Discrete choice and discrete optimization: a continuous quest for integration

Michel Bierlaire

Transport and Mobility Laboratory
School of Architecture, Civil and Environmental Engineering
Ecole Polytechnique Fédérale de Lausanne

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Outline

Optimization problem

Given...

the demand

Find...

the best configuration of the system.

Example: airline

Context

- An airline considers to propose various destinations $i = \{1, \ldots, J\}$ to its customers.
- Each potential destination i is served by an aircraft, with capacity c_i.
- The price of the ticket for destination i is p_i.
- The demand is known: W_i passengers want to travel to i.
- The fixed cost of operating a flight to destination i is F_i.
- The airline cannot invest more than a budget B.

Question

What destinations should the airline serve to maximize its revenues?

Example: airline

Decisions variables

$y_i \in \{0, 1\}$: 1 if destination i is served, 0 otherwise.

Maximize revenues

$$\max \sum_{i=1}^{J} \min(W_i, c_i) p_i y_i$$

Constraints

$$\sum_{i=1}^{J} F_i y_i \leq B$$

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Example: airline

Integer linear optimization problem

- Decision variables are integers.
- Objective function and constraints are linear.
- Here: knapsack problem.

Solving the problem

- Branch and bound
- Cutting planes

Example: airline

Pricing

- What price p_i should the airline propose?

$$\max \sum_{i=1}^{J} \min(W_i, c_i) p_i y_i$$

Issues

- Non linear objective
- Unbounded problem

Example: airline

Unbounded problem

- As demand is constant, the airline can make money with very high prices.
- We need to take into account the impact of price on demand.

Logit model

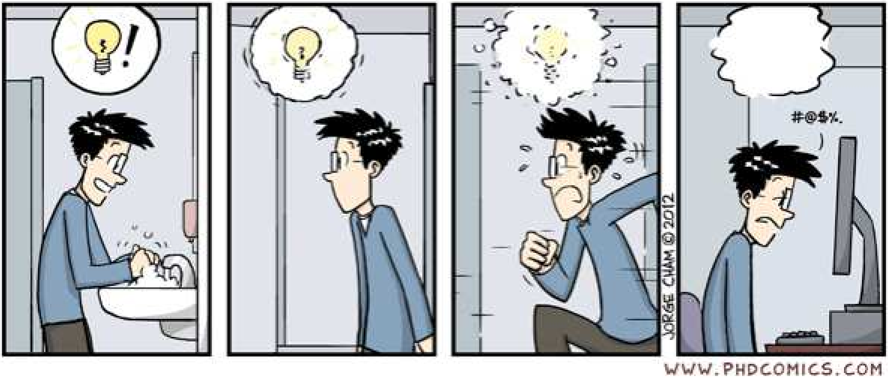
$$W_i = \sum_n P_n(i|p_i, z_{in}, s_n)$$

$$P_n(i|p_i, z_{in}, s_n) = \frac{y_i e^{V_{in}(p_i, z_{in}, s_n)}}{\sum_{j \in \mathcal{C}} y_j e^{V_{jn}(p_j, z_{jn}, s_n)}}.$$

The problem becomes highly non linear.

Outline

The main idea



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The main idea

Linearization

Hopeless to linearize the logit formula (we tried...)

First principles

Each customer solves an optimization problem

Solution

Use the utility and not the probability

A linear formulation

Utility function

$$U_{in} = V_{in} + \varepsilon_{in} = \sum_k \beta_k x_{ink} + f(z_{in}) + \varepsilon_{in}.$$

Simulation

- Assume a distribution for ε_{in}
- E.g. logit: i.i.d. extreme value
- Draw R realizations ξ_{inr}, $r = 1, \ldots, R$
- The choice problem becomes deterministic

Scenarios

Draws

- Draw R realizations ξ_{inr}, $r = 1, \ldots, R$
- We obtain R scenarios

$$U_{inr} = \sum_k \beta_k x_{ink} + f(z_{in}) + \xi_{inr}.$$

- For each scenario r, we can identify the largest utility.
- It corresponds to the chosen alternative.

Comparing utilities

Variables

$$\mu_{ijnr} = \begin{cases} 1 & \text{if } U_{inr} \geq U_{jnr}, \\ 0 & \text{if } U_{inr} < U_{jnr}. \end{cases}$$

Constraints

$$(\mu_{ijnr} - 1)M_{nr} \leq U_{inr} - U_{jnr} \leq \mu_{ijnr}M_{nr}, \forall i, j, n, r.$$

where

$$|U_{inr} - U_{jnr}| \leq M_{nr}, \forall i, j,$$

Comparing utilities

$$(\mu_{ijnr}-1)M_{nr} \leq U_{inr}-U_{jnr} \leq \mu_{ijnr}M_{nr}, \forall i,j,n,r.$$

Constraints: $\mu_{ijnr}=1$

$$0 \leq U_{inr}-U_{jnr} \leq M_{nr}, \forall i,j,n,r.$$

$$U_{jnr} \leq U_{inr}, \forall i,j,n,r.$$

Constraints: $\mu_{ijnr}=0$

$$-M_{nr} \leq U_{inr}-U_{jnr} \leq 0, \forall i,j,n,r.$$

$$U_{inr} \leq U_{jnr}, \forall i,j,n,r.$$

Comparing utilities

$$(\mu_{ijnr} - 1)M_{nr} \leq U_{inr} - U_{jnr} \leq \mu_{ijnr} M_{nr}, \forall i, j, n, r.$$

Equivalence if no tie

$$\begin{aligned}
\mu_{ijnr} = 1 &\Longrightarrow U_{inr} \geq U_{jnr} \\
\mu_{ijnr} = 0 &\Longrightarrow U_{inr} \leq U_{jnr} \\
U_{inr} > U_{jnr} &\Longrightarrow \mu_{ijnr} = 1 \\
U_{inr} < U_{jnr} &\Longrightarrow \mu_{ijnr} = 0
\end{aligned}$$

Accounting for availabilities

Motivation

- If $y_i = 0$, alternative i is not available.
- Its utility should not be involved in any constraint.

New variables: two alternatives are both available

$$\eta_{ij} = y_i y_j$$

Linearization:

$$y_i + y_j \leq 1 + \eta_{ij},$$
$$\eta_{ij} \leq y_i,$$
$$\eta_{ij} \leq y_j.$$

Comparing utilities of available alternatives

Constraints

$$M_{nr}\eta_{ij} - 2M_{nr} \leq U_{inr} - U_{jnr} - M_{nr}\mu_{ijnr} \leq (1-\eta_{ij})M_{nr}, \forall i,j,n,r.$$

$\eta_{ij} = 1$ and $\mu_{ijnr} = 1$

$$0 \leq U_{inr} - U_{jnr} \leq M_{nr}, \forall i,j,n,r.$$

$\eta_{ij} = 1$ and $\mu_{ijnr} = 0$

$$-M_{nr} \leq U_{inr} - U_{jnr} \leq 0, \forall i,j,n,r.$$

Comparing utilities of available alternatives

Constraints

$$M_{nr}\eta_{ij} - 2M_{nr} \leq U_{inr} - U_{jnr} - M_{nr}\mu_{ijnr} \leq (1-\eta_{ij})M_{nr}, \forall i,j,n,r.$$

$\eta_{ij} = 0$ and $\mu_{ijnr} = 1$

$$-M_{nr} \leq U_{inr} - U_{jnr} \leq 2M_{nr}, \forall i,j,n,r,$$

$\eta_{ij} = 0$ and $\mu_{ijnr} = 0$

$$-2M_{nr} \leq U_{inr} - U_{jnr} \leq M_{nr}, \forall i,j,n,r,$$

Comparing utilities of available alternatives

Valid inequalities

$$\mu_{ijnr} \leq y_i, \qquad \forall i, j, n, r,$$
$$\mu_{ijnr} + \mu_{jinr} \leq 1, \qquad \forall i, j, n, r.$$

The choice

Variables

$$w_{inr} = \begin{cases} 1 & \text{if } n \text{ chooses } i \text{ in scenario } r, \\ 0 & \text{otherwise} \end{cases}$$

Maximum utility

$$w_{inr} \leq \mu_{ijnr}, \forall i, j, n, r.$$

Availability

$$w_{inr} \leq y_i, \forall i, n, r.$$

The choice

One choice

$$\sum_{i \in \mathcal{C}} w_{inr} = 1, \forall n, r.$$

Demand and revenues

Demand

$$W_i = \frac{1}{R} \sum_{n=1}^{n} \sum_{r=1}^{R} w_{inr}.$$

Revenues

$$R_i = \frac{1}{R} \sum_{n=1}^{N} p_i \sum_{r=1}^{R} w_{inr}.$$

Outline

A simple example

Data

- $\mathcal{C}$: set of movies
- Population of N individuals
- Utility function:
  $U_{in} = \beta_{in} p_{in} + f(z_{in}) + \varepsilon_{in}$

Decision variables

- What movies to propose? y_i
- What price? p_{in}

Demand model

Logit model

Probability that n chooses movie i:

$$P(i|y, p_n, z_n) = \frac{y_i e^{\beta_{in} p_{in} + f(z_{in})}}{\sum_j y_j e^{\beta_{jn} p_{jn} + f(z_{jn})}}$$

Total revenue:

$$\sum_{i \in C} y_i \sum_{n=1}^{N} p_{in} P(i|y, p_n, z_n)$$

Non linear and non convex in the decision variables

Example: programming movie theaters

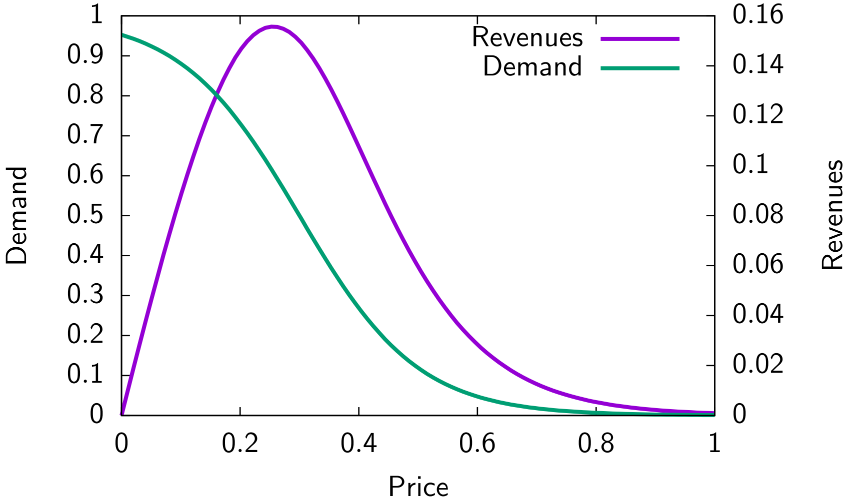
Data

- Two alternatives: my theater (m) and the competition (c)
- We assume an homogeneous population of N individuals

$$U_c = 0 + \varepsilon_c$$
$$U_m = \beta_c p_m + \varepsilon_m$$

- $\beta_c < 0$
- Logit model: ε_m i.i.d. EV

Demand and revenues



Optimization (with GLPK)

Data

- $N = 1$
- $R = 100$
- $U_m = -10p_m + 3$
- Prices: 0.10, 0.20, 0.30, 0.40, 0.50

Results

- Optimum price: 0.3
- Demand: 56%
- Revenues: 0.168

Heterogeneous population

Two groups in the population

$$U_{in} = \beta_n p_i + c_n$$

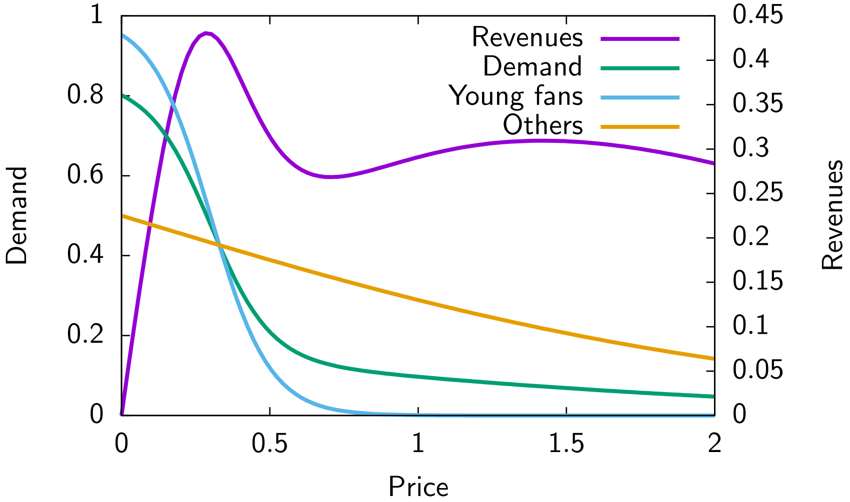
Young fans: 2/3

$\beta_1 = -10,\ c_1 = 3$

Others: 1/3

$\beta_1 = -0.9,\ c_1 = 0$

Demand and revenues



Optimization

Data

- $N = 3$
- $R = 100$
- $U_{m1} = -10p_m + 3$
- $U_{m2} = -0.9p_m$
- Prices: 0.3, 0.7, 1.1, 1.5, 1.9

Results

- Optimum price: 0.3
- Customer 1 (fan): 60% [theory: 50 %]
- Customer 2 (fan) : 49% [theory: 50 %]
- Customer 3 (other) : 45% [theory: 43 %]
- Demand: 1.54 (51%)
- Revenues: 0.48

Two theaters, different types of films

Two theaters, different types of films

Theater m

- Expensive
- Star Wars Episode VII

Theater k

- Cheap
- Tinker Tailor Soldier Spy

Heterogeneous demand

- Two third of the population is young (price sensitive)
- One third of the population is old (less price sensitive)

Two theaters, different types of films

Data

- Theaters m and k
- $N = 6$
- $R = 10$
- $U_{mn} = -10p_m + \textcircled{4}$, $n = 1, 2, 4, 5$
- $U_{mn} = -0.9p_m$, $n = 3, 6$
- $U_{kn} = -10p_k + \textcircled{0}$, $n = 1, 2, 4, 5$
- $U_{kn} = -0.9p_k$, $n = 3, 6$
- Prices m: 1.0, 1.2, 1.4, 1.6, 1.8
- Prices k: half price

Theater m

- Optimum price m: 1.6
- 4 young customers: 0
- 2 old customers: 0.5
- Demand: 0.5 (8.3%)
- Revenues: 0.8

Theater k

- Optimum price m: 0.5
- Young customers: 0.8
- Old customers: 1.5
- Demand: 2.3 (38%)
- Revenues: 1.15

Two theaters, same type of films

Theater m

- Expensive
- Star Wars Episode VII

Theater k

- Cheap
- Star Wars Episode VIII

Heterogeneous demand

- Two third of the population is young (price sensitive)
- One third of the population is old (less price sensitive)

Two theaters, same type of films

Data

- Theaters m and k
- $N = 6$
- $R = 10$
- $U_{mn} = -10p_m + \textcircled{4}$, $n = 1, 2, 4, 5$
- $U_{mn} = -0.9p_m$, $n = 3, 6$
- $U_{kn} = -10p_k + \textcircled{4}$, $n = 1, 2, 4, 5$
- $U_{kn} = -0.9p_k$, $n = 3, 6$
- Prices m: 1.0, 1.2, 1.4, 1.6, 1.8
- Prices k: half price

Theater m

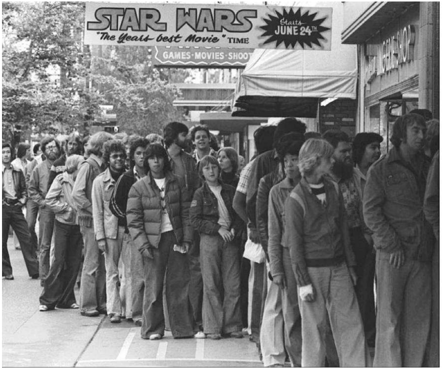
- Optimum price m: 1.8
- Young customers: 0
- Old customers: 1.9
- Demand: 1.9 (31.7%)
- Revenues: 3.42

Theater k

Closed

Extension: dealing with capacities

- Demand may exceed supply
- Not every choice can be accommodated
- Difficulty: who has access?
- Assumption: priority list is exogenous



Outline

Summary

Discrete choice in optimization

- Non linear and non convex
- Idea: use utility instead of probability
- Rely on simulation to capture stochasticity
- Not restricted to logit
- Sole assumption: endogenous variable appears linearly in the utility

Proposed formulation

- General: not designed for a specific application or context.
- Flexible: wide variety of demand and supply models.
- Scalable: the level of complexity can be adjusted.
- Integrated: not sequential.
- Operational: can be solved efficiently.

Dynamic extension

Price and capacity vary over time

Ongoing research

Revenue management

Airlines, train operators, etc.

Decomposition methods

- Scenarios are (almost) independent from each other (except objective function)
- Individuals are also loosely coupled (except for capacity constraints)

Thank you!

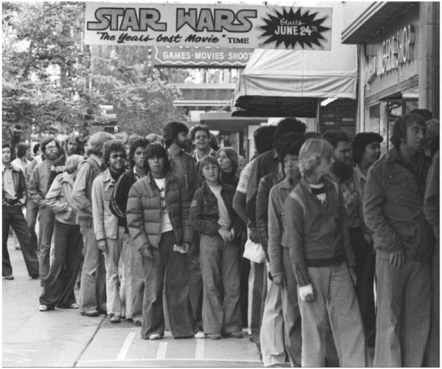
Questions?

Outline

1. Introduction
2. Integrated framework
3. A simple example
   - A linear formulation
   - Example: one theater
   - Example: two theaters
4. Summary
5. Appendix: dealing with capacities
   - Example: two theaters

Dealing with capacities

- Demand may exceed supply
- Not every choice can be accommodated
- Difficulty: who has access?
- Assumption: priority list is exogenous



Priority list

Application dependent

- First in, first out
- Frequent travelers
- Subscribers
- ...

In this framework

The list of customers must be sorted

Dealing with capacities

Variables

- y_{in}: decision of the operator
- y_{inr}: availability

Constraints

$$\sum_{n=1}^{N} w_{inr} \leq c_i$$

$$y_{inr} \leq y_{in}$$

$$y_{i(n+1)r} \leq y_{inr}$$

Constraints

$$c_i(1 - y_{inr}) \leq \sum_{m=1}^{n-1} w_{imr} + (1 - y_{in})c_{\max}$$

$y_{in} = 1,\ y_{inr} = 1$

$$0 \leq \sum_{m=1}^{n-1} w_{imr}$$

$y_{in} = 1,\ y_{inr} = 0$

$$c_i \leq \sum_{m=1}^{n-1} w_{imr}$$

$y_{in} = 0,\ y_{inr} = 0$

$$c_i \leq \sum_{m=1}^{n-1} w_{imr} + c_{\max}$$

Constraints

$$\sum_{m=1}^{n-1} w_{imr} + (1 - y_{in})c_{\max} \leq (c_i - 1)y_{inr} + \max(n, c_{\max})(1 - y_{inr})$$

$y_{in} = 1, \; y_{inr} = 1$

$$1 + \sum_{m=1}^{n-1} w_{imr} \leq c_i$$

$y_{in} = 1, \; y_{inr} = 0$

$$\sum_{m=1}^{n-1} w_{imr} \leq \max(n, c_{\max})$$

$y_{in} = 0, \; y_{inr} = 0$

$$\sum_{m=1}^{n-1} w_{imr} + c_{\max} \leq \max(n, c_{\max})$$

Two theaters, different types of films

Data

- Theaters m and k
- Capacity: 2
- $N = 6$
- $R = 5$
- $U_{mn} = -10p_m + 4, \; n = 1, 2, 4, 5$
- $U_{mn} = -0.9p_m, \; n = 3, 6$
- $U_{kn} = -10p_k + 0, \; n = 1, 2, 4, 5$
- $U_{kn} = -0.9p_k, \; n = 3, 6$
- Prices m: 1.0, 1.2, 1.4, 1.6, 1.8
- Prices k: half price

Theater m

- Optimum price m: 1.8
- Demand: 0.2 (3.3%)
- Revenues: 0.36

Theater k

- Optimum price m: 0.5
- Demand: 2 (33.3%)
- Revenues: 1.15

Example of two scenarios

Customer	Choice	Capacity m	Capacity k
1	0	2	2
2	0	2	2
3	k	2	1
4	0	2	1
5	0	2	1
6	k	2	0

Customer	Choice	Capacity m	Capacity k
1	0	2	2
2	k	2	1
3	0	2	1
4	k	2	0
5	0	2	0
6	0	2	0

Two theaters: all prices divided by 2

Data

- Theaters m and k
- Capacity: 2
- $N = 6$
- $R = 5$
- $U_{mn} = -10p_m + 4, \; n = 1, 2, 4, 5$
- $U_{mn} = -0.9p_m, \; n = 3, 6$
- $U_{kn} = -10p_k + 0, \; n = 1, 2, 4, 5$
- $U_{kn} = -0.9p_k, \; n = 3, 6$
- Prices m: 0.5, 0.6, 0.7, 0.8, 0.9
- Prices k: half price

Theater m

- Optimum price m: 0.5
- Demand: 1.4
- Revenues: 0.7

Theater k

- Optimum price m: 0.45
- Demand: 1.6
- Revenues: 0.72

Example of two scenarios

Customer	Choice	Capacity m	Capacity k
1	0	2	2
2	0	2	2
3	0	2	2
4	k	2	1
5	k	2	0
6	0	2	0

Customer	Choice	Capacity m	Capacity k
1	k	2	1
2	k	2	0
3	0	2	0
4	m	1	0
5	0	1	0
6	m	0	0