Submitted to *Transportation Science*
manuscript TS-2020-0096

A simulation-based heuristic to find approximate equilibria with disaggregate demand models

Stefano Bortolomiol
École Polytechnique Fédérale de Lausanne, School of Architecture, Civil and Environmental Engineering, Transport and Mobility Laboratory, Switzerland, stefano.bortolomiol@epfl.ch

Virginie Lurkin
Eindhoven University of Technology, Department of Industrial Engineering & Innovation Science, The Netherlands, v.j.c.lurkin@tue.nl

Michel Bierlaire
École Polytechnique Fédérale de Lausanne, School of Architecture, Civil and Environmental Engineering, Transport and Mobility Laboratory, Switzerland, michel.bierlaire@epfl.ch

Oligopolistic competition occurs in various transportation markets. In this paper, we introduce a framework to find approximate equilibrium solutions of oligopolistic markets where demand is modeled at the disaggregate level using discrete choice models, according to random utility theory. Compared with aggregate demand models, the added value of discrete choice models is the possibility to account for more complex and precise representations of individual behaviors. Due to the form of the resulting demand functions, there is no guarantee that an equilibrium solution for the given market exists, nor it is possible to rely on derivative-based methods to find one. Therefore, we propose a model-based algorithmic approach to find approximate equilibria, which is structured as follows. A heuristic reduction of the search space is initially performed. Then, a subgame equilibrium problem is solved using a mixed integer optimization model inspired by the fixed-point iteration algorithm. The optimal solution of the subgame is compared against the best responses of all suppliers over the strategy sets of the original game. Best response strategies are added to the restricted problem until all ε-equilibrium conditions are satisfied simultaneously. Numerical experiments show that our methodology can approximate the results of an exact method that finds a pure equilibrium in the case of multinomial logit model of demand with single-product offer and homogeneous demand. Furthermore, it succeeds at finding approximate equilibria for two transportation case studies featuring more complex discrete choice models, heterogeneous demand, multi-product offer by supplies, and price differentiation, for which no analytical approach exists.

Key words: competition, equilibrium, disaggregate demand, discrete choice modeling

1. Introduction

Oligopolistic competition occurs in various markets when a limited number of suppliers compete for the same pool of customers. This is often the case in transportation, due to reasons such as external regulations, limited capacity of the infrastructure and other barriers to entry (Bresnahan and Reiss 1991, Starkie 2002, Liu, Chen, and Huang 2011, Beck 2011). In oligopolies, suppliers make decisions that are influenced both by the preferences of the customers, who are considering to purchase one of the alternative services available on the market, and by the decisions of their competitors. Nowadays, these decisions are informed by both detailed consumer data, from which precise individual behavioral models are derived, and competitor insights, which allow to react in real time to market changes.

Oligopolistic competition has been extensively analyzed through static and dynamic models, with the goal of finding and evaluating market equilibria (Stigler 1964, Friedman 1971, Shaked and Sutton 1983, Maskin and Tirole 1987, 1988a,b, Brander and Zhang 1993). Mathematically, an equilibrium is guaranteed to exist only if a number of conditions related to continuity, differentiability and convexity are satisfied for the demand, cost and profit functions (Murphy, Sherali, and Soyster 1982). In particular, these requirements pose limitations on the demand function. Demand functions can be estimated using aggregate or disaggregate data. The main benefit of estimating demand at the disaggregate level is the possibility to account for product differentiation and consumer behavioral heterogeneity at the individual level (Anderson, De Palma, and Thisse 1992). Although there exists a large body of discrete choice modeling literature describing complex disaggregate choice behavior, disaggregate demand models are generally aggregated before being included in models of oligopolies (McFadden and Reid 1975, Koppelman 1976, Ben-Akiva and Lerman 1985, Berry, Levinsohn, and Pakes 1995). The reason is that only the simplest disaggregate demand models, which require several limiting assumptions, satisfy the equilibrium existence conditions. In all other general cases, equilibrium existence is not guaranteed, and no analytical approach can be used to find one (Hanson and Martin 1996, Morrow and Skerlos 2011, Gallego and Wang 2014, Li et al. 2019). While including disaggregate demand models in an equilibrium framework comes at the expense of theoretical properties that characterize the existence and uniqueness of equilibrium or ε-equilibrium solutions, the resulting market model is nevertheless valuable to practitioners, since it can provide a detailed description of consumer behavior. Therefore, in this paper, we propose a heuristic approach that allows to study a market where demand is modeled at a disaggregate level.

In the absence of an equilibrium, stable outcomes can still exist if suppliers accept to make suboptimal decisions. In this context, a state where no supplier can increase its profits to more than $1+\varepsilon$ times its current profit by unilaterally changing its own strategy is defined as ε-approximate equilibrium (Daskalakis, Mehta, and Papadimitriou 2007, Christodoulou, Koutsoupias, and Spirakis 2011).

The main contribution of this paper is a model-based algorithmic framework to find approximate equilibrium solutions of oligopolistic markets for which we have a general disaggregate representation of demand. The framework allows to exploit an estimated discrete choice model of demand and include it as such in a model of oligopolistic competition featuring heterogeneous demand, multi-product offer by suppliers and price differentiation. More specifically, the utility functions of the consumers are linearized using simulation and embedded in the supplier optimization problem (Pacheco Paneque et al. 2021), thus accounting for heterogeneity in the population. Competition among firms is modeled explicitly as a non-cooperative game in which all players optimize their own decisions while accounting for the decisions of their competitors. To the best of our knowledge, this is the first methodology which allows to effectively include general discrete choice models of demand, which can bring more behavioral realism, in models of competition by means of simulation. Our work is thus shedding light on a largely unexplored area of research with numerous applications in transportation.

The rest of the paper is organized as follows. Section 2 provides a literature review on oligopolistic competition and identifies the opportunity to enhance existing approaches by modeling demand at a disaggregate level using general discrete choice models. Section 3 outlines the three components of the modeling framework, namely demand, supply and market interactions. Section 4 presents our model-based algorithmic approach to find ε-approximate equilibrium solutions. Section 5 illustrates numerical experiments that validate the algorithmic approach. Finally, Section 6 concludes the paper and provides directions for future research.

2. Literature review

The discipline that studies competition between groups of decision-makers when individual choices jointly determine the outcome is known as game theory. Fudenberg and Tirole (1991) and Osborne and Rubinstein (1994) provide an overview of the principal concepts in game theory. For the purpose of this research, we consider here supply-supply and supply-demand interactions, which fit into the frameworks of the Nash non-cooperative game and of the Stackelberg game, respectively. The Nash game (Nash 1951) considers players that have equal status. A Nash equilibrium solution of the game is a state in which no player can improve its payoff by unilaterally changing their decision. Nisan et al. (2007) describe several algorithmic methods used to find Nash equilibria under various assumptions. The Stackelberg game (von Stackelberg 1934) features two players, identified as leader and follower, both trying to optimize their own objective function. The leader knows the follower's best responses to all the leader's strategies, and will therefore optimize its decisions accordingly. The Stackelberg game can be modeled as an optimization problem having an optimization problem in the constraints (Bracken and McGill 1973), also known as bilevel program (Colson, Marcotte, and Savard 2007).

The theory of competition in non-cooperative games has been extensively used to analyze oligopolistic markets where a small number of firms have non-negligible market power. The first contributions date back to the seminal works by Cournot (1838) and Bertrand (1883), who analyze a market where an homogeneous product is sold to an homogeneous population and where firms compete on quantity and price, respectively. Hotelling (1929) proposes a duopolistic game in which firms decide on the location of production and on the price of the product, while the spatial distribution along a line of the homogeneous and inelastic demand affects the cost of transportation from producers to customers. Gabszewicz and Thisse (1979) consider consumers having identical preferences but variable incomes who make indivisible and mutually exclusive purchases, and conclude that the existence of a Cournot equilibrium requires the continuity in the demand function and that the proof of existence based on fixed-point arguments is based on the quasi-concavity of the profit functions. Murphy, Sherali, and Soyster (1982) propose a mathematical programming approach to find market equilibria in an oligopolistic market supplying an homogeneous product, in which firms must determine their production levels. Assumptions are made on the revenue curves, which must be concave, on the demand curve, which must be continuously differentiable and on the supply curve, which must be convex and continuously differentiable. When these conditions hold, the equilibrium solution can be found by solving the Karush-Kuhn-Tucker conditions for the optimization problems of the firms.

When consumers make a choice from a set of mutually exclusive alternatives, demand can be modeled at the disaggregate level using discrete choice models. Following random utility theory, discrete choice models are probabilistic, since the utility specification cannot capture all the relevant factors that influence choice (Manski 1977). The simplest and most used discrete choice model is the logit model, which has a closed-form expression of the choice probabilities, but relies on the independence from irrelevant alternatives (IIA) assumption. More complex models, such as the nested logit model (Ben-Akiva and Lerman 1985) and the mixed logit model (McFadden and Train 2000), relax the assumptions of the logit model and allow for more realistic substitution patterns, random taste variation and correlation in unobserved factors.

Choice-based optimization models incorporate discrete choice models of consumer behavior into the optimization problem of the supplier. In the literature, applications of choice-based optimization models include revenue management (Andersson 1998, Talluri and Van Ryzin 2004, Vulcano, Van Ryzin, and Chaar 2010) and facility location (Benati and Hansen 2002, Haase 2009). Hanson and Martin (1996) discuss the use of logit models in a profit maximization problem with multiple products and multiple customer segments. The authors show that this problem is generally non-concave and propose a path-following approach that starts from a related concave logit profit function and allows to reach the global optimum of the original non-concave logit profit function. Li et al. (2019) consider optimal pricing under a discrete mixed logit model where demand is decomposed in a finite number of market segments, each with its own set of parameters.

The authors show that the mixed logit profit function is not well-behaved and propose a gradient-descent approach with randomized starting points to find stationary solutions.

In a competitive context, existence conditions for different classes of games under logit and nested logit models are provided by Milgrom and Roberts (1990), Bernstein and Federgruen (2004) and Li and Huh (2011), among others. Aksoy-Pierson, Allon, and Federgruen (2013) consider a market with single-product firms offering differentiated products to customers who can be segmented into homogeneous groups based on observable factors. Choice probabilities of each group are computed with a logit model, and a unique price is offered to all customers. The authors identify conditions on price bounds and segment market shares that guarantee the existence and uniqueness of equilibrium. Gallego and Wang (2014) note that a number of previously established equilibrium existence and uniqueness results for price-competition game with multiple products and logit demand require the price-sensitivity parameters for all the products of the same firm to be identical, an assumption that is rejected by numerous empirical studies. The authors relax this assumption for a market with homogeneous demand and nested logit models, and show that the multi-product pricing problem for each firm can be solved using as a single decision variable the adjusted markup, defined as price minus cost minus the reciprocal of the price sensitivity. Mild conditions are provided that guarantee existence and uniqueness of a Nash equilibrium. Morrow and Skerlos (2011) study numerical approaches to compute equilibrium prices of a market with multi-product offer and homogeneous prices under a general mixed logit model of demand. The authors present necessary stationarity condition and analyze numerical methods to compute equilibrium prices. A case study about a new-vehicle market is used to compare variants of Newton's method and fixed-point iterations and to analyze the effect of the sample set size used for the mixed logit simulation on the variability of choice probabilities and the resulting equilibrium prices. Morrow and Skerlos (2011) state that determining existence or uniqueness of Bertrand-Nash equilibrium prices with general discrete choice models, heterogeneous multi-product firms and heterogeneous consumers is an open problem, but they acknowledge that computational methods are a valid means to find solutions for practical applications. In a dynamic context, Lin and Sibdari (2009) propose a model of dynamic price competition between firms when each firm sells a single product in a market of substitutable products. Using the multinomial logit model to describe the discrete choice of a representative consumer, the authors show the existence of a Nash equilibrium and propose policies to find the equilibrium in case of full and partial information. Similarly, Levin, McGill, and Nediak (2009) consider a stochastic dynamic game where customers are subdivided into market segments and demonstrate the existence of subgame equilibrium solution for each decision period under a number of different assumptions with respect to information and competition, using a generalized choice model of demand.

The surveyed contributions in the field of competition under disaggregate discrete choice models of demand share the common finding that the existence of an equilibrium can only be guaranteed thanks to limiting assumptions on the demand model. Such assumptions are made in order to deal with the complexity of discrete choice formulations while retaining equilibrium existence conditions. We take a complementary stance and present a methodology that is applicable to complex discrete choice models at the expense of pure equilibrium existence conditions. In this paper, we propose a framework that accommodates advanced discrete choice models, such as the mixed logit, with heterogeneous population, multi-product offer by suppliers and price differentiation. A combination of all these features has not been found in the literature. Under these assumptions, there is no guarantee that a pure Nash equilibrium exists. Nevertheless, this should not prevent us from searching for potential market outcomes.

While the most important theoretical concept used to analyze oligopolistic markets is that of equilibrium, it is widely acknowledged in the economic literature that in non-cooperative games the utility maximization behavior is unlikely to hold true for all agents. A suboptimal non-maximizing behavior could be purposely selected by colluding firms to avoid prisoner's dilemma outcomes or could be caused by the hidden cost of discovering better strategies (Stigler 1964, Radner 1980, Rotemberg and Saloner 1986). In the problem we study, we assume that suppliers do not know whether a pure Nash equilibrium exists. Therefore, they are willing to accept ε-approximate equilibrium states where no supplier can increase its payoff to more than $1+\varepsilon$ times its current payoff by unilaterally changing its suboptimal strategy.

For the rest of this work, we concentrate on static competition models and we neglect considerations on capacity and congestion. Methodological tools to deal with congestion effects in the demand response are available in the literature, but ad-hoc computational methods must be investigated to make them operational in an equilibrium context.

3. The modeling framework

3.1. Demand modeling

We consider a market where a number of different products are offered to a population.

The notation is as follows. Let N represent the set of customers (or groups of homogeneous customers), who are assumed to be utility maximizers, and let I indicate the discrete and finite set of alternatives available in the market. Utility functions U_{in} are defined for each customer $n \in N$ and alternative $i \in I$. Each utility function takes into account the socioeconomic characteristics and the tastes of the individual as well as the attributes of the alternative. According to random utility theory (Manski 1977), U_{in} can be decomposed into a systematic component V_{in} which includes all that is observed by the analyst and a random term ε_{in} which captures the uncertainties caused by unobserved attributes and unobserved taste variations. Therefore, the resulting discrete choice models are naturally probabilistic. The probability that customer n chooses alternative i is defined as $P_{in} = \Pr[V_{in} + \varepsilon_{in} = \max_{j\in I}(V_{jn} + \varepsilon_{jn})]$. In order to be able to estimate choice probabilities, assumptions must be made about the distribution of the error term. Popular discrete choice models are the probit and the logit, built upon the assumption of normally distributed and extreme value distributed error terms, respectively.

It is important to note that the choice probabilities of most advanced models currently considered in the choice modeling literature, including mixed logit and probit, must be expressed as integrals and approximated numerically, for instance by using simulation procedures (Train 2009). As a result, while discrete choice models can accurately capture heterogeneous behavior on the demand side at a disaggregate level, their mathematical properties make it difficult to incorporate them in tractable optimization models.

Pacheco Paneque et al. (2021) propose a methodology to obtain choice probabilities by relying on simulation to draw from the distribution of the error term of the utility function. For each customer n and alternative i, a set R of draws are extracted from the known error term distribution, corresponding to different behavioral scenarios. For each scenario $r \in R$, the error term parameter ξ_{inr} is drawn and the utility becomes equal to

$$U_{inr} = V_{in} + \xi_{inr}. \tag{1}$$

Customers then deterministically choose the alternative with the highest utility in each scenario, that is

$$P_{inr} = \begin{cases} 1 & \text{if } U_{inr} = \max_{j\in I} U_{jnr}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Over multiple scenarios, the probability that customer n chooses alternative i is equal to the number of times the alternative is chosen over the number of draws, that is

$$P_{in} = \frac{1}{|R|} \sum_{r\in R} P_{inr}. \tag{3}$$

With a sufficient number of simulation draws, the obtained choice probabilities approximate the analytical formulation, where one exists. Expression (2) can be linearized and inserted as lower-level constraints in a mixed integer optimization model that optimizes supply decisions.

3.2. Supply modeling

Suppliers are modeled as profit maximizers, according to the traditional microeconomic treatment and without loss of generality, but their objective functions could also include indicators other than profit. To optimize their objective function, suppliers make strategic decisions about the availability of their products on the market and the corresponding attributes such as price and quantity. We assume that suppliers choose their strategies according to their knowledge of demand at a disaggregate level. A majority of the existing works on choice-based optimization propose non-linear formulations and estimate consumer choice probabilities with the logit model, whose advantage is the existence of a closed-form expression.

In addition to the notation used in Section 3.1, consider a supplier k participating in the market and let $I_k \subset I$ indicate the subset of alternatives controlled by the supplier. The parameters of the endogenous variables of the discrete choice model are indicated with β, while the exogenous variables and the corresponding deterministic or distributed parameters are grouped in the term q. Exogenous variables are all those which are not affected by the decisions of the optimizing supplier, such as the socioeconomic characteristics of the customers and the attributes of the alternatives $i \notin I_k$. Additionally, let $\mathcal{S}_k$ be the set of strategies that can be selected by the optimizing supplier. Each strategy $s_k \in \mathcal{S}_k$ is composed of a vector of decision variables, which we can separate into the vector p_k of all prices p_{in}, potentially differentiated for each (class of) customer $n \in N$ and alternative $i \in I_k$, and a generic vector X_k of all other decision variables, such as levels of service, which can be alternative-specific, customer-specific or both. At this point, no assumption is made on the strategy set, which could be finite or infinite and could include discrete or continuous decision variables. For the sake of simplicity, in the rest of this work we make the two following assumptions: (i) no limitation exists on the number of customers that can select any given alternative, and (ii) all utility functions are demand-independent, that is, there is no congestion effect. Pacheco Paneque (2020) shows how these assumptions can be relaxed in the case of a single optimizing supplier.

Then, the non-linear optimization problem of supplier k can be written as follows:

$$\max_{s_k=(p_k,X_k)} \quad \pi_{s_k} = \sum_{i\in I_k}\sum_{n\in N} p_{in}P_{in} - c(X_k), \tag{4}$$

$$s.t. \quad P_{in} = \Pr(V_{in}+\varepsilon_{in} \geq V_{jn}+\varepsilon_{jn} \quad \forall j \in I) \qquad \forall i \in I_k, \forall n \in N, \tag{5}$$

$$V_{in} = \beta_{p,in}p_{in} + \beta_{in}X_{in} + q_{in} \qquad \forall i \in I_k, \forall n \in N. \tag{6}$$

The objective function (4) maximizes the expected profit π of supplier k, calculated as the difference between the expected revenues obtained from the sales and the cost of offering the products. Notice that the function is non-convex due to the presence of the choice probabilities, even with a logit model (Hanson and Martin 1996). Constraints (5) are the expressions of the choice probabilities. Constraints (6) define the deterministic part of the utility functions, composed of an exogenous part q_{in} and an endogenous part which depends on the chosen strategy $s_k = (p_k, X_k)$, which links the upper-level problem with the lower-level problem.

The choice probabilities (5) of advanced discrete choice models, such as the mixed logit, cannot be expressed in closed form. In order to integrate such models in a mixed integer optimization problem, we rely on simulation to draw from the error term distribution, deriving the utilities (1) and the choice probabilities (2) (Pacheco Paneque et al. 2021). We additionally define the auxiliary variables $U_{nr} = \max_i U_{inr}$, which represents the expected maximum utility for customer n in scenario r, while the binary decision variables P_{inr} identify the alternative i chosen by each customer n in each scenario r. Now constraints (5-6) can be replaced by constraints (8-12) and the mixed integer optimization problem of supplier k can be written as follows:

$$\max_{s_k=(p_k,X_k)} \quad \pi_{s_k} = \frac{1}{|R|}\sum_{i\in I_k}\sum_{n\in N}\sum_{r\in R} p_{in}P_{inr} - c(X_k), \tag{7}$$

$$s.t. \quad U_{inr} = \beta_{p,in}p_{in} + \beta_{in}X_{in} + q_{in} + \xi_{inr} \qquad \forall i\in I, \forall n\in N, \forall r\in R, \tag{8}$$

$$U_{inr} \le U_{nr} \qquad \forall i\in I, \forall n\in N, \forall r\in R, \tag{9}$$

$$U_{nr} \le U_{inr} + M_{U_{nr}}(1-P_{inr}) \qquad \forall i\in I, \forall n\in N, \forall r\in R, \tag{10}$$

$$\sum_{i\in I} P_{inr} = 1 \qquad \forall n\in N, \forall r\in R, \tag{11}$$

$$P_{inr}\in\{0,1\} \qquad \forall i\in I, \forall n\in N, \forall r\in R. \tag{12}$$

The objective function (7) maximizes the expected profit π of supplier k. The utility functions (8) now include a drawn error term. The utility functions of the alternatives $i \notin I_k$ are parameters, since the decisions of all suppliers but k are assumed to be given. In (9-12) we use big-M constraints to ensure that in each behavioral scenario customers deterministically choose the alternative yielding the highest utility. Notice that an optimal solution might be selected so that the optimizing supplier receives marginal profit from one customer in one specific scenario where two alternatives yield the same utility. In that case, the model would automatically assign the customer to the alternative that maximizes the objective function. Experimentally, we observe that the effect of one arbitrary tie-breaking procedure out of $|N \cdot R|$ scenarios is very small, and tends to zero as the number of draws increases.

3.3. Market modeling

Let us consider an oligopolistic market where demand is modeled as in Section 3.1 and supply as in Section 3.2. Because of imperfect competition, the payoff of each supplier is a function of both the decisions of the customers and the strategies of the competitors. All suppliers simultaneously solve a choice-based optimization problem, resulting in a non-cooperative game for which we search for Nash equilibrium solutions.

The consequence of using demand functions based on disaggregate choice models, which are highly non-linear and non-convex, is that there is no guarantee of existence or uniqueness of a pure strategy Nash equilibrium for the problem, and it is not possible to rely on first-order conditions to find an equilibrium solution. As a consequence, here we focus on other methods to find ε-approximate equilibrium solutions, with ε being a value sufficiently small to justify the assumption of non-deviation from a state of stability. Notice that, in general cases, the existence of a ε-equilibrium solution cannot be proved for any given ε.

The fixed-point iteration algorithm has been commonly used as a numerical procedure to calculate Nash equilibria of simultaneous games. In transportation, examples include Fisk (1984) and Adler (2001), among others. Starting from an initial market configuration, best-response problems are solved for all players in a sequential manner, until a solution already reached in one of the previous iterations is repeated. This solution method is attractive from a computational perspective, since the complexity of the algorithm is equivalent to the complexity of the choice-based optimization model presented in Section 3.2. However, when best-response problems are non-convex, this method can only be used as a heuristic, since the algorithm could iterate cyclically over a set of strategies without the guarantee that a stable ε-equilibrium outcome is reached.

If we consider a setting in which suppliers have finite strategy sets and utility functions are linearized as proposed by Pacheco Paneque et al. (2021), we observe that the fixed-point iteration algorithm can be expressed in the form of a mixed integer optimization model which solves the simultaneous game with a one-step approach by considering only two iterations of the algorithm. Specifically, we define as distance the non-negative value measuring the sum of the profit differences between an initial unknown solution and each player's simultaneous best response to the initial solution. The objective of the optimization model is to minimize such distance, while satisfying profit maximization constraints for the suppliers and utility maximization constraints for the consumers. If we start from an equilibrium solution of the problem, the sum of the profit differences between the initial and the best-response configurations is equal to 0. Conversely, if we do not start from an equilibrium solution, the distance is greater than 0, since at least one of the suppliers changes its strategy to improve its profit, and a near-equilibrium solution is found. Other metrics, such as the deviation in supply strategies, could be utilized together with profits to measure the deviation between the initial and the best-response configurations (Bertsimas, Gupta, and Paschalidis 2015).

To formalize the mixed integer optimization model, let K represent the set of suppliers, each controlling a subset of the alternatives that are available to the customers. We impose that $\cup_{k\in K} I_k \subset I$, in order not to have a captive market and allow customers to leave it without purchasing. Each supplier $k \in K$ has a finite set of strategies $\mathcal{S}_k$ from which to choose. In the presence of variables such as prices, which are usually modeled as continuous variables, we assume that a finite subset can be derived. We define the vector of the prices p_s and of the other decisions X_s of supplier k playing strategy $s \in \mathcal{S}_k$. Additionally, let $s_{K\setminus\{k\}}$ be the observed strategies chosen by all suppliers other than k. If we define as π_s the payoff obtained by supplier k when choosing strategy s, then in order to find a Nash equilibrium solution we need to verify that

$$\pi_k^{max} = \pi_s = \max_{s\in\mathcal{S}_k} \pi_s(s, s_{K\setminus\{k\}}) \qquad \forall k \in K. \tag{13}$$

We define the binary decision variables x_s, which are equal to 1 if strategy $s \in \mathcal{S}_k$ is the best response of supplier k to the initial configuration and 0 otherwise. Finally, the superscripts $'$ and $''$ are used to indicate the variables of the initial configuration and of the best response configurations, respectively.

Then, the fixed-point optimization model with linearized choice probabilities can be written as follows:

$$\min \sum_{k \in K} (\pi_k^{''} - \pi_k^{'}), \tag{14}$$

s.t. Initial configuration:

$$U_{inr}^{'} = \beta_{p,in} p_{in}^{'} + \beta_{in} X_{in}^{'} + q_{in} + \xi_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{15}$$

$$U_{inr}^{'} \leq U_{nr}^{'} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{16}$$

$$U_{nr}^{'} \leq U_{inr}^{'} + M_{U_{nr}}(1 - P_{inr}^{'}) \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{17}$$

$$\sum_{i \in I} P_{inr}^{'} = 1 \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{18}$$

$$\pi_k^{'} = \frac{1}{|R|} \sum_{i \in I_k} \sum_{n \in N} \sum_{r \in R} p_{in}^{'} P_{inr}^{'} - c(X^{'}) \qquad \forall k \in K, \tag{19}$$

Final configuration:

$$U_{inrs}^{''} = \beta_{p,in} p_{ins}^{''} + \beta_{in} X_{ins}^{''} + q_{in} + \xi_{inr} \qquad \forall i \in I_k, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{20}$$

$$U_{inrs}^{''} = U_{inr}^{'} \qquad \forall i \notin I_k, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{21}$$

$$U_{inrs}^{''} \leq U_{nrs}^{''} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{22}$$

$$U_{nrs}^{''} \leq U_{inrs}^{''} + M_{U_{nr}}(1 - P_{inrs}^{''}) \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{23}$$

$$\sum_{i \in I} P_{inrs}^{''} = 1 \qquad \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{24}$$

Best response constraints:

$$\pi_s = \frac{1}{|R|} \sum_{i \in I_k} \sum_{n \in N} \sum_{r \in R} p_{ins}^{''} P_{inrs}^{''} - c(X^{''}) \qquad \forall s \in \mathcal{S}_k, \forall k \in K, \tag{25}$$

$$\pi_s \leq \pi_k^{''} \qquad \forall s \in \mathcal{S}_k, \forall k \in K, \tag{26}$$

$$\pi_k^{''} \leq \pi_s + M(1 - x_s) \qquad \forall s \in \mathcal{S}_k, \forall k \in K, \tag{27}$$

$$\sum_{s \in \mathcal{S}_k} x_s = 1 \qquad \forall k \in K, \tag{28}$$

$$P_{inr}^{'}, P_{inrs}^{''} \in \{0, 1\} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{29}$$

$$x_s \in \{0, 1\} \qquad \forall s \in \mathcal{S}_k, \forall k \in K. \tag{30}$$

The objective function (14) minimizes the sum over all the suppliers of the difference between the final and the initial profit. Constraints (15-18) define the utilities and impose that customers choose the alternative with the highest utility in the initial configuration. Constraints (19) calculate the profits in the initial configuration. Constraints (20-24) impose the utility maximization principle in the best response configurations. Here, utilities are evaluated for all strategies of all suppliers. In each strategic scenario, the decisions of the optimizing supplier only affect the utility of its alternatives (20), while the utilities of the competitors' alternatives remain unchanged with respect to the initial configuration (21). Finally, constraints (25-28) state that each supplier selects the best response strategy to the initial configuration.

Starting from an initial unknown configuration, the model requires a number of strategic scenarios to be solved that is equal to $\sum_{k\in K} |\mathcal{S}_k|$. Since the optimization model (14-30) is highly combinatorial, it cannot be used as a stand-alone method, since it is not suitable to solve equilibrium problems with continuous variables or with large discrete strategy spaces. However, the optimal solution of the model, found on restricted strategy spaces, can be used as a candidate approximate equilibrium solution to be verified on the original strategy spaces. An algorithmic framework which combines a heuristic search and the mixed integer optimization model (14-30) is proposed in Section 4.

4. Algorithmic solution

In this section, we propose a model-based algorithmic approach to find approximate equilibrium solutions of the problem described in Section 3. The algorithm consists of three blocks: (i) identify candidate equilibrium solutions or regions in a fast and efficient way; (ii) use the fixed-point optimization model (14-30) on restricted strategy sets to find subgame near-equilibria; (iii) verify if best response conditions are satisfied for the initial problem and, if they are not, add strategies to the restricted problem. The approximate equilibrium solutions of the problem can provide relevant insights for decision-making or regulatory purposes. The pseudocode is presented in Algorithm 1.

4.1. Heuristic reduction of the search space

We start from an equilibrium problem with non-convex profit functions, which cannot be solved using derivative-based methods. We first reduce the search space heuristically, since the decision space of each supplier is, in principle, very large. This is particularly true when the strategy chosen by the supplier is the result of multiple interconnected decisions. Notice that often there exist constraints in real-life markets which define relationships between different products, such as fare classes in airlines or discount levels on off-peak trains in railways. Including such problem-specific constraints can help reducing the search space and circumscribing potential equilibrium regions. On the demand side, using the classic non-linear formulation of the discrete choice probabilities (model 4-6) when solving the best response problem is faster than using the simulation-based linear formulation (model 7-12) only for simple choice models such as the logit. However, the computational performance of the non-linear formulation rapidly deteriorates in case of more complex choice models or discrete supply decisions, since derivative-based approaches cannot be used to find global optima. Notwithstanding this limitation, at this stage any of the two formulations as well as any other heuristic that finds near-optimal solutions of the choice-based optimization model can be used.

Algorithm 1:

Input : A set I of alternatives
A set K of suppliers, each $k \in K$ controlling alternatives $I_k \subset I$
A set $\mathcal{S}_k$ of strategies for each $k \in K$
A heterogeneous population
An estimated discrete choice model of consumer behavior

Output: A list E of approximate equilibrium solutions

$E \leftarrow \emptyset$
repeat
    Initialize $\varepsilon = \varepsilon_{start}$
    Assign a strategy $s_k \in \mathcal{S}_k$ to each competitor $k \in K$
    to create an initial market configuration $\mathbb{S} = \cup_{k \in K} s_k$
    repeat
        for $k \in K$ **do**
            Solve best response problem (7-12) for supplier k and obtain s_k^{new}
            Update solution $\mathbb{S}$ by assigning $s_k \leftarrow s_k^{new}$
    until $\mathbb{S}$ *is a previously visited solution or a stopping criterion is satisfied*
    Derive restricted strategy sets $S_k \subset \mathcal{S}_k$ from the results of the previous block
    repeat
        Solve fixed-point (FP) problem (14-30) for strategy sets S_k and
        obtain subgame near-equilibrium solution $\mathbb{S}^{FP} = \cup_{k \in K} s_k^{FP}$ and profits π_k^{FP}
        for $k \in K$ **do**
            Solve best response (BR) problem (7-12) for supplier k and
            obtain s_k^{BR} and π_k^{BR}
            if $\pi_k^{BR} > (1+\varepsilon)\pi_k^{FP}$ **then**
                Add best response strategy s_k^{BR} to the restricted strategy set S_k
        if *ε-equilibrium solution not found for i iterations* **then**
            Modify restricted strategy set S_k
            Increase ε
    until $\pi_k^{BR} \leq (1+\varepsilon)\pi_k^{FP}$ *for all* $k \in K$ *and* $\mathbb{S}^{FP}$ *significantly different from all* $\mathbb{S} \in E$
    $\mathbb{S}^{FP}$ is an ε-equilibrium solution of the problem
    Add $\mathbb{S}^{FP}$ to list E
until *stopping criterion is satisfied*

In the experiments described in Section 5, we initially solve the competitive game in a sequential manner (lines 4-9), following the approach used by Adler, Pels, and Nash (2010), among others. More specifically, we define an initial feasible market configuration (line 4) and we rely on the linear formulation (7-12) to solve best response problems based on the updated market conditions (lines 5-8) until we reach a state of the market that was already visited in one of the previous iterations (line 9). If no solution is repeated, the sequential approach is stopped when a given number of iterations is reached or when other appropriate criteria, such as deviation between consecutive iterations, are satisfied. Experimentally, we observe that the algorithm generally converges to a bounded region of the solution space, reaching a cyclic equilibrium in which the solution iterates over a finite set of best response strategies.

4.2. Exact solution of the restricted problem

The first block of the algorithm identifies a candidate equilibrium region with heuristic methods. The insights obtained from the first block, for instance in terms of bounds of the decision variables, allow us to define restricted strategy sets S_k for all competitors k (line 10). Next, we use the fixed-point optimization model (14-30) to find a subgame solution which is a candidate approximate equilibrium of the game. The size of the problems that can be solved exactly with this model is limited, since the model is combinatorial on the sets I, N, R and S. For this reason, using restricted strategy sets allows to produce tighter bounds on the utility functions of the customers. Within a limited range of supply decisions, we observe that small changes in the strategies produce small changes in the utility functions of the customers, rarely affecting the ordinal ranking of the utilities of the alternatives in their choice set.

The simulation of the error term of the utility function and the consequent use of binary variables to model choices in each scenario r make it possible to precompute the choices of customer n by comparing the lower and upper bounds of the utilities. In particular,

$$LB(U_{inr}) > \max_{j \in I : j \neq i} UB(U_{jnr}) \implies \begin{cases} P_{inr} = 1 \\ P_{jnr} = 0 \quad \forall j \in I, j \neq i. \end{cases} \tag{31}$$

If this condition is verified, customer n is captive to alternative i in scenario r. Therefore, this scenario can be removed from the optimization model. Using restricted strategy sets with tight bounds can substantially reduce the size of the fixed-point optimization model to be solved, since the optimal strategies around the equilibrium region are ultimately determined by a subset of undecided customers who select one among a subset of alternatives.

4.3. Verification of best response conditions

The solution obtained by solving the restricted problem with the fixed-point model (line 12) to optimality must be verified on the original game by solving best response problems for all competitors (lines 13-14). If no supplier can increase its profit by more than ε by solving its best response problem, that is,

$$\pi_k^{BR} \leq (1+\varepsilon)\pi_k^{FP} \qquad \forall k \in K, \tag{32}$$

then the optimal solution of the subgame is accepted as ε-equilibrium solution of the game (lines 20-21). Contrarily, if at least one competitor can increase its profits by more than ε, the optimal solution of the subgame is not accepted as game equilibrium and the best response strategy is added to the restricted set (line 16). The fixed-point model is then solved again with updated restricted strategy sets, following a column-generation-like approach. During the execution of the algorithm, if no ε-equilibrium solution is found within a given time or number of iterations, diversification strategies such as modifying the restricted strategy sets can be used. Furthermore, we allow the threshold value ε to be increased within the algorithm (lines 17-19).

The algorithm combines a heuristic exploration of the solution space with the use of an exact method to solve subproblems. After finding an ε-equilibrium solution, the algorithm is restarted with a different initial solution until either a minimum number of different approximate equilibrium solutions (not necessarily with the same value of ε) are found or a time limit is reached (line 22). Similarity conditions can be implemented in the algorithm in order to avoid finding approximate equilibrium solutions that are too similar one another (line 20). Notice that, in the presence of continuous variables, there are either zero or infinitely many ε-equilibrium solutions for any given ε, since the neighborhood of an ε-equilibrium solution is also an ε-equilibrium. However, in an empirical application such as pricing, it is of limited use to find two or more ε-equilibrium solutions where all prices differ by only a few cents.

It is also important to acknowledge the difference between the approximate equilibrium problem and the supplier optimization problem. In particular, when we take the point of view of the suppliers, an ε_1-approximate equilibrium solution could Pareto-dominate ε_2-approximate equilibrium solution even when $\varepsilon_2 < \varepsilon_1$. Provided that ε_1 is a reasonable value for which the non-deviation assumption holds for all suppliers, we consider both solutions as valid approximate equilibrium solutions that could inform the final strategic decision of each supplier, which we assume to be taken exogenously to our model.

Finally, different approximate equilibria can be compared in terms of stability, dominance, tacit collusion or social welfare, from the point of view of both the suppliers and the market regulator.

5. Numerical experiments

In this section, we present numerical experiments that illustrate the applicability of the model-based algorithmic framework outlined in Section 4.

In Section 5.1, we take a simple case study presented by Lin and Sibdari (2009). First, we show that our algorithmic approach can approximate the unique Nash equilibrium solution for the case of a logit model with homogeneous demand. Then, we modify the demand data to show the effects of including observed or unobserved heterogeneity in the demand model.

Next, we tackle two case studies of competitive markets in transportation for which real, non-trivial, disaggregate choice models are published in the literature. In Section 5.2, we look at an urban parking choice case study, for which a mixed logit model estimation is taken from Ibeas et al. (2014). In Section 5.3, we analyze an intercity mode and departure time choice case study in the context of high-speed rail competition, for which a nested logit model estimation is derived from Cascetta and Coppola (2012). The latter case study features heterogeneous demand, multi-product offer by suppliers and price differentiation.

5.1. Logit with unique Nash equilibrium

Lin and Sibdari (2009) provide numerical experiments for a duopoly where each firm offers a single product to a set N of homogeneous customers, for whom the systematic components of the utility functions are defined as follows: $V_0 = 0$ (opt-out alternative), $V_1 = 5 - 0.1p_1$ (firm 1), $V_2 = 4 - 0.1p_2$ (firm 2). A multinomial logit model is used to model the discrete choice of customers. We reproduce the results for a single-period game with unlimited inventory. Under these conditions, there exists a unique Nash equilibrium for the game (Bernstein and Federgruen 2004), which is obtained analytically using the closed-form expression of the logit probabilities. The equilibrium prices are $p_1^* = 23.02$ and $p_2^* = 16.57$, the choice probabilities are $P_0^* = 0.038$, $P_1^* = 0.566$ and $P_2^* = 0.396$, and the revenues are $\pi_1^* = 13.02|N|$ and $\pi_2^* = 6.57|N|$.

5.1.1. Approximation of choice probabilities through simulation. First, we show that the simulation technique proposed by Pacheco Paneque et al. (2021) approximates the logit choice probabilities obtained analytically. We consider the given equilibrium prices and we compute the choice probabilities using (3), that is, we draw $|R|$ times from the error term Gumbel distribution to simulate the value of the utility function (1) and we aggregate the deterministic choices made in each scenario. We replicate this experiment 100 times for $|R| = 10$, 50, 100, 500, 1000, 5000, 10000. Figure 1 presents the boxplots of the expected choice probabilities for the three alternatives. The results show that, as the number of draws becomes larger, the mean of the expected choice probability for the three alternatives tends to the analytical value, while the variance decreases considerably. This type of test can inform the choice of the value of $|R|$, which in practical applications is generally a tradeoff between accuracy and computability.

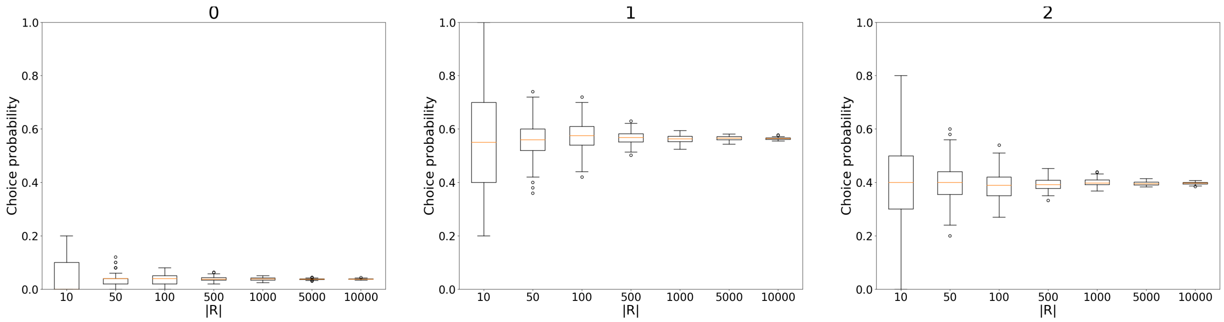


Figure 1 Boxplot of the choice probabilities obtained for the equilibrium solution by Lin and Sibdari (2009) when simulating the value of the utility function using the methodology proposed by Pacheco Paneque et al. (2021) (100 independent replications).

Equilibrium		Prices		Profits		Market shares		
#	ε	1	2	1	2	1	2	3
1	0.9%	21.77	17.63	12.89	6.54	0.037	0.592	0.371
2	0.7%	21.83	16.57	12.40	6.56	0.036	0.568	0.396
3	0.7%	22.56	17.94	13.06	6.83	0.040	0.579	0.381
4	0.7%	23.38	17.96	13.07	7.17	0.042	0.559	0.399
5	0.7%	21.90	16.65	12.44	6.59	0.036	0.568	0.396
Analytical	0	23.02	16.57	13.02	6.57	0.038	0.566	0.396

Table 1 List of ε-equilibrium solutions and comparison with the analytical solution.

5.1.2. Approximation of equilibrium solution. Secondly, we run Algorithm 1 to find approximate equilibrium solutions for the duopoly. We stop the algorithm when five different solutions are found and we use $|R| = 1000$. The solutions are presented in Table 1 and show that our methodology successfully finds ε-equilibrium solutions that approximate the unique Nash equilibrium solution found analytically by Lin and Sibdari (2009).

5.1.3. Variability due to simulation. Due to simulation, the difference ε between the profits obtained for a given solution and the best-response profits depends on the sample set. Morrow and Skerlos (2011) find the variability of equilibrium solutions across samples to be dependent on the sample set size, a result that is consistent with our analysis of the simulated choice probabilities. Once an approximate equilibrium solution is found, a sensitivity analysis can be performed by taking the approximate equilibrium prices and computing the market shares and profits for 100 independent replications of the error term draws. Then, with the same draws we solve best-response problems for the two suppliers to find the potential increase in profits. Figure 2 shows the boxplots of the distribution of ε for each of the five approximate equilibrium solutions. We can observe that the median value of ε ranges between 0.9% (solution 4) and 1.6% (solution 1) and in none of the 100 independent replications it goes above 4.0% for any of the five solutions. This ex-post verification on the approximate equilibrium solution allows to confirm the initial choice of the value of $|R|$, or to reject it. In the latter case, the value $|R|$ chosen in Section 5.1.1 should be increased and the algorithm rerun.

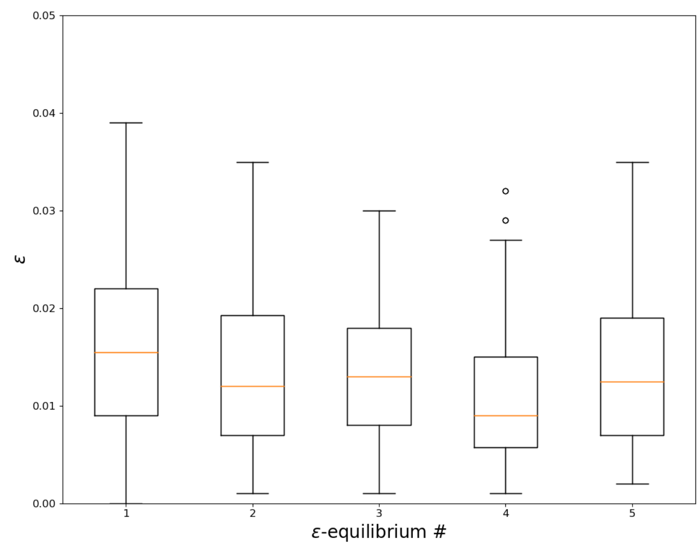


Figure 2 **Sensitivity analysis on the ε-equilibrium solutions obtained using Algorithm 1 (100 independent replications).**

5.1.4. Accounting for observed and unobserved heterogeneity. There are two operational ways to account for heterogeneity when estimating utility parameters of discrete choice models. Observed heterogeneity models define segments in the population based on socioeconomic characteristics such as income, age or place of residence, and assume that all individuals in the same segment share the same set of parameters to be estimated from data. Let us imagine to re-estimate the demand model used by Lin and Sibdari (2009) on the same dataset by defining three consumer segments, each covering a third of the population. Let us assume that the resulting systematic components of the utility functions look as follows: $V_{01} = V_{02} = V_{03} = 0$ (opt-out alternative), $V_{11} = 5 - 0.1p_1$ (firm 1, segment 1), $V_{12} = 7 - 0.1p_1$ (firm 1, segment 2), $V_{13} = 3 - 0.1p_1$ (firm 1, segment 3), $V_{21} = 4 - 0.1p_1$ (firm 2, segment 1), $V_{22} = 3 - 0.1p_1$ (firm 2, segment 2), $V_{23} = 5 - 0.1p_1$ (firm 2, segment 3). Notice that individuals in segment 1 have the same utility function of the homogeneous population of the base scenario, while individuals in segments 2 and 3 deviate symmetrically from the average behavior by exhibiting higher-than-average preferences for product 1 and product 2, respectively. If the suppliers can price differentiate across the three segments, we obtain three independent markets which can be solved analytically as above. Contrarily, if price differentiation does not follow the demand estimation segments or is not applied at all, analytical methods are generally not applicable and existence of equilibrium is not guaranteed (Aksoy-Pierson, Allon, and Federgruen 2013). Algorithm 1 can be used in all these cases. We test the case where a single price is defined for each alternative, and we obtain the results shown in Table 2.

Equilibrium		Prices		Profits		Market shares		
#	ε	1	2	1	2	1	2	3
1	0.8%	33.85	26.04	16.92	11.02	0.077	0.500	0.423
2	0.8%	34.15	27.14	17.36	11.08	0.083	0.508	0.408
3	0.6%	34.14	26.15	17.01	11.12	0.077	0.498	0.425
4	0.7%	34.14	26.18	17.07	11.08	0.077	0.500	0.423
5	0.9%	34.19	27.15	17.38	11.09	0.083	0.508	0.408

Table 2 **List of approximate equilibrium solutions (observed heterogeneity).**

Equilibrium		Prices		Profits		Market shares		
#	ε	1	2	1	2			
1	0.9%	33.69	25.68	17.55	9.86	0.095	0.521	0.384
2	0.7%	33.46	25.68	17.53	9.79	0.095	0.524	0.381
3	0.7%	33.86	25.68	17.61	9.86	0.096	0.520	0.384
4	0.7%	33.23	25.68	17.48	9.76	0.094	0.526	0.380
5	0.7%	33.94	25.43	17.41	9.97	0.095	0.513	0.392

Table 3 **List of approximate equilibrium solutions (unobserved heterogeneity).**

Unobserved heterogeneity models assume that parameters are randomly distributed in the population. Let us now imagine that the demand model used by Lin and Sibdari (2009) is re-estimated on the same dataset by allowing for randomly distributed alternative-specific parameters. Let us assume that the resulting systematic components of the utility functions look as follows: $V_0 = 0$ (opt-out alternative), $V_1 = \beta_1 - 0.1p_1$ (firm 1), $V_2 = \beta_2 - 0.1p_2$ (firm 2), where β_1 and β_2 are normally distributed parameters such that $\beta_1 =\sim \mathcal{N}(5,2)$ and $\beta_2 =\sim \mathcal{N}(4,1)$. Applying Algorithm 1 again, we obtain the results shown in Table 3.

These stylized examples show how assumptions on demand heterogeneity can lead to big differences in equilibrium prices, market shares and profits. When estimating discrete choice models from real data, specification tests such as the likelihood-ratio test are used to determine whether an unrestricted and a restricted model estimated from the same dataset are significantly different. While aggregation or simplifying assumptions on the demand side lead to tractable equilibrium problems solvable with exact methods, we show that our methodology makes it possible to use more complex and precise disaggregate choice models in an equilibrium context. Ultimately, the trade-off between the increased realism of the demand model and the complexity of the resulting market equilibrium problem that leads to approximate results must be assessed on a case-by-case basis by the analyst.

5.2. Parking choice

Ibeas et al. (2014) use a mixed logit model to study car driver's behavior when choosing among three different parking alternatives available in a small Spanish town. Recall that the choice probabilities of the mixed logit model do not have a closed form and must be estimated using numerical integration or approximation by simulation (McFadden and Train 2000). While simulation constitutes an additional

Alternative	FSP	PSP	PUP
Access time to parking (min)	10	10	5
Time to final destination (min)	10	10	10
Variable costs of operation (€)	0	0.25	0.50
Price (€)	0	p_{PSP}	p_{PUP}

Table 4 Attributes of the alternatives for the parking instance.

computational burden in non-linear formulations, it adapts well to our modeling framework, since we can apply the same technique used to linearize the utility expression (1), that is, drawing from the known distribution of the error term. By doing so, we obtain

$$U_{inr} = \beta_{p,inr} p_{in} + q_{inr} + \xi_{inr}, \tag{33}$$

where $\beta_{p,inr}$ and q_{inr} can include distributed random parameters that vary for each draw r. Randomly distributed parameters allow to better capture heterogeneity in the population and, therefore, to better inform pricing strategies. For this reason, the mixed logit is among the most common discrete choice models in transport applications (Train 2009).

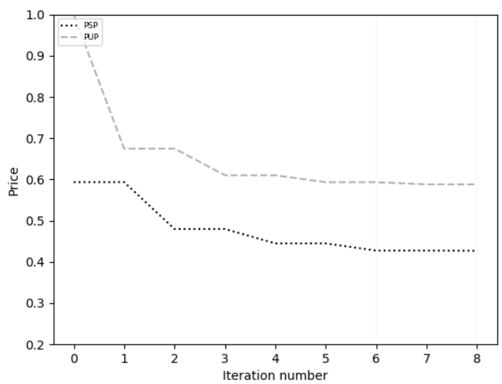
5.2.1. Instance description. In Ibeas et al. (2014), the three parking alternatives are on-street parking (FSP), paid on-street parking (PSP) and paid underground parking (PUP). In our test scenario, we assume that the two paid parking options are owned by two different suppliers, both aiming at maximizing profits, while free on-street parking is considered as the opt-out option. The two suppliers compete on price and we assume that all customers must pay the same price for the same service. Furthermore, we assume that there are variable costs that depend on the demand and are equal to 0.25 € per user for the PSP alternative and 0.50 € per user for the PUP alternative. Table 4 presents all attributes of the alternatives.

The explanatory variables considered in the discrete choice model estimated by Ibeas et al. (2014) include the following socioeconomic characteristics: trip origin (if outside town, it affects the utility of free street parking), age of the vehicle (if less than three years old, it affects the utility of paid underground parking), income of the driver (if low, it affects the utility of paid alternatives), area of residency of the driver (if in town, it affects the utility of paid alternatives). Additionally, the following attributes of the alternatives are considered: access time to destination, access time to parking and parking fee. For the latter two continuous variables, the corresponding coefficients are normally distributed in the utility function. Table 5 illustrates the parameters of the discrete choice model derived from Ibeas et al. (2014).

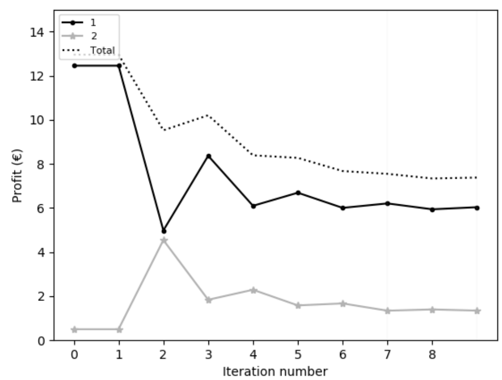
The size of the set N of heterogeneous customers is fixed and is equal to 50.

β	Value
ASC_{FSP}	0.0
ASC_{PSP}	32.0
ASC_{PUP}	34.0
Fee (€) c	$\sim \mathcal{N}(-32.328, 14.168)$
Fee PSP - low income (€)	-10.995
Fee PUP - low income (€)	-13.729
Fee PSP - resident (€)	-11.440
Fee PUP - resident (€)	-10.668
Access time to parking (min)	$\sim \mathcal{N}(-0.788, 1.06)$
Access time to destination (min)	-0.612
Age of vehicle (1/0)	4.037
Origin (1/0)	-5.762

Table 5 Model parameters derived from Ibeas et al. (2014).



(a) Prices



(b) Profits

Figure 3 Evolution of prices and profits for one run of the fixed-point iteration algorithm in the parking case study.

5.2.2. Notes on algorithm implementation. Following Algorithm 1, a strategy is initially assigned to each competitor. In this case, the strategies of suppliers 1 and 2 correspond to a single price decision on PSP and PUP, respectively. Then, best response problems are solved iteratively until a solution is repeated or a near-equilibrium region is found. Figure 3 shows the evolution of prices and profits during the iterative procedure for a given initial market configuration. Here, we observe that the fixed-point iteration method quickly converges to a well-defined area of the solution space. We conjecture that this is due to the relative simplicity of the supply strategic decision in this specific experiment.

Next, restricted sets are generated which include strategies in the candidate equilibrium region of both suppliers. Table 6 illustrate the column-generation-like technique through an example, when $\varepsilon = 0.015$. The first subgame includes 5 strategies for each supplier, derived by considering evenly spaced prices between lower and upper bounds. The fixed-point optimization model (14-30) yields a subgame optimal solution in which $PSP = 0.423$ and $PUP = 0.584$. Then, best response problems are solved on the original strategy sets, finding that the optimal solution of the subgame is not an ε-equilibrium of the game, since supplier 1 can increase its profit by 1.51% by setting $PSP = 0.432$.

Alternative	$\|S\|$	Bounds		Subgame solution		Best responses		ε
		LB	UB	Price	Profit	Price	Profit	
PSP	5	0.423	0.423	0.423	5.799	0.432	5.886	0.015
PUP	5	0.583	0.584	0.584	1.327	0.612	1.335	0.006
PSP	6	0.423	0.432	0.432	5.892	0.432	5.892	0.000
PUP	6	0.583	0.612	0.584	1.416	0.593	1.471	0.039
...	...	...	...	...	...	...	...	...
PSP	10	0.423	0.447	0.432	6.117	0.430	6.186	0.011
PUP	10	0.583	0.612	0.593	1.449	0.593	1.469	0.014

Table 6 Illustration of the algorithm to find an ε-equilibrium solution for the parking choice case study, with $\varepsilon = 0.015$.

The best response strategies are then added to the restricted sets, and the subgame will therefore have 6 strategies for each supplier in the next iteration. After a number of iterations, a game ε-equilibrium solution is found, with $PSP = 0.432$ and $PUP = 0.593$.

The algorithm is restarted until a predefined number of approximate equilibrium solutions is found, possibly with different values of ε, to verify whether different initial states lead to significantly different equilibrium regions.

5.2.3. Results. We run two scenarios for this case study. In both cases, we stop the algorithm when five different approximate equilibrium solutions are found.

In the base scenario, we assume that the price paid by a customer for the chosen alternative is equal to the revenue earned by the supplier that manages the alternative. Table 7 lists the approximate equilibrium solutions for the base scenario. We observe that all solutions are very similar. Indeed, in this case study, solving the game with the fixed-point iteration method is sufficient to provide market insights, and the following steps of the algorithmic framework provide little added value. This is generally not true for more complex markets, as we will show in Section 5.3. Table 8 shows that, at approximate equilibrium prices, town residents would strongly favor paid on-street parking, while non-residents would be almost equally split between on-street and underground parking.

In the second scenario, we assume that the municipality decides to subsidize city residents by providing them with a 25% discount on the fare of the underground parking facility. This means that, when a city resident choose the PUP alternative, the PUP supplier earns a revenue which is equal to the approximate equilibrium price p_{PUP}, but the user only pays $0.75 p_{PUP}$. By using a disaggregate demand model, both suppliers can take into account demand heterogeneity and optimally adapt their strategies to the new policy. Table 9 lists the new approximate equilibrium solutions. As expected, the strategic behavior of the two competitors changes as a result of the updated utility functions of a subset of customers. We can compare the two scenarios in terms of aggregate and disaggregate market shares through Tables 8 and 10. On the one hand, discounted PUP becomes more desirable for city residents, therefore the PUP supplier can afford increasing the price while capturing a much

Equilibrium		Prices		Profits	
#	ε	PSP	PUP	PSP	PUP
1	1.2%	0.428	0.588	5.95	1.40
2	1.5%	0.424	0.584	5.86	1.32
3	1.4%	0.432	0.593	6.12	1.45
4	1.4%	0.432	0.588	5.96	1.45
5	1.2%	0.434	0.590	6.01	1.49

Table 7 List of approximate equilibrium solutions of the base scenario of the parking choice case study.

#	Market shares			Market shares residents			Market shares non-residents		
	FSP	PSP	PUP	FSP	PSP	PUP	FSP	PSP	PUP
1	0.013	0.666	0.321	0.020	0.783	0.197	0.004	0.518	0.478
2	0.012	0.675	0.313	0.019	0.790	0.191	0.004	0.529	0.467
3	0.015	0.672	0.313	0.023	0.786	0.191	0.006	0.527	0.467
4	0.015	0.652	0.332	0.023	0.773	0.204	0.006	0.499	0.496
5	0.016	0.652	0.332	0.023	0.773	0.204	0.006	0.499	0.496

Table 8 Aggregate and disaggregate market shares for the approximate equilibrium solutions of the base scenario of the parking case study.

Equilibrium		Prices		Profits	
#	ε	PSP	PUP	PSP	PUP
1	3.3%	0.385	0.611	3.86	2.37
2	4.1%	0.382	0.630	4.19	2.35
3	3.1%	0.386	0.628	4.24	2.38
4	5.0%	0.398	0.615	3.81	2.77
5	3.3%	0.400	0.613	3.84	2.75

Table 9 List of approximate equilibrium solutions of the PUP discount scenario of the parking choice case study.

#	Market shares			Market shares residents			Market shares non-residents		
	FSP	PSP	PUP	FSP	PSP	PUP	FSP	PSP	PUP
1	0.003	0.570	0.427	0.002	0.435	0.563	0.004	0.741	0.255
2	0.003	0.636	0.361	0.002	0.521	0.477	0.004	0.782	0.214
3	0.003	0.624	0.373	0.002	0.510	0.489	0.004	0.771	0.226
4	0.004	0.514	0.482	0.004	0.361	0.634	0.004	0.708	0.288
5	0.004	0.508	0.488	0.004	0.360	0.636	0.004	0.697	0.299

Table 10 Aggregate and disaggregate market shares for the approximate equilibrium solutions of the PUP discount scenario of the parking case study.

larger segment market share than before (47-64% versus 19-21% in the base scenario). On the other hand, the PSP supplier must decrease its price in order to make profits by targeting the non-residents, for whom no discount is available.

To conclude, with this simple example we show that it is possible to include an advanced discrete choice model that does not have a closed-form expression of the choice probabilities in an equilibrium problem. Indeed, the strength of the linearized choice-based optimization model is that it overcomes the issue of having normally distributed parameters in the mixed logit utility specification by relying on simulation. In the next case study, we analyze a market with more alternatives and more complex strategic problems on the supply side, which justifies the need of a scalable model-based algorithmic framework.

5.3. Schedule-based high-speed rail competition

The ongoing liberalization of domestic and international rail passenger services in Europe led to the creation of new competitive markets on public transport routes which were traditionally covered by a monopolistic operator. A plethora of contributions exist that forecast and analyze competitive high-speed rail market in Italy (Ben-Akiva et al. 2010, Cascetta and Coppola 2012, Valeri 2013, Capurso, Hess, and Dekker 2019, Cascetta et al. 2020), Sweden (Broman and Eliasson 2019) and Germany (Ait Ali and Eliasson 2019), among others. In some of these cases and in many other high-demand international routes, rail operators also compete with intercity bus operators (Grimaldi, Augustin, and Beria 2017, Fageda and Sansano 2018) and low-cost or legacy airlines (Adler, Pels, and Nash 2010, Albalate, Bel, and Fageda 2015).

Research on public transport has shown that a frequency-based representation of a timetable is suitable to model travel choices in high-frequency urban transit systems, but it is not appropriate to model choices of intercity trips, since the schedule of the available services affects the decision of the customer (Cascetta and Coppola 2016). In the context of high-speed rail pricing in a competitive environment, it is therefore fundamental for the suppliers to take into account the heterogeneity of customers in terms of desired time of travel and willingness-to-pay to avoid early/late departures/arrivals. We follow the schedule-based approach introduced by Nuzzolo, Crisalli, and Gangemi (2000) and applied by Cascetta and Coppola (2012), which allows to investigate a railway demand model that, contrarily to frequency-based representations, explicitly models each train service. This allows to assign penalties for early departure and late arrival times with respect to the user's desired departure and arrival times, respectively. By using such detailed representation of the choice set, it is possible to evaluate the effect of pricing policies and variations in the schedule of single trains. To the best of our knowledge, there has not been any attempt to include departure time choices at a disaggregate level when modeling pricing decisions in a competitive rail market.

5.3.1. Instance description. In our case study, we consider a competitive intercity travel market with two high-speed rail (HSR) companies, each operating two direct services connecting two cities in a typical morning period. Train departure times and travel times are assumed to be exogenously given. Additionally, we include two scheduled flights, a slow intercity train and private transport as opt-out alternatives for which timetables and prices are fixed and exogenously given. We endogenously model the pricing strategies of the two train operators and we assume that they must decide on the prices at which to sell tickets for each scheduled departure time. Table 11 presents the attributes of all scheduled services.

Furthermore, we generate a synthetic population of 1000 travelers for the given OD pair. Each individual has a trip purpose (business or other), an income level (high or low), a desired arrival time

Alternative	0	1	2	3	4	5	6	7
Mode	Car	IC	Air	Air	HSR	HSR	HSR	HSR
Endogenous	No	No	Yes	Yes	Yes	Yes	Yes	Yes
Operator	-	-	-	-	1	1	2	2
Dep time	-	2:00	7:10	8:10	5:45	6:45	5:40	6:40
Arr time	-	10:00	8:20	9:20	8:45	9:45	9:00	10:00
Travel time (min)	360	480	70	70	180	180	200	200
Waiting time (min)	-	-	60	60	-	-	-	-
Access time (min)	-	0-60	30-60	30-60	0-60	0-60	0-60	0-60
Egress time (min)	-	0-30	30-60	30-60	0-30	0-30	0-30	0-30
Price (€)	100	30	60	60	p_4	p_5	p_6	p_7

Table 11 Attributes of all scheduled services for the HSR problem instance.

Group (n)	Size (θ_n)	Trip purpose	Reimbursement	Income	Origin
1	387	Other	-	Low	Rural
2	81	Other	-	Low	Urban
3	27	Other	-	High	Rural
4	11	Other	-	High	Urban
5	15	Business	No	Low	Rural
6	44	Business	Yes	Low	Rural
7	35	Business	No	Low	Urban
8	146	Business	Yes	Low	Urban
9	9	Business	No	High	Rural
10	41	Business	Yes	High	Rural
11	47	Business	No	High	Urban
12	157	Business	Yes	High	Urban

Table 12 Contingency table for the synthetic population according to socioeconomic characteristics.

at destination between 9:00 and 11:00 which follows a non-uniform distribution, and a specific origin location (urban or rural) which leads to different access times to terminals. The following demand patterns are to be mentioned: most business travelers desire to arrive at their final destination before 10:00, while most other travelers are indifferent to arrival time; there is a higher proportion of high income and business travelers among urban travelers than among rural travelers; a part of business travelers are reimbursed and are therefore less price sensitive. We categorize the synthetic population in 12 groups of consumers, each having homogeneous socioeconomic characteristics. Table 12 represents the contingency table of the synthetic population.

For the discrete choice model, we refer to the model estimated by Cascetta and Coppola (2012), which is derived from a RP/SP survey dataset collected in Italy at the national level. Table 13 illustrates the parameters used in our experiments. Two separate sets of parameters are considered for business trips and other trip purposes. Additionally, the cost parameters are mode-specific and interact with income, producing different values of travel time savings, which are reported in Table 14. A nested logit model is used where the three nests μ_{HSR1}, μ_{HSR2} and μ_{Air} capture the correlation between the scheduled services of the each of the two train operators and of the airline.

We remark that the dataset used for the experiments and the derived results are hypothetical and do not represent real scenarios that are related to choices made by existing high-speed rail operators.

[1] Non-starred values are taken as such from Cascetta and Coppola (2012). The alternative-specific constants (ASCs) are calibrated to reflect some predefined modal split estimates, following the procedure described in Hensher, Rose,

β		Business travelers			Other purpose travelers
μ_{HSR1}		1.190		1.333	
μ_{HSR2}		1.134		1.299	
μ_{Air}		1.086		1.106	
ASC_{Car}		0.000*		0.000*	
ASC_{IC}		-1.289*		-2.138*	
ASC_{Air}		-2.893*		-1.856*	
ASC_{HSR1}		-0.825*		-0.572*	
ASC_{HSR2}		-1.174*		-1.069*	
Travel time (min)		-0.0133		-0.0054	
Access/egress time (min)		-0.00555		-0.0103	
Early schedule delay (min)		-0.00188		-0.00677	
Late schedule delay (min)		-0.0130		-0.00617	
	Reimbursed	High income	Low income	High income	Low income
Cost car (euro)	-0.0222*	-0.0296*	-0.0527	-0.0228*	-0.0405
Cost Air (euro)	-0.0109	-0.0113*	-0.0201	-0.0109*	-0.0194
Cost IC (euro)	-0.0158	-0.0212*	-0.0377	-0.0097*	-0.0172
Cost HSR (euro)	-0.0120	-0.0160*	-0.0284	-0.0144*	-0.0256

Table 13 Model parameters derived from Cascetta and Coppola (2012). [1]

Value of Travel Time	Reimbursed	High income	Low income	High income	Low income
Car (euro/h)	35.88*	26.95*	15.14	14.24*	8.00
Air (euro/h)	73.21	70.67*	39.70	29.73*	16.70
IC (euro/h)	50.51	37.68*	21.17	33.54*	18.84
HSR (euro/h)	66.50	50.02*	28.10	22.53*	12.66

Table 14 Values of travel time savings

5.3.2. Notes on algorithm implementation. This case study presents two additional sources of complexity: on the demand side, the nested logit model is used to capture the similarities existing within subsets of alternatives which share observed and unobserved attributes; on the supply side, each high-speed train operator makes a simultaneous decision on the price of multiple alternatives.

We recall that, in the nested logit model, the deterministic utility V_{in} of customer n for alternative i belonging to nest m is affected by the utilities of all the other alternatives j that belong to the same nest by means of a logsum term:

$$V_{in} = V_{in}^{'} + \log(\mu \exp(V_{in}^{'}(\mu_m - 1))(\sum_{j\in m} \exp(V_{jn}^{'}\mu_m))^{\frac{\mu}{\mu_m}-1}), \tag{34}$$

where $V_{in}^{'}$ represents the logit utility, μ is a scale parameter which is commonly normalized to 1 and $\mu_m \geq 1$ is a scale parameter that expresses the correlation between alternatives in the same nest.

By inserting (34) into our linearized utility expression (1), we obtain

$$U_{inr} = V_{in}^{'} + \log(\mu \exp(V_{in}^{'}(\mu_m - 1))(\sum_{j\in m} \exp(V_{jn}^{'}\mu_m))^{\frac{\mu}{\mu_m}-1}) + \xi_{inr}, \tag{35}$$

and Greene (2005). The $\beta_{cost_{car}}$ parameter for reimbursed business customers is derived by assuming that the ratio between the values of travel time of reimbursed and non-reimbursed business travelers by car is the same as for train alternatives. We have introduced an additional distinction between high income and low income travelers, by arbitrarily assuming that β_{cost} parameters of non-reimbursed business travelers and of other travelers from Cascetta and Coppola (2012) apply to our low income segment of the population. β_{cost} parameters for high income customers are derived by assuming that the ratio between the values of travel time of high income and low income customers is the same as in the SAMPERS long-distance model developed in Sweden and reported in Börjesson (2014).

where $V_{in}^{'} = \beta_{p,in} p_i + q_{in}$ and $V_{jn}^{'} = \beta_{p,jn} p_j + q_{jn}$. Expression (35) is non-linear due to the presence of endogenous variables, i.e. the prices p_j of all high-speed train services, in the logsum term. To overcome this issue, we initially fix the values of the endogenous variables in the logsum term and only optimize for those in the logit utility $V_{in}^{'}$. By doing so, the supply optimization problem is solved as a mixed integer linear problem. Of course, the initial logsum values need to be updated and the problem solved again in an iterative fashion, until reaching convergence.

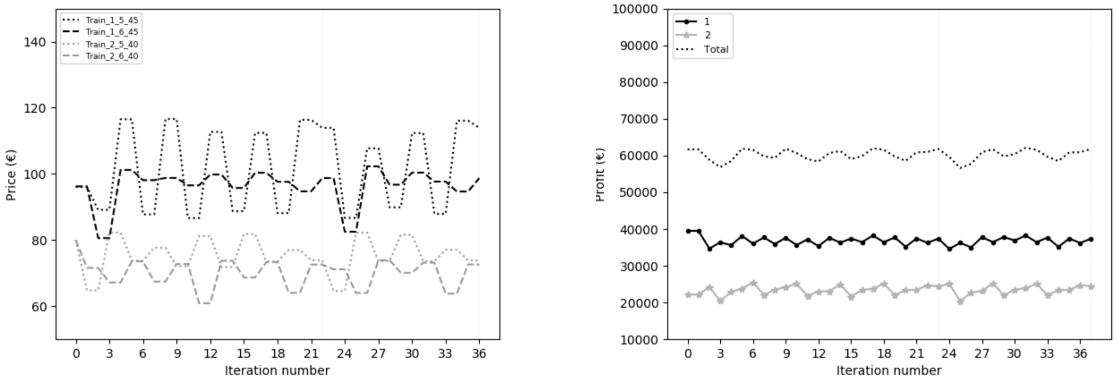
For the first heuristic block of our algorithmic framework, computational experiments show that, when all scale parameters $\mu_m < 1.5$ as in this case study, this iterative optimization procedure is computationally tractable and convergence between the logsum variables and the linear variables is reached in parallel with the convergence to an equilibrium region of the competitive game. Preliminary tests indicate that convergence might be slower to reach for higher values of μ_m, since small changes in the values of the endogenous variables have large effects on the logsum terms, thus making the fixed-point method cumbersome. We hypothesize that implementing smoothing techniques for the update of the logsum terms could be beneficial, but further research should be conducted to derive more sound conclusions on this matter.

The value of the parameters μ_m is less of an issue when solving for subgame equilibrium, provided that sufficiently tight bounds are imposed on the distance between the strategies included in the restricted sets.

Similarly to the parking choice case study, we apply Algorithm 1 as such to find approximate equilibrium solutions for the given market. We remark that in this case a supplier strategy corresponds to a bundle of price decisions on different alternatives. Therefore, tradeoffs exist between alternatives controlled by the same supplier which can lead to different strategic behaviors in different regions of the solution space.

In the first block, we solve the problem using the fixed-point iteration algorithm. Figure 4 shows the evolution of prices and profits for a run of the algorithm, given a random initial state of the market. We can notice that the algorithm converges to an equilibrium region after a number of iterations (Figure 4a). However, none of these states is a stable solution, because there is always a competitor who has an interest to deviate from the current state by best-responding to the rival to increase its profits (Figure 4b).

Using the fixed-point iteration algorithm multiple times with different initial configurations is insufficient to model the competitive behavior of the suppliers, because neither the stability of the solution nor the potential existence of tacit collusion to deviate from the purely non-cooperative outcome are taken into account. On the contrary, relying on the fixed-point optimization model, albeit on restricted strategy sets, allows to search for subgame solutions for which best response conditions are satisfied by all suppliers simultaneously. The application to this case study of the



(a) Prices (b) Profits

Figure 4 **Evolution of prices and profits for one run of the fixed-point iteration algorithm in the high-speed rail case study.**

Supplier	$\mid S \mid$	Alternative	Bounds		Subgame solution		Best responses		ε
			LB	UB	Price	Profit	Price	Profit	
HSR1	4	4	86.68	116.09	106.29	36983	114.98	37787	0.022
		5	82.51	102.28	95.69		97.28		
HSR2	4	6	64.58	82.25	76.36	24205	80.57	24660	0.019
		7	63.78	73.91	70.53		72.42		
HSR1	5	4	86.68	116.09	116.09	38112	115.27	38248	0.003
		5	82.51	102.28	102.28		103.66		
HSR2	5	6	64.58	82.25	80.57	25157	76.65	25520	0.014
		7	63.78	73.91	72.42		72.72		
HSR1	6	4	86.68	116.09	114.98	37994	114.95	38277	0.008
		5	82.51	103.66	97.28		104.14		
HSR2	6	6	64.58	82.25	80.57	24551	80.25	24852	0.012
		7	63.78	73.91	72.42		72.72		
HSR1	7	4	86.68	116.09	114.98	37609	114.58	37901	0.008
		5	82.51	104.14	97.28		98.02		
HSR2	7	6	64.58	82.25	76.65	24752	80.94	24911	0.006
		7	63.78	73.91	72.72		71.45		

Table 15 **Illustration of the algorithm to find an ε-equilibrium solution for the high-speed rail case study with uniform pricing, with $\varepsilon = 0.01$.**

column-generation-like technique that forms the second and third blocks of the algorithmic approach is illustrated in Table 15. Notice that the price bounds for the fixed-point model are updated whenever new best response strategies are added to the restricted sets. In principle, strategies can also be removed to keep the restricted sets small and the bounds tight to precompute as many captive customer choices as possible. Finally, the value of ε is subject to problem-specific considerations on competitive behavior and can be updated during execution, since there is no guarantee that ε-equilibrium solutions exist for any given ε. In these experiments,

ε is initialized to 0.01 and then progressively increased through properly tuned parameters until the desired number of solutions is found.

Equilibrium		Prices				Profits	
#	ε	$HSR1_1$	$HSR1_2$	$HSR2_1$	$HSR2_2$	HSR1	HSR2
1	0.8%	114.98	97.28	76.65	72.72	37609	24752
2	0.5%	114.58	98.02	80.94	71.45	38051	24869
3	0.4%	114.63	98.36	80.74	72.33	38129	24908
4	1.0%	114.78	98.27	77.31	72.34	37988	24723
5	0.7%	114.75	97.96	81.26	71.23	38070	24901

Table 16 List of approximate equilibrium solutions of the HSR case study with uniform pricing.

#	Market shares			Urban market shares			Rural market shares		
	Opt-out	HSR1	HSR2	Opt-out	HSR1	HSR2	Opt-out	HSR1	HSR2
1	0.321	0.353	0.326	0.232	0.425	0.343	0.403	0.286	0.311
2	0.316	0.359	0.325	0.229	0.430	0.340	0.395	0.294	0.311
3	0.321	0.359	0.320	0.231	0.430	0.339	0.403	0.294	0.304
4	0.321	0.354	0.326	0.231	0.429	0.340	0.403	0.285	0.313
5	0.324	0.355	0.322	0.229	0.430	0.340	0.410	0.286	0.305

Table 17 Aggregate and disaggregate market shares for the approximate equilibrium solutions of the HSR case study with uniform pricing.

Equilibrium		Prices						Profits	
#	ε	$HSR1_1$	$HSR1_2$	$HSR2_1^u$	$HSR2_1^r$	$HSR2_2^u$	$HSR2_2^r$	HSR1	HSR2
1	1.3%	115.02	97.73	107.39	58.91	88.69	70.66	38043	25783
2	1.5%	90.00	90.00	92.18	78.82	86.13	68.99	39021	25225
3	1.3%	110.27	97.11	105.02	63.55	82.02	72.53	38385	25726
4	1.1%	115.90	97.68	107.69	65.31	81.48	70.91	39034	25796
5	0.9%	115.33	89.99	104.95	63.21	97.35	72.90	38512	24847

Table 18 List of approximate equilibrium solutions of the HSR case study with price differentiation.

#	Market shares			Urban market shares			Rural market shares		
	Opt-out	HSR1	HSR2	Opt-out	HSR1	HSR2	Opt-out	HSR1	HSR2
1	0.318	0.359	0.323	0.258	0.465	0.277	0.373	0.262	0.365
2	0.291	0.416	0.293	0.224	0.506	0.271	0.352	0.334	0.314
3	0.319	0.366	0.316	0.251	0.465	0.285	0.381	0.275	0.345
4	0.321	0.361	0.318	0.254	0.458	0.287	0.382	0.273	0.345
5	0.317	0.386	0.297	0.258	0.490	0.252	0.372	0.290	0.338

Table 19 Aggregate and disaggregate market shares for the approximate equilibrium solutions of the HSR case study with price differentiation.

5.3.3. Results. We perform two experiments. In the first experiment, no price differentiation based on customer groups is considered. We run the algorithm until five approximate equilibrium solutions with different values of ε are found. The list of solutions is provided in Table 16. Each row corresponds to a different solution, for which the prices for all alternatives and the profits are reported. We notice that both suppliers price their first service higher than the second, reflecting the demand of business customers who are less sensitive to cost and more sensitive to arrival time. Supplier 1 can set higher prices, reflecting the higher quality of service perceived by the customers (notice the lower travel times in Table 11 and the higher alternative specific constants in Table 13) and gets higher profits. Table 17 reports the market shares, both in aggregate form and segmented according to the origin of the travelers.

In the second experiment, we consider the case where supplier 2 exploits the available disaggregate demand model to develop a selling strategy that targets urban and rural customers separately through price differentiation. This means that now supplier 2 sets four different prices p_6^{urban}, p_6^{rural}, p_7^{urban} and p_7^{rural}, while supplier 2 keeps its uniform pricing strategy, deciding on the two prices p_4 and p_5. The purpose of this test is to check whether being able to price differentiate constitutes a competitive advantage against suppliers that do not price differentiate. Table 18 lists the approximate equilibrium solutions when supplier 1 price differentiates. We can see that the equilibrium prices of supplier 2 for urban and rural customers reflect the different price sensitivity of the two groups. Compared to the approximate equilibria of the uniform pricing case, supplier 2 tends to increase the price for urban customers and decrease the price for rural customers, for both scheduled departures. Among urban customers, the first service is again priced higher than the second. In most approximate equilibria, profits are higher than in the uniform pricing case, both for supplier 1 and supplier 2. Table 19 shows the effects on market shares. Compared to the uniform pricing case, aggregate market shares remain stable, but notable differences exist in the segment market shares, with supplier 2 increasing its market share among rural customers and supplier 1 attracting more urban customers.

These considerations exemplify the type of analyses that can be carried out either by the competitors or by an external market regulator who has access to estimations of disaggregate demand data.

6. Conclusion

In this paper, we presented a general framework to find approximate equilibrium solutions of oligopolistic markets where demand is modeled at the disaggregate level using discrete choice models. A disaggregate representation of demand allows to better account for product differentiation and consumer behavioral heterogeneity at the individual level. However, this comes at the expense of having a guarantee that a pure equilibrium exists, since the supplier profit functions are non-convex. For this reason, we proposed a simulation-based heuristic approach that finds approximate equilibrium solutions. The methodology was applied to two examples of oligopolistic markets within the transportation sector, namely parking and high-speed rail, for which advanced choice models, taken as such from the literature, were used to model demand. Numerical experiments achieved to find approximate equilibrium solutions that provide meaningful information for competing firms and policy-makers alike.

The interaction between demand and supply models is a fundamental problem in the analysis of imperfect competition, and aggregation techniques have been largely utilized to obtain demand functions that could be used in equilibrium models. Our paper proposes an alternative framework which requires limited assumptions on the specification of the used discrete choice models. This means that it can accommodate a large variety of choice models available in the literature. The potential of our methodology needs to be evaluated in light of the advances in discrete choice modeling, which allow for increasingly complex and precise representations of individual behavior. In particular, there is a vast literature on choice models with latent variables, modeling subjective behavioral dimensions such as attitudes and perceptions. These models can be integrated in our framework exactly as done for the two examples provided in this paper.

Searching for a pure equilibrium of a problem that has an aggregate demand model and searching for an approximate equilibrium of a problem with a disaggregate demand model are two valid approaches to find market equilibria. In general, since the inputs of the two problems are different, the outputs cannot be compared. Both demand models are approximations of the real demand. The reason for investigating equilibrium models with disaggregate demand is that there exist specification tests that allow to say whether a demand model is better than another model. In practice, the trade-off between the realism of the assumptions (or lack thereof) and the complexity of the demand model must be assessed on a case-by-case basis by the analyst.

The following research directions should be further investigated.

The modeling framework could be extended to take into account congestion effects and capacity constraints, both of which are important phenomena in transportation. Congestion can be included in the utility function by adding a demand-dependent component. Due to the highly non-linear relation between demand and congestion in transportation, an inner fixed-point iteration approach should be added to the framework in order to reach lower-level user equilibrium. To include capacity constraints, exogenous priority rules that simulate the arrival process of customers could be implemented, as proposed by Binder, Maknoon, and Bierlaire (2017). Such a methodology would also allow for price differentiation based on the time of booking, in a revenue management fashion.

We acknowledge that the main limitation to the use of our algorithmic framework is the computational complexity caused by the multidimensional solution space that must be evaluated to solve the best response conditions to verify approximate equilibrium solutions. At this point, the applicability of our framework to large-scale problem depends on the capability to efficiently exploit the problem structure and find tight bounds or ad-hoc algorithms to such hard combinatorial problem. Pacheco Paneque (2020) investigates the challenges associated with choice-based optimization using mixed integer optimization models and propose a Lagrangian decomposition scheme that finds near-optimal solutions fast. Future work could be undertaken to extend these results to a competitive setting.

In the context of transportation, the framework could also incorporate the role of a regulator, which actively influences the market towards a social welfare maximization outcome.

Finally, from a decision-making perspective, it would be interesting to analyze the stability of equilibrium with regard not only to competitor behavior, but also to demand stochasticity, which can be caused by uncertainties in the phase of demand estimation.

Acknowledgments

This research is supported by the Swiss National Science Foundation [Grant 175483 "New generation of demand-supply interaction models"]. We would like to express our gratitude to Nicolas Pradignac and Théophile Thiery for their contribution to the project.

References

Adler N, 2001 *Competition in a deregulated air transportation market. European Journal of Operational Research* 129(2):337–345.

Adler N, Pels E, Nash C, 2010 *High-speed rail and air transport competition: Game engineering as tool for cost-benefit analysis. Transportation Research Part B: Methodological* 44(7):812–833.

Ait Ali A, Eliasson J, 2019 *Railway capacity allocation: a survey of market organizations, allocation processes and track access charges.* Technical report, Swedish National Road & Transport Research Institute (VTI).

Aksoy-Pierson M, Allon G, Federgruen A, 2013 *Price competition under mixed multinomial logit demand functions. Management Science* 59(8):1817–1835.

Albalate D, Bel G, Fageda X, 2015 *Competition and cooperation between high-speed rail and air transportation services in Europe. Journal of Transport Geography* 42:166–174.

Anderson SP, De Palma A, Thisse JF, 1992 *Discrete choice theory of product differentiation* (MIT Press).

Andersson SE, 1998 *Passenger choice analysis for seat capacity control: A pilot project in Scandinavian Airlines. International Transactions in Operational Research* 5(6):471–486.

Beck A, 2011 *Barriers to entry in rail passenger services: empirical evidence for tendering procedures in Germany. European Journal of Transport and Infrastructure Research* 11(1).

Ben-Akiva ME, Cascetta E, Coppola P, Papola A, Velardi V, 2010 *High speed rail demand forecasting in a competitive market: the Italian case study. Proceedings of the World Conference of Transportation Research (WCTR), Lisbon, Portugal.*

Ben-Akiva ME, Lerman SR, 1985 *Discrete choice analysis: theory and application to travel demand* (MIT Press).

Benati S, Hansen P, 2002 *The maximum capture problem with random utilities: Problem formulation and algorithms. European Journal of Operational Research* 143(3):518–530.

Bernstein F, Federgruen A, 2004 *Dynamic inventory and pricing models for competing retailers. Naval Research Logistics (NRL)* 51(2):258–274.

Berry S, Levinsohn J, Pakes A, 1995 *Automobile prices in market equilibrium. Econometrica: Journal of the Econometric Society* 841–890.

Bertrand J, 1883 *Théorie des richesses: revue de 'Théorie mathématique de la richesse sociale' par Léon Walras et 'Recherches sur les principes mathématiques de la richesse' par Augustin Cournot. Journal des savants* 499–508.

Bertsimas D, Gupta V, Paschalidis IC, 2015 *Data-driven estimation in equilibrium using inverse optimization. Mathematical Programming* 153(2):595–633.

Binder S, Maknoon Y, Bierlaire M, 2017 *Exogenous priority rules for the capacitated passenger assignment problem. Transportation Research Part B: Methodological* 105:19–42.

Börjesson M, 2014 *Forecasting demand for high speed rail. Transportation Research Part A: Policy and Practice* 70:81–92.

Bracken J, McGill JT, 1973 *Mathematical programs with optimization problems in the constraints. Operations Research* 21(1):37–44.

Brander JA, Zhang A, 1993 *Dynamic oligopoly behaviour in the airline industry. International Journal of Industrial Organization* 11(3):407–435.

Bresnahan TF, Reiss PC, 1991 *Entry and competition in concentrated markets. Journal of Political Economy* 99(5):977–1009.

Broman E, Eliasson J, 2019 *Welfare effects of open access competition on railway markets. Transportation Research Part A: Policy and Practice* 129:72–91.

Capurso M, Hess S, Dekker T, 2019 *Modelling the role of consideration of alternatives in mode choice: an application on the Rome-Milan corridor. Transportation Research Part A: Policy and Practice* 129:170–184.

Cascetta E, Cartenì A, Henke I, Pagliara F, 2020 *Economic growth, transport accessibility and regional equity impacts of high-speed railways in Italy: ten years ex post evaluation and future perspectives. Transportation Research Part A: Policy and Practice* 139:412–428.

Cascetta E, Coppola P, 2012 *An elastic demand schedule-based multimodal assignment model for the simulation of high speed rail (HSR) systems. EURO Journal on Transportation and Logistics* 1(1-2):3–27.

Cascetta E, Coppola P, 2016 *Assessment of schedule-based and frequency-based assignment models for strategic and operational planning of high-speed rail services. Transportation Research Part A: Policy and Practice* 84:93–108.

Christodoulou G, Koutsoupias E, Spirakis PG, 2011 *On the performance of approximate equilibria in congestion games. Algorithmica* 61(1):116–140.

Colson B, Marcotte P, Savard G, 2007 *An overview of bilevel optimization. Annals of Operations Research* 153(1):235–256.

Cournot AA, 1838 *Recherches sur les principes mathématiques de la théorie des richesses* (L. Hachette).

Daskalakis C, Mehta A, Papadimitriou CH, 2007 *Progress in approximate nash equilibria. Proceedings of the 8th ACM Conference on Electronic Commerce*, 355–358.

Fageda X, Sansano S, 2018 *Factors influencing prices and frequencies in the interurban bus market: evidence from Europe. Transportation Research Part A: Policy and Practice* 111:266–276.

Fisk C, 1984 *Game theory and transportation systems modelling. Transportation Research Part B: Methodological* 18(4-5):301–313.

Friedman JW, 1971 *A non-cooperative equilibrium for supergames. The Review of Economic Studies* 38(1):1–12.

Fudenberg D, Tirole J, 1991 *Game theory* (MIT Press).

Gabszewicz JJ, Thisse JF, 1979 *Price competition, quality and income disparities. Journal of economic theory* 20(3):340–359.

Gallego G, Wang R, 2014 *Multiproduct price optimization and competition under the nested logit model with product-differentiated price sensitivities. Operations Research* 62(2):450–461.

Grimaldi R, Augustin K, Beria P, 2017 *Intercity coach liberalisation. The cases of Germany and Italy. Transportation Research Procedia* 25:474–490.

Haase K, 2009 *Discrete location planning*. Technical report, Institute of Transport and Logistics Studies, University of Sydney.

Hanson W, Martin K, 1996 *Optimizing multinomial logit profit functions. Management Science* 42(7):992–1003.

Hensher DA, Rose JM, Greene WH, 2005 *Applied choice analysis: a primer* (Cambridge University Press).

Hotelling H, 1929 *Stability in competition. Economic Journal* 39:41–57.

Ibeas A, Dell'Olio L, Bordagaray M, Ortúzar JdD, 2014 *Modelling parking choices considering user heterogeneity. Transportation Research Part A: Policy and Practice* 70:41–49.

Koppelman FS, 1976 *Guidelines for aggregate travel prediction using disaggregate choice models. Transportation Research Record* 610:19–24.

Levin Y, McGill J, Nediak M, 2009 *Dynamic pricing in the presence of strategic consumers and oligopolistic competition. Management Science* 55(1):32–46.

Li H, Huh WT, 2011 *Pricing multiple products with the multinomial logit and nested logit models: Concavity and implications. Manufacturing & Service Operations Management* 13(4):549–563.

Li H, Webster S, Mason N, Kempf K, 2019 *Product-line pricing under discrete mixed multinomial logit demand. Manufacturing & Service Operations Management* 21(1):14–28.

Lin KY, Sibdari SY, 2009 *Dynamic price competition with discrete customer choices. European Journal of Operational Research* 197(3):969–980.

Liu TL, Chen J, Huang HJ, 2011 *Existence and efficiency of oligopoly equilibrium under toll and capacity competition. Transportation Research Part E: Logistics and Transportation Review* 47(6):908–919.

Manski CF, 1977 *The structure of random utility models. Theory and decision* 8(3):229–254.

Maskin E, Tirole J, 1987 *A theory of dynamic oligopoly, III: Cournot competition. European Economic Review* 31(4):947–968.

Maskin E, Tirole J, 1988a *A theory of dynamic oligopoly, I: Overview and quantity competition with large fixed costs. Econometrica: Journal of the Econometric Society* 549–569.

Maskin E, Tirole J, 1988b *A theory of dynamic oligopoly, II: Price competition, kinked demand curves, and Edgeworth cycles. Econometrica: Journal of the Econometric Society* 571–599.

McFadden D, Reid F, 1975 *Aggregate travel demand forecasting from disaggregated behavioral models* (Institute of Transportation and Traffic Engineering, University of California).

McFadden D, Train K, 2000 *Mixed MNL models for discrete response. Journal of Applied Econometrics* 15(5):447–470.

Milgrom P, Roberts J, 1990 *Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica: Journal of the Econometric Society* 1255–1277.

Morrow WR, Skerlos SJ, 2011 *Fixed-point approaches to computing Bertrand-Nash equilibrium prices under mixed-logit demand. Operations Research* 59(2):328–345.

Murphy FH, Sherali HD, Soyster AL, 1982 *A mathematical programming approach for determining oligopolistic market equilibrium. Mathematical Programming* 24(1):92–106.

Nash J, 1951 *Non-cooperative games. Annals of mathematics* 286–295.

Nisan N, Roughgarden T, Tardos E, Vazirani VV, 2007 *Algorithmic game theory*, volume 1 (Cambridge University Press).

Nuzzolo A, Crisalli U, Gangemi F, 2000 *A behavioural choice model for the evaluation of railway supply and pricing policies. Transportation Research Part A: Policy and Practice* 34(5):395–404.

Osborne MJ, Rubinstein A, 1994 *A course in game theory* (MIT Press).

Pacheco Paneque M, 2020 *A general framework for the integration of complex choice models into mixed integer optimization*. Ph.D. thesis, École Polytechnique Fédérale de Lausanne.

Pacheco Paneque M, Bierlaire M, Gendron B, Sharif Azadeh S, 2021 *Integrating advanced discrete choice models in mixed integer linear optimization. Transportation Research Part B: Methodological* 146:26–49.

Radner R, 1980 *Collusive behavior in noncooperative epsilon-equilibria of oligopolies with long but finite lives. Journal of economic theory* 22(2):136–154.

Rotemberg J, Saloner G, 1986 *A supergame-theoretic model of price wars during booms. New Keynesian Economics* 2:387–415.

Shaked A, Sutton J, 1983 *Natural oligopolies. Econometrica: journal of the Econometric Society* 1469–1483.

Starkie D, 2002 *Airport regulation and competition. Journal of air Transport management* 8(1):63–72.

Stigler GJ, 1964 *A theory of oligopoly. Journal of Political Economy* 72(1):44–61.

Talluri K, Van Ryzin G, 2004 *Revenue management under a general discrete choice model of consumer behavior. Management Science* 50(1):15–33.

Train KE, 2009 *Discrete choice methods with simulation* (Cambridge University Press).

Valeri E, 2013 *Air and rail transport in the Rome-Milan corridor: competition policy implications based on a discrete choice analysis.* Ph.D. thesis, Università degli Studi di Trieste.

von Stackelberg H, 1934 *Marktform und gleichgewicht* (Julius Springer).

Vulcano G, Van Ryzin G, Chaar W, 2010 *Choice-based revenue management: An empirical study of estimation and optimization. Manufacturing & Service Operations Management* 12(3):371–392.