Formulating discrete choice models as Mixed Integer Linear Problems (MILP)

Michel Bierlaire

Transport and Mobility Laboratory
School of Architecture, Civil and Environmental Engineering
Ecole Polytechnique Fédérale de Lausanne

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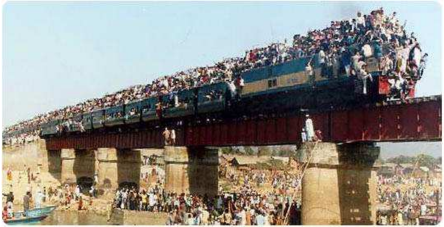
Outline

Demand models

- Supply = infrastructure
- Demand = behavior, choices
- Congestion = mismatch

Demand models



- Usually in OR:
- optimization of the supply
- for a given (fixed) demand

Aggregate demand

- Homogeneous population
- Identical behavior
- Price (P) and quantity (Q)
- Demand functions: $P = f(Q)$
- Inverse demand: $Q = f^{-1}(P)$

Disaggregate demand

- Heterogeneous population
- Different behaviors
- Many variables:
  - Attributes: price, travel time, reliability, frequency, etc.
  - Characteristics: age, income, education, etc.
- Complex demand/inverse demand functions.

Demand-supply interactions

Operations Research

- Given the demand...
- configure the system

Behavioral models

- Given the configuration of the system...
- predict the demand

Johnson City Enterprise.

Published Every Saturday,

$1. per year—Advance Payment.

SATURDAY, APRIL 7, 1883.

TIME TABLE

E. T., V. & G. R. R.

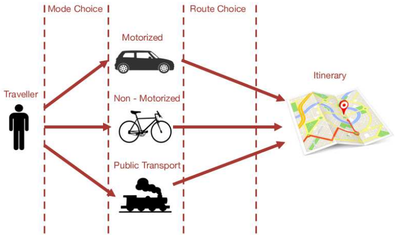
PASSENGER,	ARRIVES,
No. 1, West,	6:37, a. m.
No. 2, East,	9:45, p. m.
No. 3, West,	11:51, p.m.
No. 4, East,	3:56, a. m.
LOCAL FREIGHT,	ARRIVES,
No. 5,	7:20, a. m.
No. 8,	6:20, p. m.

JNO. W. EAKIN, Agent.

E. T. & W. N. C. R. R.

Passenger, leaves,	7, a. m.
" arrives,	6, p. m.

J. C. HARDIN, Agent.



Demand-supply interactions

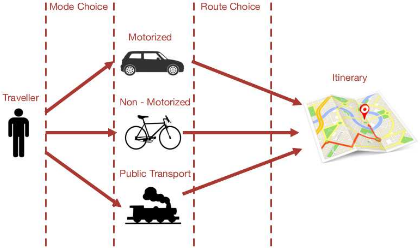
Multi-objective optimization

Minimize costs

Maximize satisfaction

Outline

Choice models



Behavioral models

- Demand = sequence of choices
- Choosing means trade-offs
- In practice: derive trade-offs from choice models

Choice models

Theoretical foundations

- Random utility theory
- Choice set: $\mathcal{C}_n$
- $y_{in} = 1$ if $i \in \mathcal{C}_n$, 0 if not
- Logit model:

$$P(i|\mathcal{C}_n) = \frac{y_{in} e^{V_{in}}}{\sum_{j \in \mathcal{C}} y_{jn} e^{V_{jn}}}$$

2000

Logit model

Utility

$$U_{in} = V_{in} + \varepsilon_{in}$$

Choice probability

$$P_n(i|\mathcal{C}_n) = \frac{y_{in} e^{V_{in}}}{\sum_{j \in \mathcal{C}} y_{jn} e^{V_{jn}}}.$$

- Decision-maker n
- Alternative $i \in \mathcal{C}_n$

Variables: $x_{in} = (p_{in}, z_{in}, s_n)$

Attributes of alternative i: z_{in}

- Cost / price (p_{in})
- Travel time
- Waiting time
- Level of comfort
- Number of transfers
- Late/early arrival
- etc.

Characteristics of decision-maker n: s_n

- Income
- Age
- Sex
- Trip purpose
- Car ownership
- Education
- Profession
- etc.

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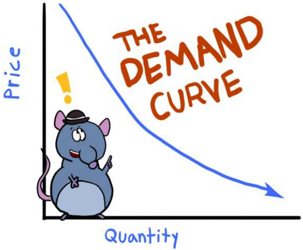
Demand curve



Disaggregate model

$$P_n(i|p_{in}, z_{in}, s_n)$$

Total demand

$$D(i) = \sum_n P_n(i|p_{in}, z_{in}, s_n)$$

Difficulty

Non linear and non convex in p_{in} and z_{in}

Outline

Choice-Based Optimization Models

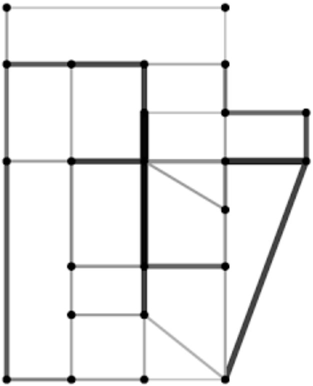
Benefits

- Merging supply and demand aspect of planning
- Accounting for the heterogeneity of demand
- Dealing with complex substitution patterns
- Investigation of demand elasticity against its main driver (e.g. price)

Challenges

- Nonlinearity and nonconvexity
- Assumptions for simple models (logit) may be inappropriate
- Advanced demand models have no closed-form
- Endogeneity: same variable(s) both in the demand function and the cost function

Stochastic traffic assignment



Features

- Nash equilibrium
- Flow problem
- Demand: path choice
- Supply: capacity

Selected literature

- [Dial, 1971]: logit
- [Daganzo and Sheffi, 1977]: probit
- [Fisk, 1980]: logit
- [Bekhor and Prashker, 2001]: cross-nested logit
- and many others...

Revenue management

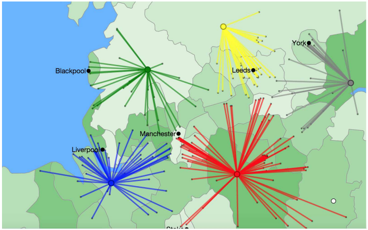
Features

- Stackelberg game
- Bi-level optimization
- Demand: purchase
- Supply: price and capacity

Selected literature

- [Labbé et al., 1998]: bi-level programming
- [Andersson, 1998]: choice-based RM
- [Talluri and Van Ryzin, 2004]: choice-based RM
- [Gilbert et al., 2014a]: logit
- [Gilbert et al., 2014b]: mixed logit
- [Azadeh et al., 2015]: global optimization
- and many others...

Facility location problem



Features

- Competitive market
- Opening a facility impact the costs
- Opening a facility impact the demand
- Decision variables: availability of the alternatives

$$P_n(i|\mathcal{C}_n) = \frac{y_{in} e^{V_{in}}}{\sum_{j \in \mathcal{C}} y_{jn} e^{V_{jn}}}.$$

Selected literature

- [Hakimi, 1990]: competitive location (heuristics)
- [Benati, 1999]: competitive location (B & B, Lagrangian relaxation, submodularity)
- [Serra and Colomé, 2001]: competitive location (heuristics)
- [Marianov et al., 2008]: competitive location (heuristic)
- [Haase and Müller, 2013]: school location (simulation-based)

Outline

A linear formulation

Utility function

$$U_{in} = V_{in} + \varepsilon_{in} = \sum_k \beta_k x_{ink} + f(z_{in}) + \varepsilon_{in}.$$

Simulation

- Assume a distribution for ε_{in}
- E.g. logit: i.i.d. extreme value
- Draw R realizations ξ_{inr}, $r = 1, \ldots, R$
- The choice problem becomes deterministic

Scenarios

Draws

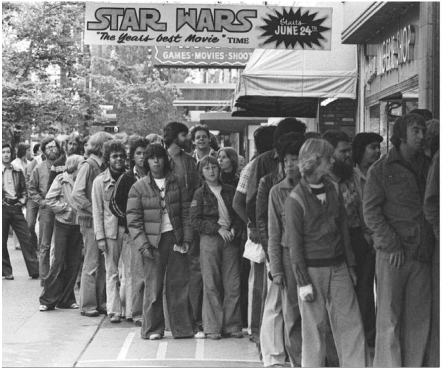
- Draw R realizations ξ_{inr}, $r = 1, \ldots, R$
- We obtain R scenarios

$$U_{inr} = \sum_k \beta_k x_{ink} + f(z_{in}) + \xi_{inr}.$$

- For each scenario r, we can identify the largest utility.
- It corresponds to the chosen alternative.

Capacities

- Demand may exceed supply
- Each alternative i can be chosen by maximum c_i individuals.
- An exogenous priority list is available.
- The numbering of individuals is consistent with their priority.



Priority list

Application dependent

- First in, first out
- Frequent travelers
- Subscribers
- ...

In this framework

The list of customers must be sorted

References

- Technical report: [Bierlaire and Azadeh, 2016]
- TRISTAN presentation: [Pacheco et al., 2016]
- STRC proceeeding: [Pacheco et al., 2017]

Demand model

- Population of N customers (n)
- Choice set $\mathcal{C}$ (i)
- $\mathcal{C}_n \subseteq \mathcal{C}$: alternatives considered by customer n

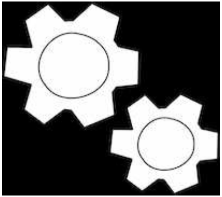
Behavioral assumption

- $U_{in} = V_{in} + \varepsilon_{in}$
- $V_{in} = \sum_k \beta_{ink} x^e_{ink} + q^d(x^d)$
- $P_n(i|\mathcal{C}_n) = \Pr(U_{in} \geq U_{jn}, \forall j \in \mathcal{C}_n)$

Simulation

- Distribution ε_{in}
- R draws $\xi_{in1}, \ldots, \xi_{inR}$
- $U_{inr} = V_{in} + \xi_{inr}$

Supply model



- Operator selling services to a market
  - Price p_{in} (to be decided)
  - Capacity c_i
- Benefit (revenue – cost) to be maximized
- Opt-out option ($i = 0$)

Price characterization

- Lower and upper bound
- Discretization: price levels
- Binary representation ($\lambda_{in\ell}$)

Capacity allocation

- Exogenous priority list of customers
- Here it is assumed as given
- Capacity as decision variable

MILP (in words)

MILP

max	benefit
subject to	utility definition
	availability
	discounted utility
	choice
	capacity allocation
	price selection

Variables

Availability

$y_i \in \{0,1\}$	services proposed by the operator
$y_{in} \in \{0,1\}$	$y_i = 1$ and services considered by customers
$y_{inr} \in \{0,1\}$	capacity restrictions

Utility and choice

U_{inr}	utility
z_{inr}	discounted utility
U_{nr}	maximum discounted utility
$w_{inr} \in \{0,1\}$	choice

Pricing

$\lambda_{in\ell} \in \{0,1\}$	binary representation of the price
$\alpha_{inr\ell} \in \{0,1\}$	linearization of the product $w_{inr}\lambda_{in\ell}$

MILP

MILP

max	benefit
subject to	**utility definition**
	availability
	discounted utility
	choice
	capacity allocation
	price selection

Utility

$$U_{inr} = \overbrace{\beta_{in} p_{in} + q_d(x_d)}^{V_{in}} + \xi_{inr} \quad \forall i, n, r \quad (1)$$

p_{in}	endogenous variable
β_{in}	associated parameter ($\beta_{0n} = 0$)
$q_d(x_d)$	exogenous demand variables

MILP

MILP

max	benefit
subject to	utility definition
	availability
	discounted utility
	choice
	capacity allocation
	price selection

$$y_{in}^d = \begin{cases} 1 & \text{if } i \in \mathcal{C}_n \\ 0 & \text{otherwise} \end{cases} \quad \forall i, n$$

Product of decisions

$$y_{in} = y_{in}^d y_i \quad \forall i, n \tag{2}$$

Availability at operator and scenario level

$$y_{inr} \leq y_{in} \quad \forall i, n, r \tag{3}$$

MILP

MILP

max	benefit
subject to	utility definition
	availability
	discounted utility
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	capacity allocation
	price selection

$$z_{inr} = \begin{cases} U_{inr} & \text{if } y_{inr} = 1 \\ \ell_{nr} & \text{if } y_{inr} = 0 \end{cases} \qquad \forall i, n, r$$

(ℓ_{nr} smallest lower bound)

Discounted utility

$$\ell_{nr} \leq z_{inr} \qquad \forall i, n, r \quad (4)$$

$$z_{inr} \leq \ell_{nr} + M_{inr} y_{inr} \qquad \forall i, n, r \quad (5)$$

$$U_{inr} - M_{inr}(1 - y_{inr}) \leq z_{inr} \qquad \forall i, n, r \quad (6)$$

$$z_{inr} \leq U_{inr} \qquad \forall i, n, r \quad (7)$$

MILP

MILP

max	benefit
subject to	utility definition
	availability
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$$U_{nr} = \max_{i \in \mathcal{C}} z_{inr} \qquad \forall n, r$$

$$w_{inr} = \begin{cases} 1 & \text{if } i = \operatorname{argmax}\{U_{nr}\} \\ 0 & \text{otherwise} \end{cases} \qquad \forall i, n, r$$

Choice

$$z_{inr} \leq U_{nr} \qquad \forall i, n, r \qquad (8)$$

$$U_{nr} \leq z_{inr} + M_{nr}(1 - w_{inr}) \qquad \forall i, n, r \qquad (9)$$

$$\sum_i w_{inr} = 1 \qquad \forall n, r \qquad (10)$$

$$w_{inr} \leq y_{inr} \qquad \forall i, n, r \qquad (11)$$

MILP

MILP

max	benefit
subject to	utility definition
	availability
	discounted utility
	choice
	capacity allocation
	price selection

Capacity allocation

- Priority list
- Two sets of constraints $\forall i > 0$
  - Capacity cannot be exceeded ($\Rightarrow y_{inr} = 1$)
  - Capacity has been reached ($\Rightarrow y_{inr} = 0$)

Price: linearization

$$\mathrm{LP}_{in} w_{inr} \leq \alpha_{inr} \leq \mathrm{UP}_{in} w_{inr}$$

$$p_{in} - (1 - w_{inr})\mathrm{UP}_{in} \leq \alpha_{inr} \leq p_{in} - (1 - w_{inr})\mathrm{LP}_{in}$$

MILP

MILP

max	**benefit**
subject to	utility definition
	availability
	discounted utility
	choice
	capacity allocation
	price selection

$$\max \sum_{i>0} (R_i - C_i)$$

Revenue

$$R_i = \frac{1}{R} \sum_n \sum_r p_{in} w_{inr},$$

Cost

$$C_i = (f_i + v_i c_i) y_i$$

Outline

A simple example

Context

- $\mathcal{C}$: set of movies
- Population of N individuals
- Competition: stying home watching TV

One theater – homogenous population

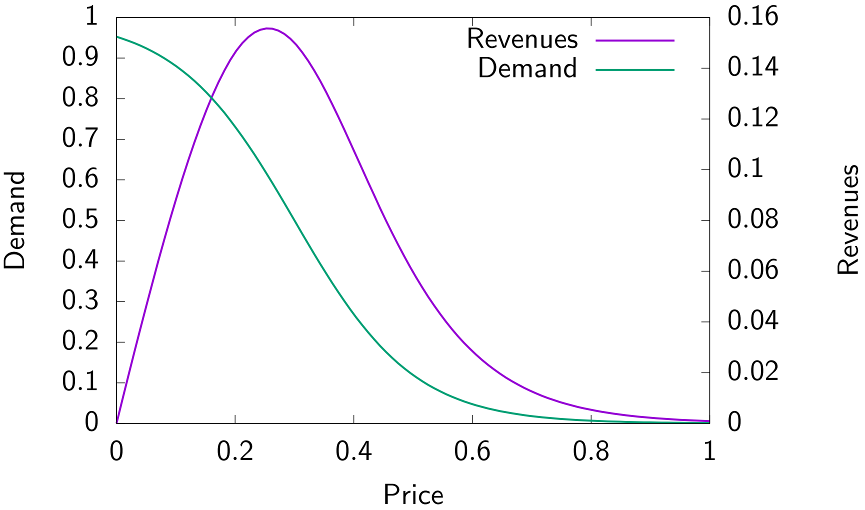
Alternatives

- Staying home: $U_{cn} = 0 + \varepsilon_{cn}$
- My theater: $U_{mn} = -10.0 p_m + 3 + \varepsilon_{mn}$

Logit model

ε_m i.i.d. EV(0,1)

Demand and revenues



Optimization

Solver

GLPK v4.61 under PyMathProg

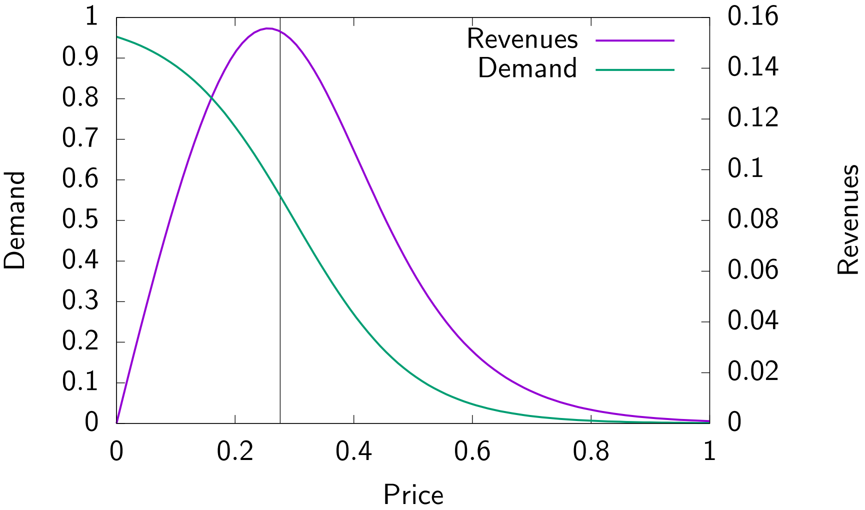
Data

- $N = 1$
- $R = 1000$

Results

- Optimum price: 0.276
- Demand: 57.4%
- Revenues: 0.159

Demand and revenues



Heterogeneous population

Two groups in the population

$$U_{mn} = -\beta_n p_m + c_n$$

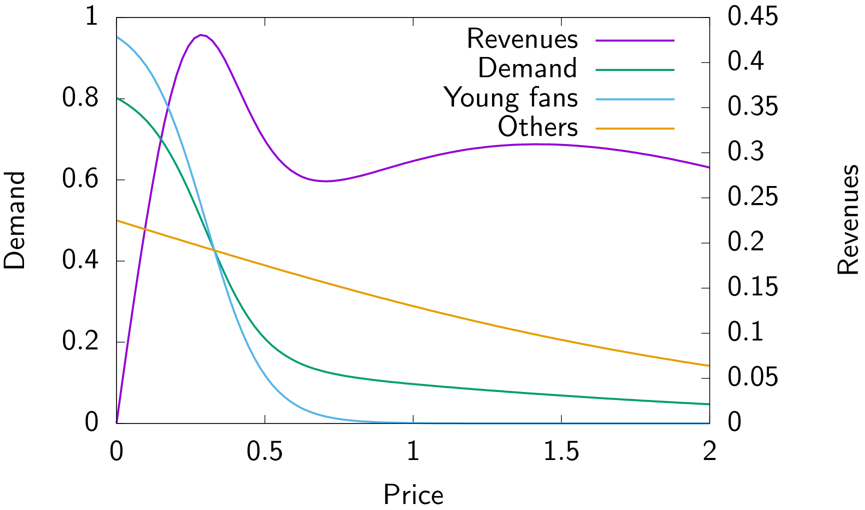
Young fans: 2/3

$\beta_1 = -10,\ c_1 = 3$

Others: 1/3

$\beta_2 = -0.9,\ c_2 = 0$

Demand and revenues



Optimization

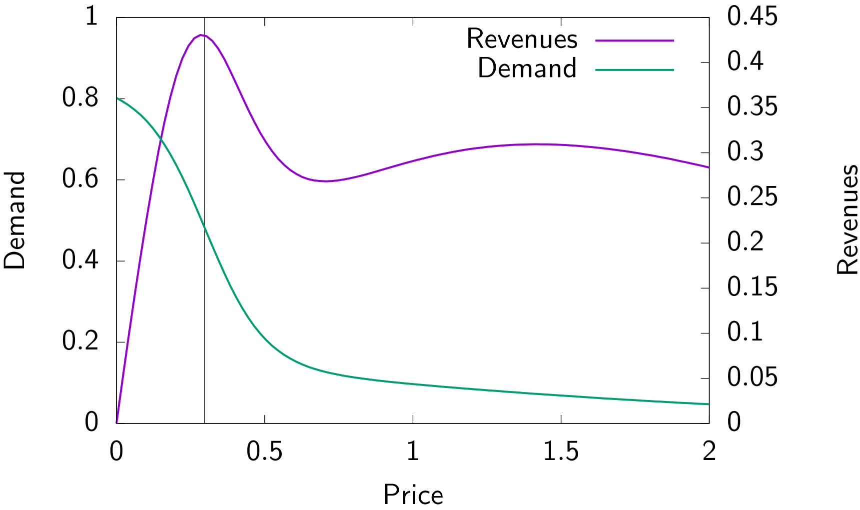
Data

- $N = 3$
- $R = 500$

Results

- Optimum price: 0.297
- Customer 1 (fan): 52.4% [theory: 50.8 %]
- Customer 2 (fan) : 49% [theory: 50.8 %]
- Customer 3 (other) : 45.8% [theory: 43.4 %]
- Demand: 1.472 (49%)
- Revenues: 0.437

Demand and revenues



Two theaters, different types of films

Two theaters, different types of films

Theater m

- Attractive for young people
- Star Wars Episode VII

Theater k

- Not particularly attractive for young people
- Tinker Tailor Soldier Spy

Heterogeneous demand

- Two third of the population is young (price sensitive)
- One third of the population is not (less price sensitive)

Two theaters, different types of films

Data

- Theaters m and k
- $N = 9$
- $R = 50$
- $U_{mn} = -10p_m + 4$, n =young
- $U_{mn} = -0.9p_m$, n =others
- $U_{kn} = -10p_k + 0$, n =young
- $U_{kn} = -0.9p_k$, n =others

Theater m

- Optimum price m: 0.390
- Young customers: 3.48 / 6
- Other customers: 1.08 / 3
- Demand: 4.56 (50.7%)
- Revenues: 1.779

Theater k

- Optimum price k: 1.728
- Young customers: 0.0 / 6
- Other customers: 0.38 / 3
- Demand: 0.38 (4.2%)
- Revenues: 0.581

Two theaters, same type of films

Theater m

- Expensive
- Star Wars Episode VII

Theater k

- Cheap (half price)
- Star Wars Episode VIII

Heterogeneous demand

- Two third of the population is young (price sensitive)
- One third of the population is not (less price sensitive)

Two theaters, same type of films

Data

- Theaters m and k
- $N = 9$
- $R = 50$
- $U_{mn} = -10p + 4$, $n =$young
- $U_{mn} = -0.9p$, $n =$others
- $U_{kn} = -10p/2 + 4$, $n =$young
- $U_{kn} = -0.9p/2$, $n =$others

Theater m

- Optimum price m: 3.582
- Young customers: 0
- Other customers: 1.9
- Demand: 1.9 (31.7%)
- Revenues: 3.42

Theater k

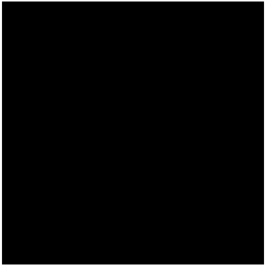
Closed

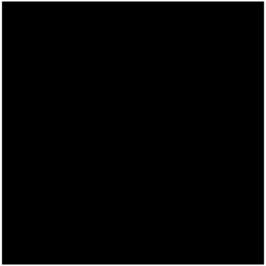
Outline

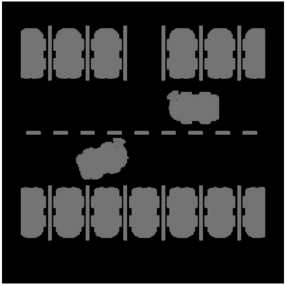
Challenge

- Select a real choice model from the literature
- Integrate it in an optimization problem.

Parking choices







- $N = 50$ customers
- $\mathcal{C} = \{\text{PSP}, \text{PUP}, \text{FSP}\}$
- $\mathcal{C}_n = \mathcal{C} \quad \forall n$

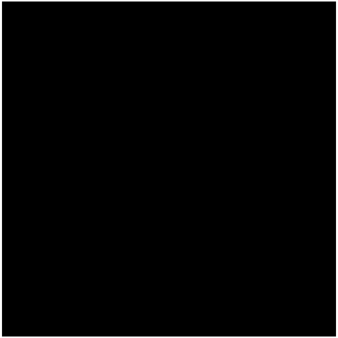
- PSP: $0.50, 0.51, \ldots, 0.65$ (16 price levels)
- PUP: $0.70, 0.71, \ldots, 0.85$ (16 price levels)
- Capacity of 20 spots

Choice model: mixtures of logit model [Ibeas et al., 2014]

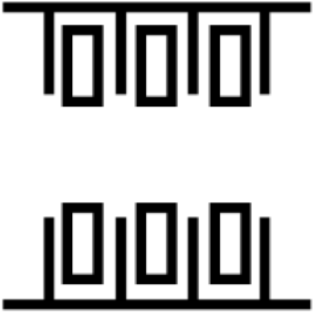
$$
\begin{aligned}
V_{FSP} &= \beta_{AT} AT_{FSP} + \boxed{\beta_{TD}} TD_{FSP} + \boxed{\beta_{Origin_{INT\_FSP}}} Origin_{INT\_FSP} \\
V_{PSP} &= \boxed{ASC_{PSP}} + \beta_{AT} AT_{PSP} + \boxed{\beta_{TD}} TD_{PSP} + \beta_{FEE} \mathbf{FEE_{PSP}} \\
&\quad + \boxed{\beta_{FEE_{PSP(LowInc)}}} \mathbf{FEE_{PSP}} LowInc + \boxed{\beta_{FEE_{PSP(Res)}}} \mathbf{FEE_{PSP}} Res \\
V_{PUP} &= \boxed{ASC_{PUP}} + \beta_{AT} AT_{PUP} + \boxed{\beta_{TD}} TD_{PUP} + \beta_{FEE} \mathbf{FEE_{PUP}} \\
&\quad + \boxed{\beta_{FEE_{PUP(LowInc)}}} \mathbf{FEE_{PUP}} LowInc + \boxed{\beta_{FEE_{PUP(Res)}}} \mathbf{FEE_{PUP}} Res \\
&\quad + \boxed{\beta_{AgeVeh_{\leq 3}}} AgeVeh_{\leq 3}
\end{aligned}
$$

- Parameters
  - Circle: distributed parameters
  - Rectangle: constant parameters
- Variables: all given but FEE (in bold)

Experiment 1: uncapacitated vs capacitated case (1)



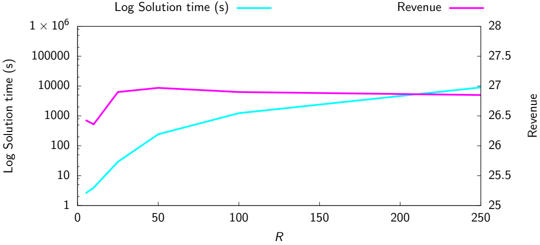
- Capacity constraints are ignored
- Unlimited capacity is assumed



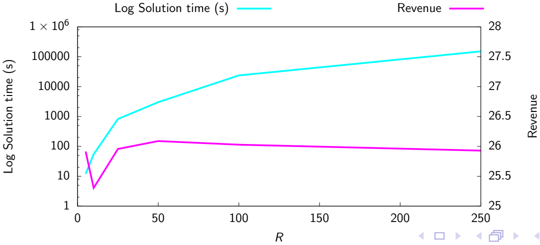
- 20 spots for PSP and PUP
- Opt-out has unlimited capacity

Experiment 1: uncapacitated vs capacitated case (2)

Uncapacitated

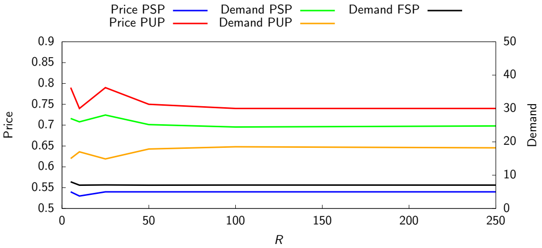


Capacitated

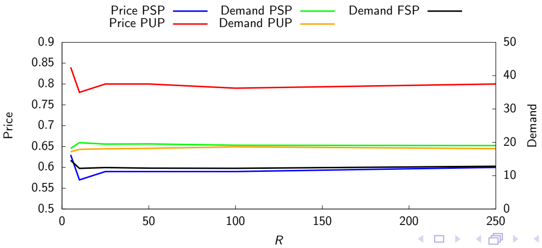


Experiment 1: uncapacitated vs capacitated case (3)

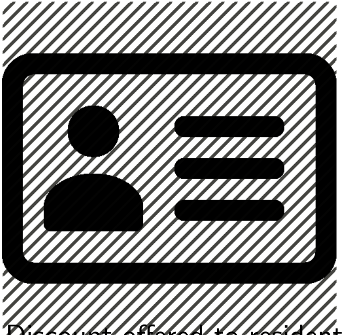
Uncapacitated

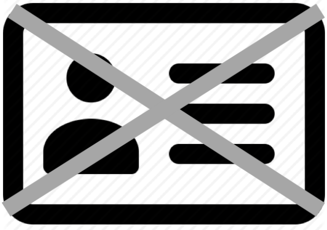


Capacitated



Experiment 2: price differentiation by segmentation (1)

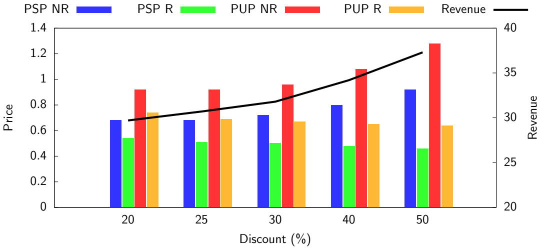




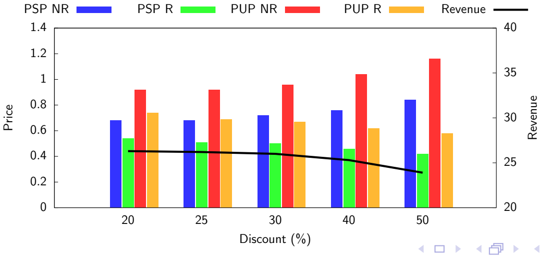
- Discount offered to residents
- Two scenarios (municipality)
  1. Subsidy offered by the municipality
  2. Operator obliged to offer reduced fees
- We expect the price to increase
  - PSP: $\{0.60, 0.64, \ldots, 1.20\}$
  - PUP: $\{0.80, 0.84, \ldots, 1.40\}$

Experiment 2: price differentiation by segmentation (2)

Scenario 1

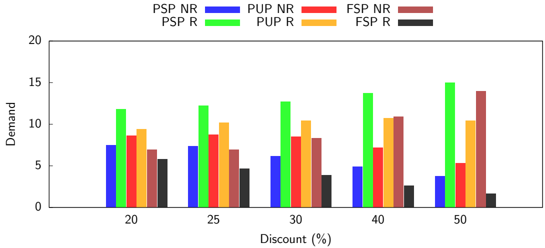


Scenario 2

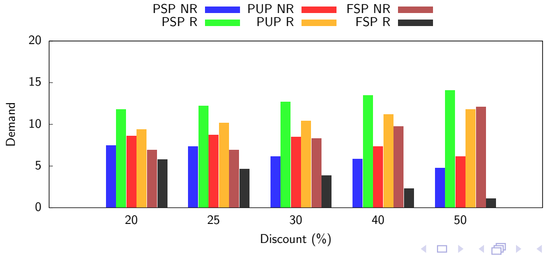


Experiment 2: price differentiation by segmentation (3)

Scenario 1



Scenario 2



Other experiments

Impact of the priority list

- Priority list = order of the individuals in the data (i.e., random arrival)
- 100 different priority lists
- Aggregate indicators remain stable across random priority lists

Benefit maximization through capacity allocation

- 4 different capacity levels for both PSP and PUP: 5, 10, 15 and 20
- Optimal solution: PSP with 20 spots and PUP is not offered
- Both services have to be offered: PSP with 15 and PUP with 5

Outline

Summary

Demand and supply

- Supply: prices and capacity
- Demand: choice of customers
- Interaction between the two

Discrete choice models

- Rich family of behavioral models
- Strong theoretical foundations
- Great deal of concrete applications
- Capture the heterogeneity of behavior
- Probabilistic models

Optimization

Discrete choice models

- Non linear and non convex
- Idea: use utility instead of probability
- Rely on simulation to capture stochasticity

Proposed formulation

- Linear in the decision variables
- Large scale
- Fairly general

Ongoing research

- Decomposition methods
- Scenarios are (almost) independent from each other (except objective function)
- Individuals are also loosely coupled (except for capacity constraints)

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