} \cdot C_{km} + C_h}{VoT_d} + \alpha_{IV}\right) \tag{17}$$

By representing the unitary costs we obtain the results shown in Figure 7 below, where the minimum and optimum fleet are represented using the formulations presented above (equation (14)). The figure represents the total costs (axis z – contour lines) of the system depending on the demand (axis x) and the supply (axis y) for a fixed area size.

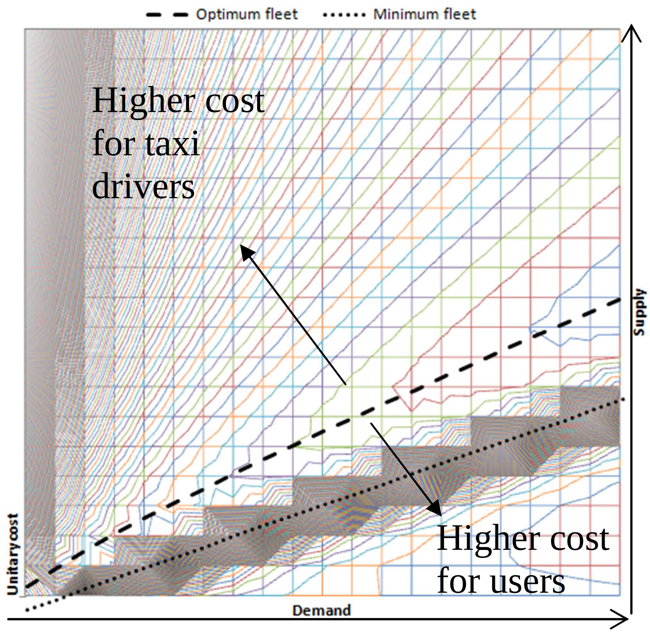

Figure 7 System unitary cost of each demand and supply configurations for the dispatching-stand market

Calculating the unitary costs, the results shown in the below figure are obtained, where a minimum global cost can be observed.

As observed in the dispatching market, each demand level has a supply level (or range M) for which the total unitary cost is minimum.

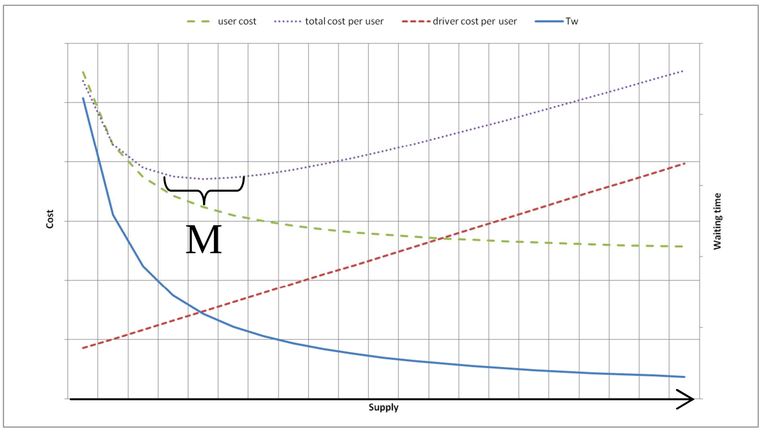

Figure 8 Waiting time, user, driver and system unitary costs of different demand levels for the dispatching-stand market

4.3. Comparison between the hailing and the dispatching taxi markets

The above formulations are compared for estimating the demand level and the area size where a dispatching market has better performance than a hailing market and vice versa.

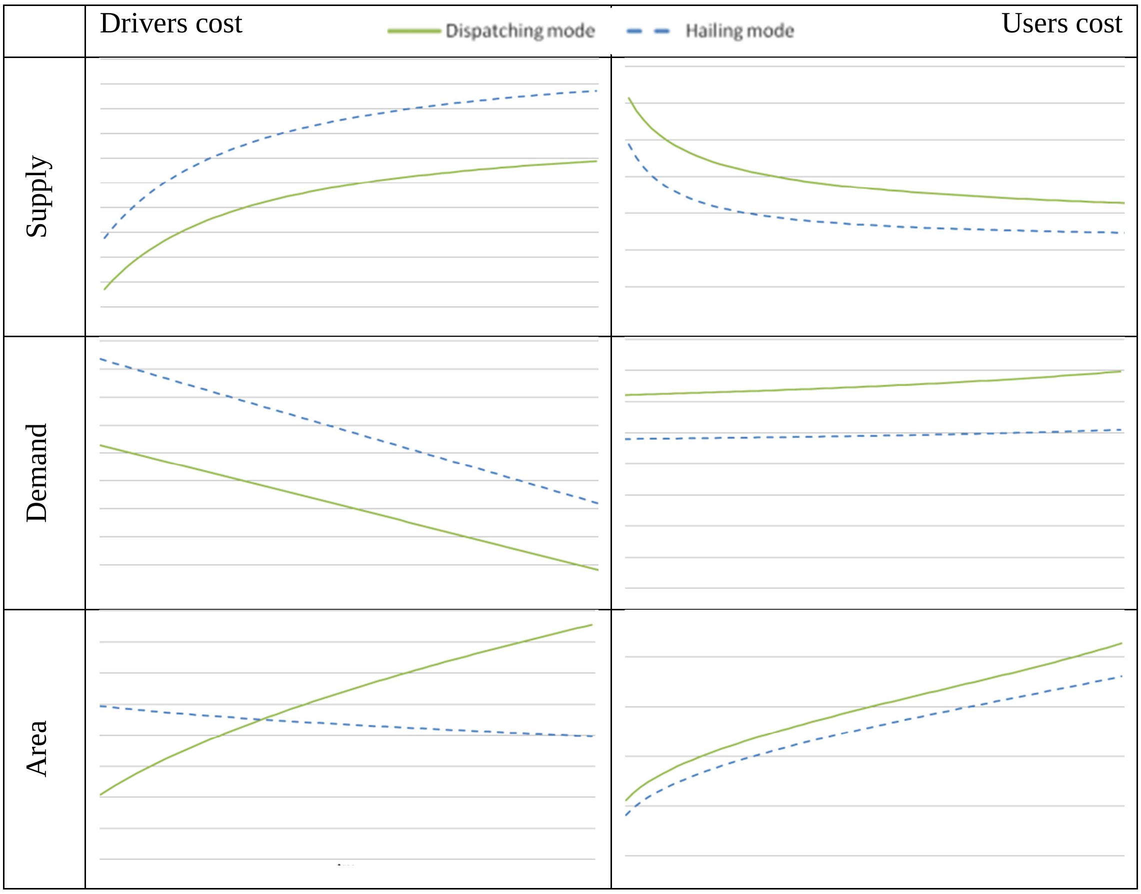

Figure 9 User and driver costs for each area, demand and supply level for the hailing and the dispatching markets

The above graph show that the user costs of the hailing market are lower than the ones of the dispatching market, independently of the area, demand or supply levels. At the same time, the driver costs are higher in the hailing market (independently of the demand and area levels, but depending on the city size).

The above result means that there will be a demand, supply and area levels where the unitary costs of the dispatching and the hailing modes are equal, defining the demand, supply or area for what both systems will have the same unitary cost. Therefore the bounds of the demand, supply and area values where one operation mode is better than the other ones can be observed in the below graphs.

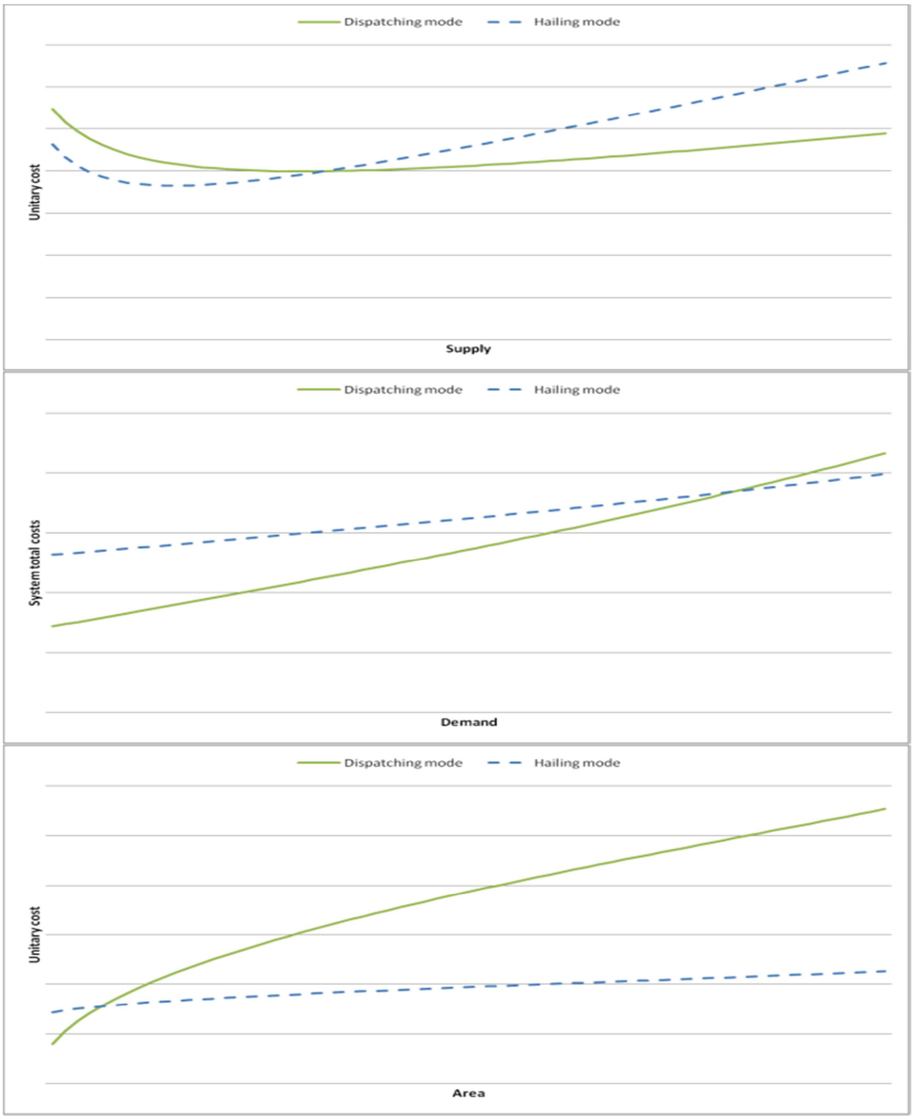

Figure 10 Unitary and system costs for each area, demand and supply level for the hailing and the dispatching market

The boundaries of each one of the three above variables (demand, supply and city area) are defined by the other two. For the same demand and supply level there is an equilibrium area value where the unitary costs of the hailing and dispatching modes are equal. Dispatching mode is recommended for smaller cities while hailing mode is recommended for higher cities than this equilibrium area. The same conclusion can be obtained for the other two variables (demand and supply). This equilibrium value is the value where the unitary costs of the same supply, demand or city are equal when operating in a hailing or a dispatching mode. Any change of one of the three variables will increase the unitary cost of both operation modes, but they will not be equal anymore, and one of the two will have lower unitary cost and it will be the desired one.

It can be also concluded from the figure above that the optimum supply for the same area and demand levels is smaller in the hailing mode than in the dispatching mode. The hailing mode prefers fewer taxis because they are circulating constantly, increasing in this way the waiting time of users, but reducing the vacant time of drivers. The dispatching mode prefers more taxis because they are waiting at taxi stands, reducing their operational costs and earnings, but reducing also the customers waiting time.

Three parameters have great influence on the above equilibrium points: the operational cost, the average speed and the Value of Time. The table below shows the qualitative impact of each one of the parameters to the three equilibrium points named above (demand, area and supply).

Table 2 Impact of the operational cost, the average speed and the Value of Time on the equilibrium points related to demand, supply and area.

Parameter	Change in the parameter	Impact on the equilibrium point		
		Demand	Supply	Area
Value of Time (VoT)	Reduction	Higher	Higher	Higher
Operational costs (C_h, C_{km})	Increase	Higher	Lower	Higher
Speed ($\bar{v}$)	Increase	Higher	Lower	Higher

The above results can be explained as follows:

- A reduction in the VoT means that higher area and demand levels can be served better by the dispatching market because users are less sensitive to waiting time. In relation to the supply means that higher supply will have fewer unitary costs in the hailing mode than in the dispatching one.
- An increase in the operational costs or in the speed means that higher area and demand can be served better by the dispatching mode, because the number of kilometers is lower than in the hailing market. In this case the supply equilibrium

point is lower, what means that higher supply will have higher unitary costs in the hailing mode than in the dispatching mode.

Finally, two curiosities of the obtained results are presented below.

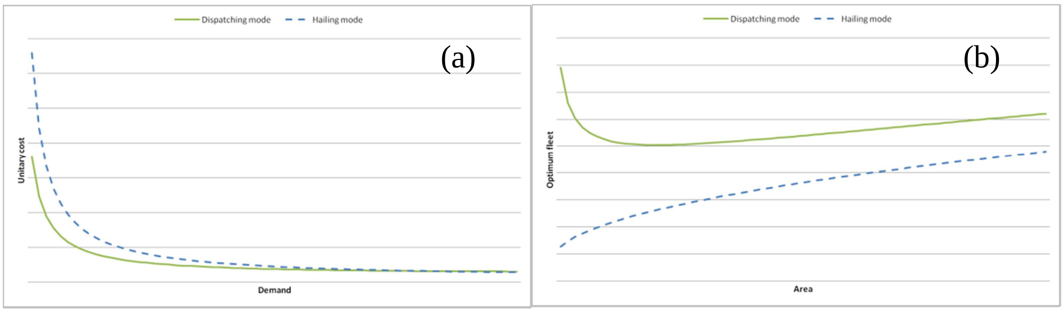

Figure 11 Unitary cost depending on the demand (a) and optimum fleet depending on the area (b)

The first graph shows how the unitary costs converge when the demand is increasing for both modes of operation. The second shows that the optimum fleet of the dispatching market presents a minimum in relation to the area while in the hailing market it grows continuously.

5. Agent based modeling of taxi services

The authors (Salanova et al. 2012) are currently working in an agent based model for estimating the variables of the three modes of operation. The graphs below are a brief presentation of the obtained results.

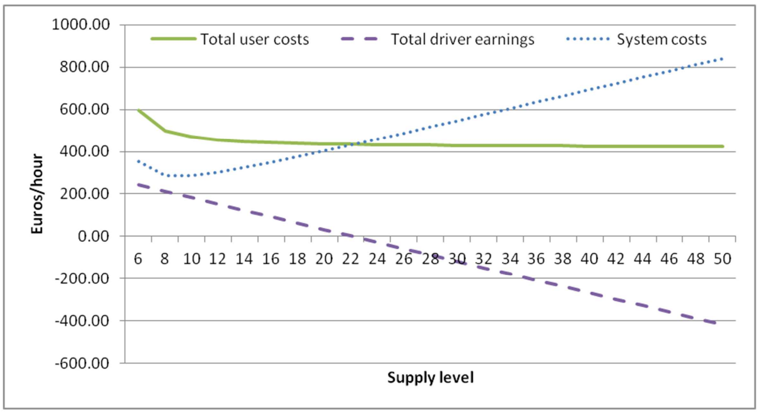

Figure 12 Passenger and driver costs related to different supply levels and to a fix demand for the dispatching mode model.

The above graph shows the systems costs of different fleet sizes for a fixed demand level and area. It can be observed that the systems cost curve is very similar to the one obtained presented in Figure 10.

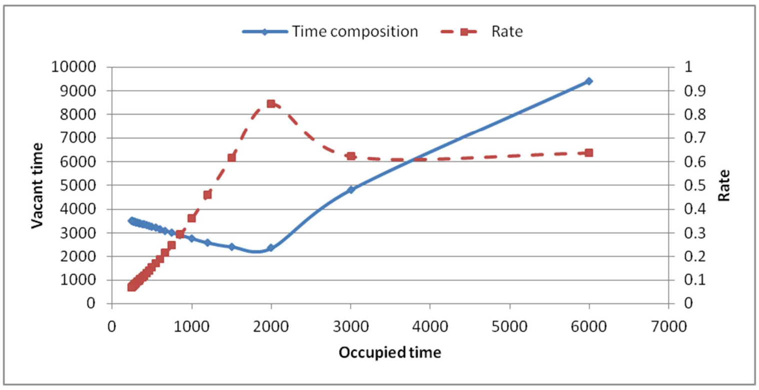

Figure 13 Relation between the empty and the busy distances for the dispatching mode model.

The above graph shows the relation between the occupied and vacant distances for different supply levels. This relation is 0.65 for the supply related to the minimum system cost, which means that the 60% of the time the taxis are empty. The table below shows the results for the three operation modes obtained by the agent-based simulation model.

Table 3 Simulation results of the three operation modes.

	Dispatching*	Stand	Hailing**
Optimum fleet for system cost (vehicles)	8	10	14
Average occupied distance (m)	6.100	4.900	3.500
Average vacant distance (m)	4.200	330	14.000
Average occupied time (min)	25	20	14,5
Average vacant moving time (min)	35	16	90
Average vacant stopped time (min)	0	24	0
Income of drivers (euros/h)	67	54	38
Driver unitary earnings (euros/hour)	32,5	30	-8
Rate occupied/vacant moving time	0.65 (35-40%)	1.3 (55-60%)	0.16 (10-15%)
Rate occupied/vacant distance	1.5 (60%)	15 (94%)	0.25 (20%)
Average user waiting time (sec)	136	65	392
User unitary cost (euros/hour)	6,45	6,25	7,5
System cost (euros/hour)	385	325	865

*In the agent based model the dispatching taxis are circulating continuously waiting for a call
**In the hailing case more than one hour was needed most of the times for completing all trips since the taxis are randomly looking for users.

6. Conclusions

An aggregated mathematical model for the estimation of the costs of the taxi services in urban areas has been presented in this paper. The model is able to take into account externalities, such as environmental issues or delays.

Formulations for the dispatching and hailing taxi markets have been reviewed and analyzed, concluding in minimum and optimum taxi fleets for different city sizes and demand levels. While the minimum taxi fleet ensures that all trips will be served, the optimum fleet ensures a minimum LoS to travellers while providing positive benefits to taxi drivers. This extra fleet size is directly proportional to the value of time of users and inversely proportional to the speed and hourly operation costs of taxi drivers. As it is obvious, larger cities need larger taxi fleets for satisfying mobility needs of the same demand level. The relation between demand and supply has been presented for different city sizes.

The model presents the methodology for obtaining the optimum fleet and the related waiting time for each demand level. If the relation between the waiting time and the demand is known, the demand can be calculated for the obtained waiting time, and using an iterative procedure demand elasticity can be introduced into the model. Demand fluctuations can be also taken into account, dividing the day into time intervals and solving for each time interval, obtaining in this way the optimum fleet for each interval and therefore the supply distribution along the day.

The proposed model is based on aggregated mathematical approximations, where spatial distribution of demand and supply cannot be taken into account as it is done in the equilibrium models. Moreover, the agent based model being developed by the authors is able to take into account this spatial distribution, showing the effects of different networks geometries and demand distribution in comparison with this aggregated model.

Applying the formulation to the different operation modes they can be characterized and compared, obtaining guidelines on the implementation of taxi services to different cities (size and demand level). The presented results indicate that small cities or cities with low demand for taxi services must have dispatching taxi market, rather than hailing market. At the same time, if the taxi fleet is big, it is recommended to have a hailing market rather than a dispatching one. This results are obtained without taking into account the externalities of the circulating taxis. If they are introduced into the models the above conclusions may change due to the impact of the circulating taxis on the average speed network.

All the presented models need to be calibrated with real world or simulation data in order to understand the behavior of the variables and to validate their hypotheses. Agent models like the presented should be developed for simulating the drivers and users behavior in the network and support the proposed models with simulation data.

Further research is also needed in the demand estimation methods, since the waiting time depends on the demand and at the same time the demand depends on the waiting time. The relation between demand and waiting time should be analyzed.

In order to better simulate the real world, mixed models must be developed, combining the different taxi operation modes, such as the model presented by Bai and Wang (2012).

Finally, real time data and forecasting procedures must be applied to the taxi models in order to evaluate the impact on the level of service and the income of taxi drivers. This new characteristics of the taxi services will anticipate user requests and traffic condition in the city (As proposed by Wong and Bell (2006)), optimizing the management of the taxi fleet.

7. Proofs

7.1. Minimum waiting time, optimum fleet and unitary cost in the dispatching market

Introducing the formulations presented by Zamora and Cornadó (1996) (for the waiting time and the average trip distance and duration) to the generalized costs presented in the model:

$$Z_u = \lambda_u \cdot A \cdot \left[\alpha_A \cdot 0 + \alpha_W \cdot \frac{0{,}4\mathrm{r}}{\bar{v}\sqrt{\lambda_d - \lambda_u \frac{\mathrm{rA}^{1/2}}{2\bar{v}}\varepsilon}} + \alpha_{IV} \cdot \frac{\mathrm{rA}^{1/2}}{2\bar{v}} + \frac{D + \frac{\mathrm{rA}^{1/2}}{2} \cdot \tau_{km}}{VoT_u} \right] \tag{18}$$

$$\begin{aligned} Z_d &= \frac{\lambda_d \cdot A}{VoT_u}\left[-\frac{\lambda_u}{\lambda_d} \cdot \left(D + \frac{\mathrm{rA}^{1/2}}{2} \cdot \tau_{km}\right) + \frac{\lambda_u}{\lambda_d} \cdot \frac{\mathrm{rA}^{1/2}}{2\bar{v}} \cdot C_{km} + C_h \right] \\ &= \frac{A}{VoT_d}\left[-\lambda_u \cdot \left(D + \frac{\mathrm{rA}^{1/2}}{2} \cdot \tau_{km}\right) + \lambda_u \cdot \frac{\mathrm{rA}^{1/2}}{2\bar{v}} \cdot C_{km} + \lambda_d \cdot C_h \right] \end{aligned} \tag{19}$$

Finding the derivative of the above formulations in respect to the supply:

$$\frac{\partial Z_u}{\partial \lambda_d} = -\frac{\lambda_u \cdot A\alpha_W \cdot 0{,}4\mathrm{r}}{2\bar{v}\sqrt{\left(\lambda_d - \lambda_u \frac{\mathrm{rA}^{1/2}}{2\bar{v}}\varepsilon\right)^3}} \tag{20}$$

$$\frac{\partial Z_d}{\partial \lambda_d} = \frac{A \cdot C_h}{VoT_d} \tag{21}$$

And the optimum supply is the following:

$$\frac{A \cdot C_h}{VoT_d} = \frac{\lambda_u A\alpha_W 0{,}4\mathrm{r}}{2\bar{v}\sqrt{\left(\lambda_d - \lambda_u \frac{\mathrm{rA}^{1/2}}{2\bar{v}}\varepsilon\right)^3}} \tag{22}$$

$$\lambda_d = \lambda_u \frac{\mathrm{rA}^{1/2}}{2\bar{v}}\varepsilon + \left(\frac{\lambda_u \alpha_W VoT_d 0{,}4\mathrm{r}}{2\bar{v}C_h}\right)^{2/3} \tag{23}$$

The associated waiting time to the optimum supply is the following:

$$\begin{aligned} T_W &= \frac{0{,}4\mathrm{r}}{\bar{v}\sqrt{\lambda_d - \lambda_u \frac{\mathrm{rA}^{1/2}}{2\bar{v}}\varepsilon}} = \frac{0{,}4\mathrm{r}}{\bar{v}}\left(\frac{2C_h\bar{v}}{\lambda_u \alpha_W VoT_d 0{,}4\mathrm{r}}\right)^{1/3} = \left(\frac{0{,}32 \cdot C_h \mathrm{r}^2}{\lambda_u \bar{v}^2 \alpha_W VoT_d}\right)^{1/3} \\ &= \left(\frac{\alpha_W VoT_d}{0{,}32 \cdot C_h \mathrm{r}^2}\lambda_u \bar{v}^2\right)^{-1/3} \end{aligned} \tag{24}$$

Finally, introducing all the above into the generalized cost function:

$$Z_u = \lambda_u \cdot A \cdot \left[\alpha_W \cdot \left(\frac{\alpha_W VoT_d}{0{,}32 \cdot C_h \mathrm{r}^2}\lambda_u \bar{v}^2\right)^{-1/3} + \alpha_{IV} \cdot \frac{\mathrm{rA}^{1/2}}{2\bar{v}} + \frac{D + \frac{\mathrm{rA}^{1/2}}{2} \cdot \tau_{km}}{VoT_u} \right] \tag{25}$$

$$Z_d = \frac{A}{VoT_d}\left[-\lambda_u \cdot (D + \frac{\mathrm{rA}^{1/2}}{2} \cdot \tau_{km}) + \lambda_u \cdot \frac{\mathrm{rA}^{1/2}}{2\bar{v}} \cdot C_{km} + \left(\lambda_u \frac{\mathrm{rA}^{1/2}}{2\bar{v}} \varepsilon + \left(\frac{\lambda_u \alpha_W VoT_d 0{,}4\mathrm{r}}{2\bar{v}C_h}\right)^{2/3}\right) \cdot C_h\right] \tag{26}$$

$$\begin{aligned}\frac{Z_u + Z_d}{\lambda_u \cdot A} &= \left(\frac{\mathrm{rA}^{1/2}}{2\bar{v}} \varepsilon + \left(\frac{\alpha_W VoT_d 0{,}4\mathrm{r}}{2\bar{v}C_h\sqrt{\lambda_u}}\right)^{2/3}\right) \cdot \frac{C_h}{VoT_d} + \frac{\mathrm{rA}^{1/2}}{2\bar{v}} \cdot \left(\frac{C_{km}}{VoT_d} + \alpha_{IV}\right) \\ &\quad + \alpha_W \left(\frac{\alpha_W VoT_d}{0{,}32 \cdot C_h \mathrm{r}^2} \lambda_u \bar{v}^2\right)^{-1/3} \\ &= \frac{C_h}{VoT_d} \frac{\mathrm{rA}^{1/2}}{2\bar{v}} \varepsilon + \left(\frac{\bar{v}^2 VoT_d \lambda_u}{(0{,}2\alpha_W \mathrm{r})^2 C_h}\right)^{-1/3} + \frac{\mathrm{rA}^{1/2}}{2\bar{v}} \cdot \left(\frac{C_{km}}{VoT_d} + \alpha_{IV}\right) \\ &\quad + \left(\frac{\lambda_u VoT_d{}^3 \bar{v}^2}{0{,}32 \cdot C_h (\alpha_W \mathrm{r})^2}\right)^{-1/3} \\ &= \left(28{,}125 \frac{\bar{v}^2 VoT_d{}^3 \lambda_u}{(\alpha_W \mathrm{r})^2 C_h}\right)^{-1/3} + \frac{\mathrm{rA}^{1/2}}{2\bar{v}} \cdot \left(\frac{C_{km}}{VoT_d} + \alpha_{IV} + \frac{C_h}{VoT_d} \varepsilon\right)\end{aligned} \tag{27}$$

7.2. Minimum waiting time, optimum fleet and unitary cost in the hailing market

Introducing the formulations presented by Zamora and Cornadó (1996) (for the waiting time and the average trip distance and duration) to the generalized costs presented in the model:

$$Z_u = \lambda_u \cdot A \cdot \left[\alpha_A \cdot 0 + \alpha_W \cdot \frac{\mathrm{A}}{\bar{v}\left(\mathrm{A}\lambda_d - A\lambda_u \frac{\mathrm{rA}^{1/2}}{2\bar{v}}\right)} + \alpha_{IV} \cdot \frac{\mathrm{rA}^{1/2}}{2\bar{v}} + \frac{D + \frac{\mathrm{rA}^{1/2}}{2} \cdot \tau_{km}}{VoT_u}\right] \tag{28}$$

$$Z_d = \frac{\lambda_d \cdot A}{VoT_u}\left[-\frac{D + \frac{\mathrm{rA}^{1/2}}{2} \cdot \tau_{km}}{VoT_u} + \bar{v} \cdot C_{km} + C_h\right] \tag{29}$$

Finding the derivative of the above formulations in respect to the supply:

$$\frac{\partial Z_u}{\partial \lambda_d} = -\frac{\lambda_u \cdot A\alpha_W \cdot \bar{v}}{\left(\bar{v}\lambda_d - \lambda_u \frac{\mathrm{rA}^{1/2}}{2}\right)^2} \tag{30}$$

$$\frac{\partial Z_d}{\partial \lambda_d} = \frac{A}{VoT_d}\left[-\lambda_u \cdot \left(D + \frac{\mathrm{rA}^{1/2}}{2} \cdot \tau_{km}\right) + \bar{v} \cdot C_{km} + C_h\right] \tag{31}$$

And the optimum supply is the following:

$$\frac{A}{VoT_d}[\bar{v} \cdot C_{km} + C_h] = \frac{\lambda_u \cdot A\alpha_W \cdot \bar{v}}{\left(\bar{v}\lambda_d - \lambda_u \frac{\mathrm{rA}^{1/2}}{2}\right)^2} \tag{32}$$

$$\lambda_d = \lambda_u \frac{\mathrm{rA}^{1/2}}{2\bar{v}} + \left(\frac{\lambda_u \alpha_W VoT_d}{\bar{v}(\bar{v} \cdot C_{km} + C_h)}\right)^{1/2} \tag{33}$$

The associated waiting time to the optimum supply is the following:

$$\begin{aligned}T_W &= \frac{1}{\bar{v}\left(\lambda_u \frac{rA^{1/2}}{2} + \left(\frac{\sqrt{\bar{v}}\lambda_u \alpha_W VoT_d}{(\bar{v} \cdot C_{km} + C_h)}\right)^{1/2} - \lambda_u \frac{rA^{1/2}}{2}\right)} = \left(\frac{\bar{v} \cdot C_{km} + C_h}{\lambda_u \bar{v} \alpha_W VoT_d}\right)^{1/2} \\ &= \left(\frac{\alpha_W VoT_d}{\bar{v} \cdot C_{km} + C_h} \lambda_u \bar{v}\right)^{-1/2}\end{aligned} \tag{34}$$

Finally, introducing all the above into the generalized cost function:

$$Z_u = \lambda_u \cdot A \cdot \left[\alpha_W \cdot \left(\frac{\alpha_W VoT_d}{\bar{v} \cdot C_{km} + C_h} \lambda_u \bar{v}\right)^{-1/2} + \alpha_{IV} \cdot \frac{\mathrm{rA}^{1/2}}{2\bar{v}} + \frac{D + \frac{\mathrm{rA}^{1/2}}{2} \cdot \tau_{km}}{VoT_u}\right] \tag{35}$$

$$Z_d = \left(\lambda_u \frac{rA^{1/2}}{2\bar{v}} + \left(\frac{\lambda_u \alpha_W VoT_d}{\bar{v}(\bar{v}\cdot C_{km} + C_h)}\right)^{1/2}\right)\frac{A}{VoT_d}\left[-\lambda_u \cdot \left(D + \frac{rA^{1/2}}{2}\cdot \tau_{km}\right) + \bar{v}\cdot C_{km} + C_h\right] \quad (36)$$

$$\begin{aligned}\frac{Z_u + Z_d}{\lambda_u \cdot A} &= \left(\frac{rA^{1/2}}{2\bar{v}} + \left(\frac{\alpha_W VoT_d}{\bar{v}\lambda_u(\bar{v}\cdot C_{km} + C_h)}\right)^{1/2}\right)\cdot \frac{\bar{v}\cdot C_{km} + C_h}{VoT_d} + \alpha_W \\ &\cdot \left(\frac{\alpha_W VoT_d}{\bar{v}\cdot C_{km} + C_h}\lambda_u \bar{v}\right)^{-1/2} + \alpha_{IV}\cdot \frac{rA^{1/2}}{2\bar{v}} \\ &= 2\left(\frac{\alpha_W(\bar{v}\cdot C_{km} + C_h)}{VoT_d \bar{v}\lambda_u}\right)^{1/2} + \frac{rA^{1/2}}{2\bar{v}}\cdot\left(\frac{\bar{v}\cdot C_{km} + C_h}{VoT_d} + \alpha_{IV}\right)\end{aligned} \quad (37)$$

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