A disaggregate choice-based approach to find ε-equilibria of oligopolistic markets

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November 7, 2019

Report TRANSP-OR 191107
Transport and Mobility Laboratory
School of Architecture, Civil and Environmental Engineering
Ecole Polytechnique Fédérale de Lausanne
http://transp-or.epfl.ch

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Abstract

We present a general framework to find ε-equilibrium solutions of oligopolistic markets in which demand is modeled at the disaggregate level using discrete choice models. Consumer choices are modeled according to random utility theory, and the choice probabilities are linearized and embedded as lower-level constraints in the supply optimization problems. To model competition, we introduce a mixed integer optimization model based on the fixed-point iteration algorithm, which can find an optimal equilibrium or near-equilibrium solution of a finite game with small strategy sets. To solve larger equilibrium problems, a model-based algorithmic approach is proposed. First, a heuristic reduction of the search space is performed. Then, an iterative procedure solves a subgame equilibrium problem with restricted strategy sets using the fixed-point optimization model, compares the optimal solution against the best responses of all suppliers over the original strategy sets, and adds best response strategies to the restricted problem until all ε-equilibrium conditions are satisfied simultaneously. Numerical experiments show the applicability of our algorithm to find oligopolistic ε-equilibria for two transportation case studies.

Keywords: competition, equilibrium, industrial organization, disaggregate demand, discrete choice modeling

1 Introduction

Oligopolistic competition occurs in various markets when a limited number of suppliers compete for the same pool of customers. In oligopolies, suppliers make decisions that are influenced both by the preferences of the customers, who are considering to purchase one of the alternative services available on the market, and by the decisions of their competitors. Nowadays, these decisions are informed by both detailed customer data, from which precise individual behavioral models are derived, and competitor insights, which allow to react in real time to market changes.

Oligopolistic competition has been extensively analyzed through static and dynamic models, with the goal of finding and evaluating market equilibria (Stigler, 1964, Friedman, 1971, Shaked and Sutton, 1983, Maskin and Tirole, 1987, 1988a,b, Brander and Zhang, 1993). However, from a mathematical perspective, an equilibrium is guaranteed to exist only if a number of conditions related to continuity, differentiability and convexity are satisfied for the demand, cost and profit functions (Murphy et al., 1982). In particular, these requirements pose limitations on the demand function, which needs to be such that all resulting profit functions are concave. Even the simplest disaggregate demand model cannot satisfy this condition. Hence, disaggregate demand is generally aggregated before being included in models of oligopolies (Berry et al., 1995, Goldberg, 1995) using existing aggregation techniques (McFadden and Reid, 1975, Koppelman, 1976, Ben-Akiva and Lerman, 1985).

The main benefit of estimating demand at the disaggregate level is the possibility to account for product differentiation and consumer behavioral heterogeneity at the individual level (Anderson et al., 1992). However, even though there exists a large body of choice modeling literature describing complex disaggregate choice behavior, these models are never included as such in equilibrium problems.

In this paper, we introduce a framework which allows to take a previously estimated discrete choice model and include it as such in a model of oligopolistic competition. In our work, the preferences of the customers are modeled at a disaggregate level according to random utility theory and are embedded as utility functions in the supplier optimization problem. Competition among market players is modeled explicitly as a non-cooperative game in which all operators optimize their own decisions based on the decisions of their competitors. This results in a choice-based multi-leader-follower game, for which a modeling and an algorithmic framework are proposed to find and analyze oligopolistic ε-equilibrium solutions. To the best of our knowledge, this is the first attempt to keep demand at a disaggregate level in a model of oligopolistic competition.

The rest of the paper is organized as follows. Section 2 provides a literature review on oligopolistic competition, from both a theoretical and an applied perspective, and identifies the opportunity to enhance existing approaches by modeling demand at a disaggregate level. Section 3 presents the three components of the modeling framework, namely demand, supply and market interactions. Section 4 proposes a model-based algorithmic approach to find ε-equilibrium solutions in oligopolistic markets. Section 5 illustrates numerical experiments performed on two transportation case studies to validate the models and algorithms. Finally, Section 6 concludes the paper and provides directions for future research.

2 Literature review

2.1 Competition

The discipline that studies competition between groups of decision-makers when individual choices jointly determine the outcome is known as game theory. Fudenberg and Tirole (1991) and Osborne and Rubinstein (1994) provide an overview of the principal concepts in game theory. For the purpose of this research, we consider here supply-supply and supply-demand interactions, which fit into the frameworks of the Nash non-cooperative game and of the Stackelberg game, respectively. The Nash game (Nash, 1951) considers players that have equal status. A Nash equilibrium solution of the game is a state in which no player can improve its payoff by unilaterally changing his decision. Nisan et al. (2007) describe several algorithmic methods used to find Nash equilibria under various circumstances. The Stackelberg game (von Stackelberg, 1934) features two players, identified as leader and follower, both trying to optimize their own objective function. The leader knows the follower's best responses to all the leader's strategies, and will therefore optimize its decisions accordingly. The Stackelberg game can be modeled as an optimization problem having an optimization problem in the constraints (Bracken and McGill, 1973), also known as bilevel program. An overview of bilevel optimization is given by Colson et al. (2007).

While the most important theoretical concept used to analyze both perfectly competitive and oligopolistic markets is that of equilibrium, in the economic literature it is widely acknowledged that in non-cooperative games the utility maximization behavior is unlikely to hold true for all agents. The state in which no non-maximizing agent would gain a significant amount by becoming a maximizer is defined as near-rational equilibrium or ε-equilibrium. Inertia and the use of rules of thumb in the decision-making process are reasons that explain non-rational behavior (Akerlof et al., 1985). However, an apparently non-maximizing behavior could also be purposely selected by firms that are implicitly colluding to avoid prisoner's dilemma outcomes (Stigler, 1964, Radner, 1980, Rotemberg and Saloner, 1986).

The theory of competition in non-cooperative games has been extensively used to analyze oligopolistic markets where a small number of firms are active and have non-negligible market power. The first contributions date back to the seminal works by Cournot (1838) and Bertrand (1883), who analyze a market where an homogeneous product is sold to an homogeneous population and where firms compete on quantity and price, respectively. Hotelling (1929) proposes a duopolistic game in which firms decide on the location of production and on the price of the product, while the spatial distribution along a line of the homogeneous and inelastic demand affects the cost of transportation from producers to customers. By using a spatial model, Hotelling questions the assumption used by Edgeworth (1925) that consumers abruptly change their product choice when a seller marginally decreases its price and argues that stability is achieved when there is little product differentiation across producers. Gabszewicz and Thisse (1979) consider consumers having identical preferences but variable incomes who make indivisible and mutually exclusive purchases. In a duopolistic market, income differentiation is shown to support product differentiation, contrarily to what Hotelling's model seemed to suggest. The authors also notice that the existence of a Cournot equilibrium requires the continuity in the demand function and that the proof of existence based on fixed-point arguments is based on the quasi-concavity of the profit functions. Murphy et al. (1982) propose a mathematical programming approach to find market equilibria in an oligopolistic market supplying an homogeneous product, in which firms must determine their production levels. Assumptions are made on the revenue curves which must be concave, on the demand curve which must be continuously differentiable and on the supply curve which must be convex and continuously differentiable. When these conditions hold, the equilibrium solution can be found by solving the Karush-Kuhn-Tucker conditions for the optimization problems of the firms. Moving from static to dynamic models, Maskin and Tirole (1988a) propose a class of alternating-move models of duopoly in which firms are committed to a certain action in the short-term, thus allowing time for the other firm to react. Firms can choose actions from a bounded set and profits are exclusively dependent on the current actions. The latter assumption is in contrast with the repeated game treatment of oligopolies, proposed for instance by Rotemberg and Saloner (1986). Due to the time-independence assumption, equilibrium solutions in the resulting game are Markov perfect equilibria.

The researches presented above, together with more recent contributions, share the common finding that the existence of an equilibrium is only guaranteed thanks to assumptions made on the demand side which ensure the concavity of the profit function. In particular, the existence of an equilibrium is not guaranteed if demand is modeled at the disaggregate level using discrete choice models. For this reason, in our work we retain the microeconomic foundations on which the presented stream of literature is grounded, while exploring alternative mathematical models and approximation algorithms that can accommodate a disaggregate representation of demand.

2.2 Competition in transportation

Oligopolistic competition in transportation occurs due to reasons such as high barriers to entry (Bresnahan and Reiss, 1991, Beck, 2011) and limited capacity of the infrastructure (Starkie, 2002, Liu et al., 2011). Historically, the airline industry has originated the largest amount of research on competition, since before the 1978 United States Airline Deregulation Act, but intercity train and bus operators and urban transport service providers also operate in oligopolistic conditions in an increasing number of cases.

Belobaba et al. (2015) provide a complete discussion on airline markets and on the effects of competition on the industry as a whole. Early research topics include non-price competition in price-constrained markets (Douglas and Miller, 1974), relation between market equilibria and social welfare (Panzar, 1979), market entry and barriers to entry for hub-and-spoke and point-to-point networks (Reiss and Spiller, 1989, Berry, 1992, Oum et al., 1995). Later, more attention has been dedicated to modeling passenger demand to understand how it affects market shares and revenues. Pels et al. (2000) model passenger choice with a nested logit model to analyze airfares, frequencies and passenger charges in a large metropolitan area that hosts more than one airport. In Adler (2001), competition in a deregulated air transport market is modeled by means of a two-stage Nash best response game. In the first stage, an integer linear program is used to model the simultaneous choice airlines make to choose their hub-and-spoke network. In the second stage, airlines aim at maximizing profits by choosing fares, frequencies and aircraft sizes for their network of routes. Passenger choices are estimated by using a logit model and are used to derive market shares. This results in a non-linear mathematical program to be solved for all competitors iteratively until either a sub-game Nash equilibrium or a quasi-equilibrium solution, that is, two or more possible solutions around which the program cycles, are found. The open questions are related to the existence of multiple Nash equilibrium solutions, to the relation between the concepts of Nash equilibrium and quasi-equilibrium, and to the possibility of not reaching equilibrium or convergence through the iterative process.

In recent years, thanks to regulatory changes that were passed in several countries, researchers' interest in other competitive transport markets has increased, together with the desire to study both competitive and collaborative interactions between different transport modes. Román et al. (2007) use revealed preference (RP) and stated preference (SP) data to estimate willingness-to-pay measures to study the competition between high-speed train and air transport between Madrid and Barcelona. Adler et al. (2010) study a competitive game that includes high-speed rail and air transport options to evaluate the potential economic impact of the Trans-European high-speed rail network. Schedules and prices are endogenous variables in the model, which has a non-linear objective function. Behrens and Pels (2012) use cross-sectional RP data to estimate mixed logit models to study passenger behavior in the London-Paris passenger market, where intermodal competition exists involving a high-speed rail operator and several airlines. The models are used to investigate passenger preferences over time and elasticities for both business and leisure travellers with respect to travel times, fares and service frequencies.

2.3 Competition in railways

The approval of a set of so-called Railway Packages by the European Union has set forward a path towards the liberalization of domestic and international passenger services in Europe (European Commission, 2019), allowing open access to tracks to companies that do not own the rail infrastructure (Nash, 2008). One of the effects of the new regulatory framework is the progressive creation of new competitive markets on public transport routes which were traditionally covered by a monopolistic operator.

Italy was the first country to open to high-speed rail competition in 2012, when the new operator NTV started offering passenger services on the high-speed rail corridor connecting Milan, Bologna, Florence, Rome and Naples, which was previously exploited only by the incumbent company Trenitalia. Since then, competition has extended to other main corridors as well as some feeder lines. A plethora of contributions exist that forecast and analyze the competitive high-speed rail market in Italy using different approaches (Ben-Akiva et al., 2010, Cascetta and Coppola, 2012, Valeri, 2013, Mancuso, 2014, Cascetta and Coppola, 2015, Beria et al., 2016, Capurso et al., 2019, Beria and Bertolin, 2019). Other countries which already feature cases of rail competition include Sweden (Broman and Eliasson, 2019), Czech Republic (Tomeš et al., 2016) and Germany (Ait Ali and Eliasson, 2019). In some of these cases and in many other high-demand international routes, rail operators also compete for customers with intercity bus operators (Grimaldi et al., 2017, Fageda and Sansano, 2018) and low-cost or legacy airlines (Adler et al., 2010, Albalate et al., 2015). Further examples of multimodal competitive market in Europe are predicted to arise in the upcoming years, when the European Union directives will become effective in all member states.

The above-mentioned literature can be categorized in two groups: (i) ex-ante analyses, which use discrete choice models to estimate customer preferences and then forecast future market shares by relying on elasticities; (ii) ex-post analyses, which develop empirical regression models to identify the determinants of prices and frequencies. With the example we illustrate in Section 5.2, we want to show that our research can fill the gap currently existing in the literature by explicitly modeling competition between different operators while retaining a behavioral characterization of demand at a disaggregate level. To the best of our knowledge, no similar attempt has been proposed as of today.

3 The modeling framework

3.1 Demand modeling

We consider a market where a number of different products are offered to a population. Customers are assumed to be utility maximizers who can only make a unitary and mutually exclusive purchase.

The notation is as follows. Let N represent the set of customers and let I indicate the set of alternatives available in the market. Utility functions U_{in} are defined for each customer $n \in N$ and alternative $i \in I$. Each utility function takes into account the socio-economic characteristics and the tastes of the individual as well as the attributes of the alternative. According to random utility theory (Manski, 1977), U_{in} can be decomposed into a systematic component V_{in} which includes all that is observed by the analyst and a random term ε_{in} which captures the uncertainties caused by unobserved attributes and unobserved taste variations. Therefore, the resulting discrete choice models are naturally probabilistic. The probability that customer n chooses alternative

i is defined as $P_{in} = \Pr[V_{in} + \varepsilon_{in} = \max_{j \in I}(V_{jn} + \varepsilon_{jn})]$. In order to be able to estimate choice probabilities, assumptions must be made about the distribution of the error term. The two most used classes of discrete choice models, the probit and the logit, are built upon the assumption of normally distributed and extreme value distributed error terms, respectively. It is important to note that the choice probabilities of most advanced models currently considered in the choice modeling literature, including mixed logit and probit, must be expressed as integrals and approximated numerically, for instance by using simulation procedures (Train, 2009). As a result, while discrete choice models can accurately capture heterogeneous behavior on the demand side at a disaggregate level, their mathematical properties make it difficult to incorporate them in tractable optimization models.

Pacheco Paneque et al. (2017) propose a linear formulation of the choice probabilities obtained by relying on simulation to draw from the distribution of the error term of the utility function. For each customer n and alternative i, a set R of draws are extracted from the known error term distribution, corresponding to different behavioral scenarios. For each scenario $r \in R$, the error term parameter ξ_{inr} is drawn and the utility becomes equal to

$$U_{inr} = V_{in} + \xi_{inr}. \tag{1}$$

Customers then deterministically choose the alternative with the highest utility in each scenario, that is

$$P_{inr} = 1 \text{ if } U_{inr} = \max_{j \in I} U_{jnr} \text{ and } P_{inr} = 0 \text{ otherwise.} \tag{2}$$

Over multiple scenarios, the probability that customer n chooses alternative i is equal to the number of times the alternative is chosen over the number of draws, that is

$$P_{in} = \frac{1}{|R|} \sum_{r \in R} P_{inr}. \tag{3}$$

With a sufficient number of simulation draws, the obtained choice probabilities approximate the analytical formulation within a confidence interval. Expression (2) can be linearized and inserted as lower-level constraint in a mixed integer optimization model that optimizes supply decisions.

3.2 Supply modeling

Suppliers are modeled as profit maximizers, according to the traditional microeconomic treatment and without loss of generality, but their objective functions could also include indicators other than profit. To optimize their objective function, suppliers make strategic decisions about the availability of their products on the market and the corresponding attributes such as price and quantity. We assume that suppliers choose their strategies according to their knowledge of demand at a disaggregate level.

Choice-based optimization models have been proposed to incorporate discrete choice models of consumer behavior into the optimization problem of the suppliers. In the literature, applications of choice-based optimization models include revenue management (Andersson, 1998, Talluri and Van Ryzin, 2004, Vulcano et al., 2010) and facility location (Benati and Hansen, 2002, Haase, 2009), among others. A majority of these works propose non-linear formulations and estimate consumer choice probabilities with the logit model, whose advantage is the existence of a closed-form expression.

Here, we introduce both the traditional non-linear formulation and the linearized formulation proposed by Pacheco Paneque et al. (2017). The latter is also applicable to models that do not have a closed-form expression of the choice probabilities, such as the mixed logit. In addition to the notation used in Section 3.1, consider a supplier k participating in the market and let $I_k \subset I$ indicate the subset of alternatives controlled by the supplier. The parameters of the endogenous variables of the discrete choice model are indicated with β, while the exogenous variables and the corresponding parameters can be grouped in the term q_{in} for each alternative and individual. Exogenous variables are all those which are not affected by the decisions of the supplier, such as the socio-economic characteristics of the customers and the attributes of the alternatives $i \notin I_k$. Additionally, let $\mathcal{S}_k$ be the set of strategies that can be selected by the supplier. Each strategy $s \in \mathcal{S}_k$ is composed of a vector of decision variables, which we can separate into the vector p of all prices p_{in}, potentially differentiated for each (class of) customer $n \in N$ and alternative $i \in I_k$, and a generic vector X of all other decision variables, which can also be alternative-specific, customer-specific or both. At this point, no assumption is made on the strategy set, which could be finite or infinite, or on the type of decision variables, which could be discrete or continuous.

Then, the non-linear version of the supplier's optimization problem can be written as follows:

$$
\begin{aligned}
\max_{s} \quad & \pi_s = \sum_{i \in I_k} \sum_{n \in N} p_{in} P_{in} - c(X), && (4)\\
\text{s.t.} \quad & P_{in} = \Pr(V_{in} + \varepsilon_{in} \geq V_{jn} + \varepsilon_{jn} \quad \forall j \in I) \qquad \forall i \in I, \forall n \in N, && (5)\\
& V_{in} = \beta_{p,in} p_{in} + \beta_{in} X_{in} + q_{in} \qquad \forall i \in I, \forall n \in N. && (6)
\end{aligned}
$$

The objective function (4) maximizes the profit π of the supplier, calculated as the difference between the expected revenues obtained from the sales and the cost of offering the products. Notice that the function is non-convex due to the presence of the choice probabilities, even with a logit model. Constraints (5) are the expressions of the choice probabilities. Constraints (6) define the deterministic utility functions, composed of an exogenous part q_{in} and an endogenous part which depends on the chosen strategy $s = (p, X)$, which links the upper-level problem with the lower-level problem.

For the linearized version of the model, we additionally define the auxiliary variables $U_{nr} = \max_i U_{inr}$, which capture the value of the highest utility for customer n in scenario r, while the binary decision variables P_{inr} identify the alternative i chosen by each customer n in each scenario r. Now constraints (5-6) can be replaced by constraints

(8-12) and the supplier's optimization problem can be written as follows:

$$\max_{s} \quad \pi_s = \frac{1}{|R|} \sum_{i \in I_k} \sum_{n \in N} \sum_{r \in R} p_{in} P_{inr} - c(X), \tag{7}$$

$$\text{s.t.} \quad U_{inr} = \beta_{p,in} p_{in} + \beta_{in} X_{in} + q_{in} + \xi_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{8}$$

$$U_{inr} \leq U_{nr} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{9}$$

$$U_{nr} \leq U_{inr} + M_{U_{nr}}(1 - P_{inr}) \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{10}$$

$$\sum_{i \in I} P_{inr} = 1 \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{11}$$

$$P_{inr} \in \{0, 1\} \qquad \forall i \in I, \forall n \in N, \forall r \in R. \tag{12}$$

The utility functions (8) now include a drawn error term. In (9-11) we use big-M constraints to ensure that in each behavioral scenario customers deterministically choose the alternative yielding the highest utility. The resulting model is discrete due to the binary choice variables P_{inr} defined in (12).

3.3 Market modeling

Let us consider an oligopolistic market where demand is modeled as in Section 3.1 and supply as in Section 3.2. Because of imperfect competition, the payoff of each supplier is a function of both the decisions of the customers and the strategies of the competitors, and all suppliers simultaneously solve a choice-based optimization problem. The result is a non-cooperative multi-leader-follower game, for which we search for pure strategy Nash equilibrium solutions.

The fixed-point iteration algorithm is a common approach to search for Nash equilibria of simultaneous games in competitive markets. In transportation, examples include Fisk (1984) and Adler (2001), among others. Starting from an initial market configuration, a numerical procedure is used which solves best response problems in a sequential manner, until a solution already reached in one of the previous iterations is repeated. The outcome of this procedure is either a pure strategy Nash equilibrium for the game or a set of strategies for each player which would continue to be played cyclically. Modeling competition in a sequential manner is attractive from a computational perspective, since the complexity of the algorithm is equivalent to the complexity of the Stackelberg game presented in Section 3.2. However, the convergence proof of the algorithm depends on conditions that include having convex payoff functions, which are not verified in a multi-leader-follower game with a disaggregate demand model. Therefore, when solving the problem with a fixed-point iteration algorithm, there is no guarantee that a pure strategy Nash equilibrium exists or, if one is found, that it is unique, since different initial configurations could lead to different equilibria.

We introduce here a mixed integer optimization model inspired by the fixed-point iteration algorithm. The purpose of this model is to solve the simultaneous game with a one-step model by considering only two iterations of the fixed-point algorithm. We define as *distance* the non-negative value measuring the sum of the profit differences between an initial unknown solution and each player's simultaneous best response to

the initial solution. If we start from an equilibrium solution of the problem, the distance between the initial profits and the best response profits is equal to 0. On the other hand, if we do not start from an equilibrium solution, the distance is greater than 0, since at least one of the players changes its strategy to improve its profit. In principle, other metrics could be utilized instead of profits to measure the deviation from the initial configuration, such as supply decisions or customer decisions.
To formalize, let K represent the set of suppliers, each controlling a subset of the alternatives that are available to the customers. We impose that $\cup_{k\in K} I_k \subset I$, in order not to have a captive market and allow customers to leave it without purchasing. Each supplier $k \in K$ has a finite set of strategies $\mathcal{S}_k$ from which to choose. In presence of variables such as prices, which are usually modeled as continuous variables, we assume that a finite subset can be derived. We define the vector parameters p_s and X_s of the prices and of the other decisions of supplier k playing strategy $s \in \mathcal{S}_k$. Additionally, let $s_{K\setminus\{k\}}$ be the observed strategies chosen by all suppliers other than k. If we define as π_s the payoff obtained by supplier k when choosing strategy s, then in order to find a Nash equilibrium solution we need to verify that

$$\pi_k^{max} = \pi_s = \max_{s\in\mathcal{S}_k} \pi_s(s, s_{K\setminus\{k\}}) \qquad \forall k \in K. \tag{13}$$

We define the binary decision variables x_s, which are equal to 1 if strategy $s \in \mathcal{S}_k$ is the best response of supplier k to the initial configuration and 0 otherwise. Finally, the superscripts $'$ and $''$ are used to indicate the variables of the initial configuration and of the best response configurations, respectively.
Then, the fixed-point optimization model with linearized choice probabilities can be written as follows:

$$\min \sum_{k\in K} (\pi_k^{''} - \pi_k^{'}), \tag{14}$$

s.t. Initial configuration:

$$U_{inr}^{'} = \beta_{p,in} p_{in}^{'} + \beta_{in} X_{in}^{'} + q_{in} + \xi_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{15}$$

$$U_{inr}^{'} \leq U_{nr}^{'} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{16}$$

$$U_{nr}^{'} \leq U_{inr}^{'} + M_{U_{nr}}(1 - P_{inr}^{'}) \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{17}$$

$$\sum_{i\in I} P_{inr}^{'} = 1 \qquad \forall i \in I, \forall n \in N, \forall r \in R, \tag{18}$$

$$\pi_k^{'} = \frac{1}{|R|} \sum_{i\in I_k} \sum_{n\in N} \sum_{r\in R} p_{in}^{'} P_{inr}^{'} - c(X^{'}) \qquad \forall k \in K, \tag{19}$$

Final configuration:

$$U_{inrs}^{''} = \beta_{p,in} p_{ins}^{''} + \beta_{in} X_{ins}^{''} + q_{in} + \xi_{inr} \quad \forall i \in I_k, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{20}$$

$$U_{inrs}^{''} = U_{inr}^{'} \qquad \forall i \notin I_k, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{21}$$

$$U_{inrs}^{''} \leq U_{nrs}^{''} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{22}$$

$$U_{nrs}^{''} \leq U_{inrs}^{''} + M(1 - P_{inrs}^{''}) \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{23}$$

$$\sum_{i\in I} P_{inrs}^{''} = 1 \qquad \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \tag{24}$$

Best response constraints:

$$\pi_s = \frac{1}{|R|} \sum_{i \in I_k} \sum_{n \in N} \sum_{r \in R} p''_{ins} P''_{inrs} - c(X'') \qquad \forall s \in \mathcal{S}_k, \forall k \in K, \quad (25)$$

$$\pi_s \leq \pi''_k \qquad \forall s \in \mathcal{S}_k, \forall k \in K, \quad (26)$$

$$\pi''_k \leq \pi_s + M(1 - x_s) \qquad \forall s \in \mathcal{S}_k, \forall k \in K, \quad (27)$$

$$\sum_{s \in \mathcal{S}_k} x_s = 1 \qquad \forall k \in K, \quad (28)$$

$$P'_{inr}, P''_{inrs} \in \{0, 1\} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K, \quad (29)$$

$$x_s \in \{0, 1\} \qquad \forall s \in \mathcal{S}_k, \forall k \in K. \quad (30)$$

The objective function (14) minimizes the sum over all the suppliers of the difference between the final and the initial profits. Constraints (15-18) define the utilities and impose that customers choose the alternative with the highest utility in the initial configuration. Constraints (19) calculate the profits in the initial configuration. Constraints (20-24) impose the utility maximization principle in the best response configurations. Here, utilities are evaluated for all strategies of all suppliers. In each strategic scenario, the decisions of the optimizing supplier only affect the utility of its alternatives (20), while the utilities of the competitors' alternatives remain unchanged with respect to the initial configuration (21). Finally, constraints (25-28) state that each supplier selects the best response strategy to the initial configuration.
This model is a one-step approach to find Nash equilibrium solutions for a competitive market with finite strategy sets by means of any mixed integer optimization solver. Starting from an initial unknown configuration, the model requires a number of strategic scenarios to be solved that is equal to $\sum_{k \in K} |\mathcal{S}_k|$. The model aims at identifying an equilibrium solution, if any exists, that satisfies both utility maximization and profit maximization principles. If no Nash equilibrium exists, the model finds an optimal near-equilibrium solution that minimizes the sum of the sum of the profit increase of the players. Notice that it is possible to discriminate between different equilibrium and near-equilibrium solutions by modifying the objective function to consider other indicators such as total profits or social welfare.

4 Algorithmic solution

Because of its complexity, it is not realistic to solve the fixed-point optimization model (14-30) to optimality to find equilibrium or near-equilibrium solutions of oligopolistic markets with large strategy sets and disaggregate demand. Indeed, the problem is highly combinatorial. Therefore, we propose a model-based algorithmic approach that consists of three blocks: (i) identify candidate equilibrium solutions or regions in a fast and efficient way; (ii) use the fixed-point optimization model introduced in Section 3.3 on restricted strategy sets to find subgame equilibria; (iii) verify if best response conditions are satisfied for the initial problem and, if they are not, add strategies to the restricted problem. This algorithmic approach finds one or more ε-equilibrium solutions of the problem, which can then be analyzed for decision-making or regulatory purposes. The pseudocode of the proposed solution is presented in Algorithm 1.

Algorithm 1: Algorithmic solution

Input : A set I of alternatives
A set K of suppliers
A heterogeneous population
An estimated discrete choice model of consumer behavior

Output: A list E of ε-equilibrium solutions

$E \leftarrow \emptyset$
repeat
  Assign a strategy $s_k \in \mathcal{S}_k$ to each competitor $k \in K$
  to create an initial market configuration $\mathbb{S} = \cup_{k \in K} s_k$
  repeat
    for $k \in K$ **do**
      Solve best response problem (7-12) for supplier k and obtain s_k^{new}
      Update solution $\mathbb{S}$ by assigning $s_k \leftarrow s_k^{new}$
  until $\mathbb{S}$ *is a previously visited solution*
  Define restricted strategy sets $S_k \subset \mathcal{S}_k$ from the strategies in $\mathbb{S}$
  repeat
    Solve fixed-point (FP) problem (14-30) for strategy sets S_k and
    obtain subgame equilibrium solution $\mathbb{S}^{FP} = \cup_{k \in K} s_k^{FP}$ and profits π_k^{FP}
    for $k \in K$ **do**
      Solve best response (BR) problem (7-12) for supplier k and
      obtain s_k^{BR} and π_k^{BR}
      if $\pi_k^{BR} > (1+\varepsilon)\pi_k^{FP}$ **then**
        Add best response strategy s_k^{BR} to the restricted strategy set S_k
  until $\pi_k^{BR} \leq (1+\varepsilon)\pi_k^{FP}$ *for all* $k \in K$
  $\mathbb{S}^{FP}$ is an ε-equilibrium solution of the problem
  Add $\mathbb{S}^{FP}$ to list E
until *stopping criterion is satisfied*

4.1 Heuristic reduction of the search space

Starting from a computationally intractable equilibrium problem, we first want to heuristically reduce the search space. On the supply side, the decision space of each supplier is, in principle, very large. This is particularly true when the strategy chosen by the supplier is the result of multiple interconnected decisions. However, in real-life markets constraints usually exist which define relationships between different

products, such as fare classes in airlines or discount levels on off-peak scheduled trains in railways. Including such problem-specific constraints can help reducing the search space and circumscribing potential equilibrium regions. On the demand side, using the classic non-linear formulation of the discrete choice probabilities (model 4-6) when solving the best response problem is faster than using the simulation-based linear formulation (model 7-12) only for simple choice models such as the logit. However, the computational performance of the non-linear formulation rapidly deteriorates in case of more complex choice models or discrete supply decisions, since derivative-based approaches are not adequate to prove optimality. Notwithstanding this limitation, at this stage any of the two formulations as well as any other heuristic that finds near-optimal solutions of the choice-based optimization model can be used.
In the experiments described in Section 5, we initially solve the competitive game in a sequential manner (lines 3-8), following the approach used by Adler et al. (2010), among others. More specifically, we define an initial feasible market configuration (line 3) and we rely on the linear formulation to solve best response problems based on the updated market conditions (lines 4-7) until we reach a state of the market that was already visited in one of the previous iterations (line 8). Experimentally, we observe that the algorithm converges to a bounded region of the solution space within few iterations, reaching a cyclic equilibrium in which the solution iterates over a finite set of best response strategies.

4.2 Exact solution of the restricted problem

Given one or more candidate equilibrium regions, we use the fixed-point optimization model (14-30) to find subgame equilibria. The size of the problems that can be solved exactly with this model is limited, since the model is combinatorial on the sets I, N, R and S. However, the results of the first block allow us to define restricted strategy sets S_k for all competitors k (line 9), which in turn produce tighter bounds on the utility functions of the customers. Within a limited range of supply decisions, we observe that small changes in the strategies produce small changes in the utility functions of the customers, rarely affecting the ordinal ranking of the utilities of the alternatives in their choice set.
The simulation of the error term of the utility function and the consequent use of binary variables to model choices in each scenario r make it possible to precompute choices of customer n by comparing the lower and upper bounds of the utilities. In particular,

$$LB(U_{inr}) > \max_{j \in I: j \neq i} UB(U_{jnr}) \implies \begin{cases} w_{inr} = 1 \\ w_{jnr} = 0 \quad \forall j \in I, j \neq i. \end{cases} \tag{31}$$

If this condition is verified, customer n is captive to alternative i in scenario r. Therefore, this scenario can be removed from the optimization model. Using restricted strategy sets with tight bounds can substantially reduce the size of the fixed-point optimization model to be solved, since the optimal strategies at equilibrium are ultimately determined by a subset of undecided customers who select one among a subset of alternatives.

4.3 Verification of best response conditions

The solution obtained by solving the restricted problem to optimality with the fixed-point model is a subgame equilibrium or near-equilibrium (line 11), which must be verified on the original game by solving best response problems for all competitors (lines 12-13). If no supplier can increase its profit by more than a predefined tolerance value ε by solving its best response problem, that is,

$$\pi_k^{BR} \leq (1+\varepsilon)\pi_k^{FP} \qquad \forall k \in K, \tag{32}$$

then the subgame equilibrium is accepted as ε-equilibrium solution of the game (lines 17-18). Contrarily, if at least one competitor can increase its profits by more than ε, the subgame equilibrium is not accepted as game equilibrium and the best response strategy is added to the restricted set (line 15). The fixed-point model is then solved again with updated restricted strategy sets, following a column-generation-like approach.
The three blocks of the algorithmic solution combine a heuristic exploration of the solution space with the use of exact methods to solve subproblems. After finding an ε-equilibrium solution, the algorithm is restarted until either a minimum number of ε-equilibrium solutions (not necessarily with the same value of ε) are found, a time limit is reached or the whole solution space is explored (line 19). Finally, different ε-equilibrium solutions can be compared in terms of stability, dominance, tacit collusion or social welfare, both from the point of view of the suppliers and from that of a regulator.

5 Case studies

In this section, we show the applicability of the model-based algorithmic framework outlined in Section 4 to two case studies, for which real, non-trivial, disaggregate choice models are published in the literature: (i) an urban parking choice case study, for which a mixed logit model estimation is taken from Ibeas et al. (2014); (ii) an intercity mode and departure time choice case study in the context of high-speed rail competition in Italy, for which a nested logit model estimation is taken from Cascetta and Coppola (2012).

5.1 Parking choice

Ibeas et al. (2014) use a mixed logit model to study car driver's behavior when choosing among three different parking alternatives available in a small Spanish town: free on-street parking (FSP), paid on-street parking (PSP) and paid underground parking (PUP). The explanatory variables considered to estimate the discrete choice model include trip origin, age of the vehicle, income and area of residency of the driver, access time to destination, access time to parking and parking fee. For the latter two variables, the corresponding coefficients are normally distributed in the utility function.

Recall that the choice probabilities of the mixed logit model do not have a closed form and must be estimated using numerical integration or approximation by simulation (McFadden and Train, 2000). While simulation constitutes an additional computational burden in non-linear formulations, it adapts well to our modeling framework, since we can apply the same technique used to linearize the utility expression (1), that is, drawing from the known distribution of the error term.
By doing so, we obtain

$$U_{inr} = \beta_{p,inr} p_{in} + q_{inr} + \xi_{inr}, \tag{33}$$

where $\beta_{p,inr}$ and q_{inr} now vary for each draw r, since the parking fee and the access time to parking parameters are associated with the endogenous price variable and with the exogenous access time to parking variable, respectively.
In our test scenario, we assume that the two paid parking options are owned by two different suppliers, while free parking is considered as the opt-out option. The two suppliers compete on price and we assume that all customers must pay the same price for an alternative. Fixed and variable costs exist to operate the two alternatives, and both suppliers aim at maximizing profits. The size of the set of customers N is fixed and is equal to 50.
Following Algorithm 1, a strategy is initially assigned to each competitor. In this case, the strategies of suppliers 1 and 2 correspond to a single price decision on PSP and PUP, respectively. Then, best response problems are solved iteratively until a solution is repeated, meaning that an equilibrium or a cyclic equilibrium is found. Figure 1 shows the evolution of prices and profits during the iterative procedure for two different initial market configurations. Here, we observe that the fixed-point iteration method converges to a well-defined area of the solution space, irrespective of the starting point. We conjecture that this is due to the relative simplicity of the supply strategic decision.
Next, restricted sets are generated which include strategies in the candidate equilibrium region of both suppliers. The column-generation-like technique solves the restricted problem to find a subgame equilibrium, verifies whether it is a ε-equilibrium of the original problem and, if that is not the case, adds new strategies to the restricted problem. An example of how this procedure works is illustrated in Table 1. The first subgame includes 6 strategies for each supplier. The fixed-point optimization model yields a subgame equilibrium in which $PSP = 0.531$ and $PUP = 0.683$. Then, best response problems are solved on the original strategy sets, finding that the subgame equilibrium is not a game ε-equilibrium, since supplier 2 can increase its profit by 2.2% by setting $PUP = 0.667$. The best response strategies are then added to the restricted sets, and the subgame will therefore have 7 strategies for each supplier in the next iteration. Finally, a game ε-equilibrium solution is found after the third iteration, with $PSP = 0.519$ and $PUP = 0.662$.
The algorithm is restarted until a predefined number of ε-equilibrium solutions is found, with different values of ε, to verify whether different initial states lead to significantly different equilibrium regions. Table 2 lists these solutions. We observe that all ε-equilibrium solutions of this competitive market are very similar. Indeed, in this case study, solving the game with the fixed-point iteration method is sufficient to

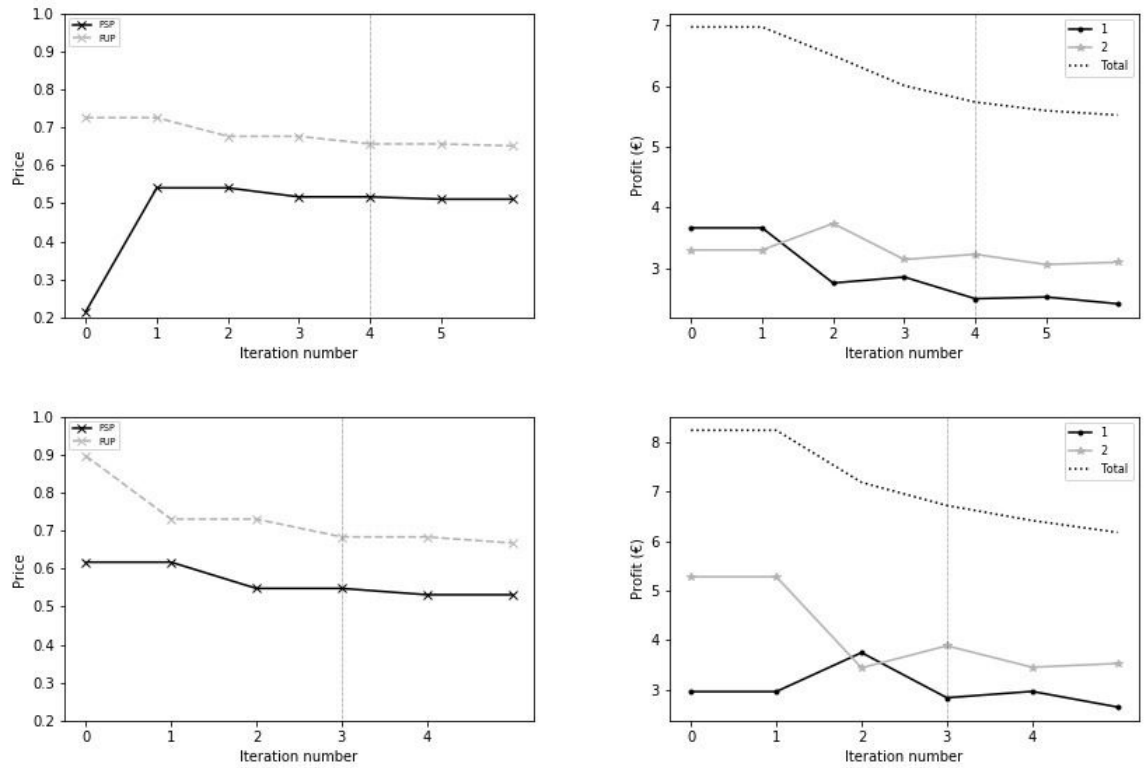

Figure 1: Evolution of prices and profits when solving best response problems iteratively for two different initial states of the market in the parking choice case study.

provide market insights, and the following steps of the algorithmic framework provide little added value. This is generally not true for more complex markets, as the next case study shows.

To conclude, with this simple example we show that it is possible to include an advanced discrete choice model without closed-form expressions of the choice probabilities in an equilibrium problem. Indeed, the strength of the linearized choice-based optimization model is that it copes with the presence of normally distributed parameters in the mixed logit utility specification without increases in computational times with respect to the logit model.

In the next case study, we analyze a market with more alternatives and more complex strategic problems on the supply side, which justifies the need of a scalable model-based algorithmic framework.

5.2 Schedule-based high-speed rail competition

We consider a competitive high-speed rail market with two companies which operate four and three direct scheduled services connecting Milan to Rome in a typical morning peak period, respectively. Train departure times and travel times are assumed to be exogenously given. Trains depart from Milan between 6:00 and 7:30 and arrive in Rome between 3 hours and 3 hours and 30 minutes later. We endogenously model the pricing

Alternative	| S |	Bounds		Subgame eq.		Best response		ε
		LB	UB	Price	Profit	Price	Profit	
PSP	6	0.531	0.548	0.531	2.959	0.531	2.964	0.002
PUP	6	0.672	0.683	0.683	3.453	0.667	3.528	0.022
PSP	7	0.531	0.548	0.535	2.662	0.519	2.716	0.020
PUP	7	0.667	0.683	0.667	3.562	0.662	3.602	0.011
PSP	8	0.519	0.548	0.519	2.601	0.517	2.626	0.009
PUP	8	0.662	0.683	0.662	3.265	0.658	3.273	0.002

Table 1: Illustration of the algorithm to find an ε-equilibrium solution for the parking choice case study, with $\varepsilon = 0.01$.

Equilibrium		Prices		Profits		Market shares		
#	ε	PSP	PUP	PSP	PUP	FSP	PSP	PUP
1	0.3%	0.510	0.663	2.565	3.119	0,090	0,466	0,444
2	0.4%	0.511	0.656	2.446	3.138	0,093	0,441	0,466
3	0.4%	0.517	0.663	2.560	3.275	0,099	0,438	0,463
4	0.9%	0.519	0.662	2.604	3.265	0,098	0,438	0,465
5	0.0%	0.508	0.645	2.322	3.031	0,083	0,430	0,487

Table 2: List of ε-equilibrium solutions of the parking choice case study.

strategies of the two operators and we assume that they must decide on a price at which to sell tickets for each scheduled departure time. For the sake of simplicity, no price differentiation based on customer groups or time of booking is considered. Additionally, we include two scheduled flights, a slow intercity train and private transport as opt-out alternatives for which prices are fixed and exogenously given.

Research on public transport has shown that frequency-based representations of timetables could be suitable to model travel choices in high-frequency urban transit systems, but they are not appropriate to model choices of intercity trips, since the schedule of the available services affects the decision of the customer (Cascetta and Coppola, 2016). In the context of high-speed rail pricing in a competitive environment, it is therefore fundamental for the suppliers to take into account the desired arrival times of customers and to estimate their willingness-to-pay to avoid early/late departures/arrivals. We follow the schedule-based approach introduced by Nuzzolo et al. (2000) and applied by Cascetta and Coppola (2012), which allows to investigate a railway demand model that, contrarily to frequency-based representations, explicitly models each train service. This allows to assign penalties for early departure and late arrival times with respect to the user's desired departure and arrival times, respectively. By using such detailed representation of the choice set, it is possible to evaluate the effect of pricing policies and variations in the schedule of single trains. To the best of our knowledge, there has not been any attempt to model departure time choices at a disaggregate level when modeling pricing decisions in a competitive rail market.

We generate a synthetic population of about 1000 travelers for the Milan-Rome OD pair. Each individual has a trip purpose (business or other), specific origin and destination locations which lead to different access and egress times to and from terminals, and a desired arrival time at destination between 9:00 and 11:00 which follow a non-homogeneous Poisson process. For the discrete choice model, we refer to the model estimated by Cascetta and Coppola (2012), which is derived from a RP/SP survey dataset collected at the national level. The authors estimate the parameters separately for business and other trip purpose. In both cases, they use a nested logit model in which the three nests capture the correlation between the scheduled services of the each of the two train operators and of the airline. Since the choice model was estimated on SP data in 2010 before the market entry of the second high-speed rail operator, we calibrated the alternative-specific constants of the choice model to reflect more recent modal split estimates (International Rail Journal, 2019). For the calibration, we follow the procedure described in Hensher et al. (2005), that is, we constrain the parameters for all attributes and socio-demographic characteristics to be equal to those obtained by Cascetta and Coppola (2012) and we let the alternative-specific constants free to vary to reflect the current observed modal split. We remark that the dataset used for the experiments and the derived results are hypothetical and do not represent real scenarios that are related to choices made by the current high-speed rail operators.

Aside from problem-specific constraints and parameter setting, this case study presents two additional sources of complexity: on the demand side, the nested logit model is used to capture the similarities existing within subsets of alternatives which share observed and unobserved attributes; on the supply side, each high-speed train operator makes a simultaneous decision on the price of multiple alternatives.

We recall that, in the nested logit model, the deterministic utility V_{in} of customer n for alternative i belonging to nest m is affected by the utilities of all the other alternatives j that belong to the same nest by means of a logsum term:

$$V_{in} = V'_{in} + \log(\mu \exp(V'_{in}(\mu_m - 1)))(\sum_{j \in m} \exp(V'_{jn}\mu_m))^{\frac{\mu}{\mu_m} - 1}), \tag{34}$$

where V'_{in} represents the logit utility, μ is a scale parameter which is commonly normalized to 1 and $\mu_m \geq 1$ is a scale parameter that expresses the correlation between alternatives in the same nest.

By inserting (34) into our linearized utility expression (1), we obtain

$$U_{inr} = V'_{in} + \log(\mu \exp(V'_{in}(\mu_m - 1)))(\sum_{j \in m} \exp(V'_{jn}\mu_m))^{\frac{\mu}{\mu_m} - 1}) + \xi_{inr}, \tag{35}$$

where $V'_{in} = \beta_{p,in} p_i + q_{in}$ and $V'_{jn} = \beta_{p,jn} p_j + q_{jn}$. Expression (35) is non-linear due to the presence of endogenous variables, i.e. the prices p_j of all high-speed train services, in the logsum term. To overcome this issue, we initially fix the values of the endogenous variables in the logsum term and only optimize for those in the logit term. By doing so, the supply optimization problem is solved as a mixed integer linear problem. Of course, the initial logsum values need to be updated and the problem solved again in an iterative fashion, until reaching convergence.

For the first heuristic block of our algorithmic framework, computational experiments show that, when all scale parameters $\mu_m < 1.5$ as in this case study, this iterative

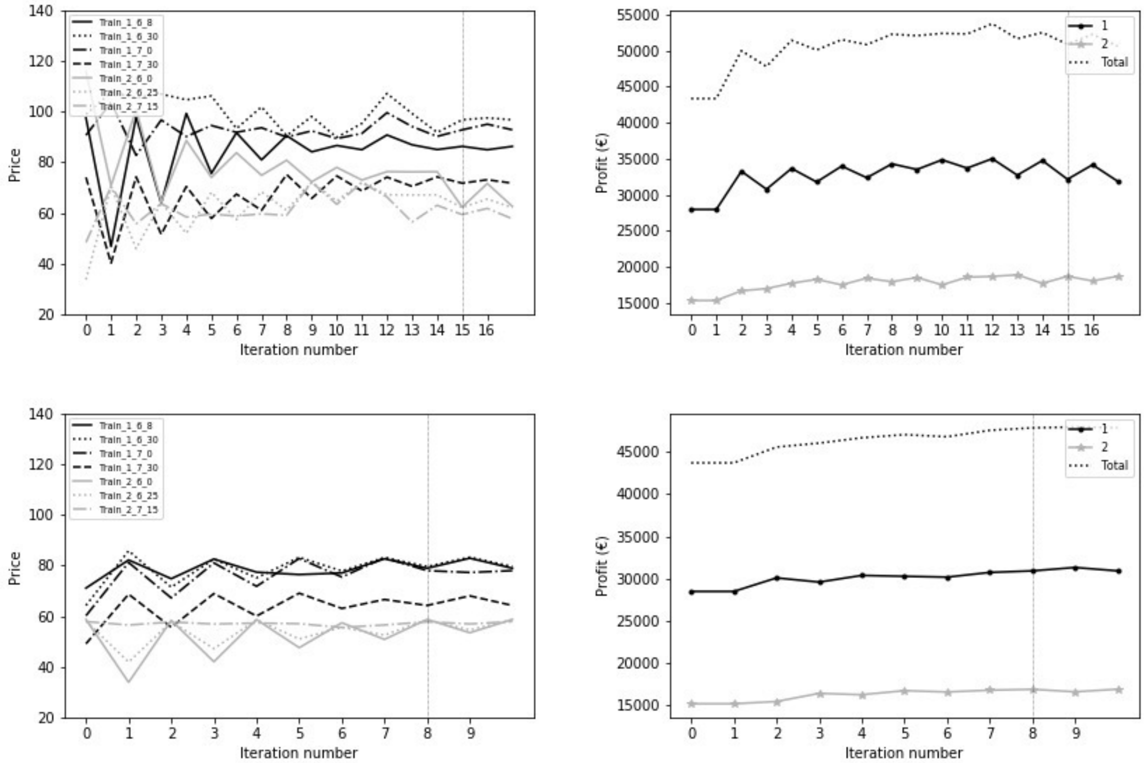

Figure 2: Evolution of prices and profits when solving best response problems iteratively for two different initial states of the market in the high-speed rail case study.

optimization procedure is computationally tractable and convergence between the logsum variables and the linear variables is reached in parallel with the convergence to an equilibrium region of the competitive game. Preliminary tests indicate that convergence might be slower to reach for higher values of μ_m, since small changes in the values of the endogenous variables have large effects on the logsum terms, thus making the fixed-point method cumbersome. We hypothesize that implementing smoothing techniques for the update of the logsum terms could be beneficial, but further research should be conducted to derive more sound conclusions on this matter. The value of the parameters μ_m is less of an issue when solving for subgame equilibrium, provided that sufficiently tight bounds are imposed on the distance between the strategies included in the restricted sets.

Similarly to the parking choice case study, we apply Algorithm 1 as such to find ε-equilibrium solutions for the given market. We remark that in this case a supplier strategy corresponds to a bundle of price decisions on different alternatives. Therefore, tradeoffs exist between alternatives controlled by the same supplier which can lead to different strategic behaviors and different regions of equilibrium in the solution space.

In the first block, we solve the problem using the fixed-point iteration algorithm. Figure 2 shows the evolution of prices and profits for two different initial states of the market. For the first one, we can notice that the algorithm converges to an equilibrium region after a number of iterations (top-left graph). However, this is an unstable cyclic equilibrium, because each competitor has an interest to deviate from

Supplier	| S |	Alternative	Bounds		Subgame eq.		Best response		ε
			LB	UB	Price	Profit	Price	Profit	
HSR1	5	6:08	103.02	103.76	103.02	33955	96.44	35034	0.032
		6:30	111.42	112.35	111.42		105.04		
		7:00	100.70	103.53	100.70		97.59		
		7:30	86.18	88.68	86.18		73.50		
HSR2	5	6:00	73.46	80.91	73.46	20163	71.97	20891	0.036
		6:25	72.57	75.42	72.57		67.61		
		7:15	70.69	82.08	70.69		61.80		
HSR1	6	6:08	96.44	103.76	96.44	35182	85.91	35907	0.021
		6:30	105.04	112.35	105.04		88.17		
		7:00	97.59	103.53	97.59		85.55		
		7:30	73.50	88.68	73.50		78.30		
HSR2	6	6:00	71.97	80.91	79.04	19249	65.34	19835	0.030
		6:25	67.61	75.42	74.71		65.34		
		7:15	61.80	82.08	79.23		66.11		
...	...	...	...	...	...	...	...	...	...
HSR1	11	6:08	83.66	103.76	85.91	33880	91.87	34112	0.007
		6:30	87.75	112.35	88.17		92.99		
		7:00	83.21	103.53	85.55		90.11		
		7:30	73.50	88.68	78.30		76.43		
HSR2	11	6:00	58.67	80.91	66.76	18323	67.41	18562	0.013
		6:25	58.50	75.42	66.76		67.31		
		7:15	58.05	82.08	65.71		65.07		

Table 3: Illustration of the algorithm to find an ε-equilibrium solution for the high-speed rail case study, with $\varepsilon = 0.015$.

the current state by best-responding to the rival to increase its profits (top-right graph). For the second one, a seemingly more stable equilibrium is reached with lower average prices and lower profits for both players.

Using the fixed-point iteration algorithm multiple times with different initial configurations, however, is insufficient to model the competitive behavior of the suppliers, because neither the stability of the equilibria nor the potential existence of tacit collusion to deviate from the purely non-cooperative outcome are taken into account. On the other hand, relying on the fixed-point optimization model, albeit on restricted strategy sets, allows to search for subgame equilibrium solutions in which best response conditions are satisfied by all suppliers simultaneously. The application to this case study of the column-generation-like technique that forms the second and third blocks of the algorithmic approach is illustrated in Table 3. Notice that the price bounds for the fixed-point model are updated whenever new best response strategies are added to the restricted sets. In principle, strategies can also be removed to keep the restricted sets small and the bounds tight to precompute as many captive customer choices as possible. Finally, the value of ε is subject to problem-specific considerations on competitive behavior and can be updated during execution. In these experiments, ε is set between 0 and 0.02.

Finally, ten ε-equilibrium solutions with different values of ε are found by restarting the algorithm to explore different candidate equilibrium regions. The list, which is not an exhaustive list of all possible ε-equilibria, is provided in Table 4. Each row

Equilibrium		Prices										
		Opt-out				HSR Operator 1				HSR Operator 2		
#	ε	Car	Air1	Air2	IC	6:08	6:30	7:00	7:30	6:00	6:25	7:15
1	1.3%	100.00	80.00	80.00	30.00	89.08	104.63	95.29	73.98	72.05	67.47	60.47
2	1.5%	100.00	80.00	80.00	30.00	97.83	97.83	93.46	71.27	58.69	58.57	58.57
3	1.3%	100.00	80.00	80.00	30.00	85.78	87.36	85.51	78.41	73.32	67.05	65.27
4	1.1%	100.00	80.00	80.00	30.00	86.96	99.57	93.48	73.13	70.31	70.21	69.60
5	0.9%	100.00	80.00	80.00	30.00	84.87	96.68	92.73	71.70	64.76	64.41	57.85
6	0.7%	100.00	80.00	80.00	30.00	87.87	88.15	83.00	78.58	72.00	70.16	65.03
7	1.3%	100.00	80.00	80.00	30.00	85.92	88.17	85.55	78.30	66.76	66.76	65.71
8	0.9%	100.00	80.00	80.00	30.00	84.99	103.72	95.74	72.42	65.12	65.12	65.01
9	1.0%	100.00	80.00	80.00	30.00	97.14	99.14	91.94	87.86	65.38	65.30	61.74
10	1.4%	100.00	80.00	80.00	30.00	101.78	101.78	89.53	86.29	65.71	65.44	64.78

Table 4: List of ε-equilibrium solutions of the HSR case study.

Equilibrium		Avg ticket cost		Revenues		Market shares		
#	ε	HSR1	HSR2	HSR1	HSR2	Opt-out	HSR1	HSR2
1	1.3%	88.53	66.35	34179	19636	0.344	0.369	0.287
2	1.5%	88.72	58.62	32010	19051	0.315	0.360	0.325
3	1.3%	84.46	68.74	34375	18450	0.341	0.407	0.252
4	1.1%	86.86	70.03	34236	18769	0.340	0.393	0.267
5	0.9%	85.38	62.13	33053	18536	0.329	0.385	0.286
6	0.7%	84.47	69.06	34717	18457	0.339	0.411	0.250
7	1.3%	84.70	66.35	33880	18323	0.331	0.400	0.269
8	0.9%	87.63	65.08	33914	19005	0.339	0.387	0.274
9	1.0%	94.18	63.92	33434	19796	0.353	0.355	0.292
10	1.4%	94.37	65.26	34160	19726	0.353	0.362	0.285

Table 5: Aggregate results for the ε-equilibrium solutions of the HSR case study.

corresponds to a different solution, for which the prices for all alternatives are reported. We observe that the price of an alternative can change up to 25% for different ε-equilibrium solutions. Table 5 presents aggregate prices, revenues and market shares for the ten ε-equilibrium solutions. We can observe that solutions 2, 5, 7 and 8 are Pareto dominated by solutions 1 and 10. The latter two require all players to accept at most 1.3% and 1.4% of revenue loss compared to their best response strategy, which however might lead the market further away from stability and produce a prisoner's dilemma outcome. It is interesting to notice that the dominated solutions have lower-than-average ticket costs and market shares for opt-out alternatives, meaning that, while not appealing to the oligopolists, they could be more desirable outcomes from the point of view of both the consumers and the regulator whose objective might be to reduce congestion or other externalities such as CO_2 emissions. These considerations exemplify the type of ex-post analyses that can be carried out either by the competitors or by an external market regulator who has access to estimations of disaggregate demand data.

6 Conclusion

In this paper, we presented a general framework to find ε-equilibrium solutions of oligopolistic markets in which demand is modeled at the disaggregate level using discrete choice models. Consumer choices are modeled according to random utility theory, and the choice probabilities are linearized by simulating the error term of the utility function and embedded as lower-level constraints in the bilevel programs that model the supply optimization problems. To model competition, we introduce a mixed integer optimization model based on the fixed-point iteration algorithm that finds an equilibrium or near-equilibrium solution of a finite game. The formulation is combinatorial and unsuitable to solve real-life problems with large strategy sets. Therefore, we propose a model-based algorithmic approach in which a heuristic reduction of the search space is first performed to identify candidate equilibrium regions. Restricted strategy sets are then generated for the suppliers and a subgame equilibrium problem is solved using the fixed-point model. Finally, the solution of the subgame is checked against the original strategy sets by solving best response problems for all suppliers. The algorithm can be restarted multiple times to explore different regions of the solution space and find several ε-equilibrium solutions. The algorithmic framework is applied to two examples of oligopolistic markets within the transport sector, namely parking and high-speed rail, for which advanced choice models, taken as such from the literature, are used to model demand. Numerical experiments achieve to find ε-equilibrium solutions that provide meaningful information for competing firms and policy-makers alike.

The interaction between demand and supply models is a fundamental problem in the analysis of imperfect competition, and aggregation techniques have been largely utilized to obtain demand functions that could be used in equilibrium models. This research shows that including disaggregate demand in a model of oligopolistic competition is a largely unexplored area of research with numerous applications in sectors such as transportation. The potential of our methodology needs to be evaluated in light of the advances in discrete choice modeling, which allow for increasingly complex and precise representations of individual behavior. In particular, there is a vast literature on choice models with latent variables, modeling subjective behavioral dimensions such as attitudes and perceptions. These models can be integrated in our framework exactly as done for the two examples provided in this paper.

The following research directions should be further investigated. The modeling framework can be tested on cases where price and product differentiation across customers are considered. Similarly, it is possible to adapt the models to include a time dimension, to let suppliers endogenously decide on capacity levels and fare classes based on the forecast demand and the customer time of arrival in the system to purchase a product, in a revenue management manner. The main constraint related to these extensions is the computational complexity caused by the multidimensional solution space that must be evaluated to solve the best response conditions to verify ε-equilibrium solutions. An ongoing parallel research stream is investigating the challenges associated with choice-based optimization using mixed integer optimization models. At this point, the applicability of our framework to large-scale problem depends on the capability to efficiently find exact solutions or tight bounds to such hard combinatorial problem. Furthermore, in the context of transportation, the framework can be extended to incorporate the role of a regulator, which could actively influence the market towards a social welfare maximization outcome. Finally, from a decision-making perspective, it would be interesting to analyze the stability of equilibrium with regard not only to competitor behavior, but also to demand stochasticity, which can be caused by uncertainties in the phase of demand estimation.

Acknowledgements

This research is supported by the Swiss National Science Foundation [Grant 175483 "New generation of demand-supply interaction models"]. We would like to express our gratitude to Nicolas Pradignac and Théophile Thiery for their contribution to the project.

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