Mathematical Modeling of Irrational Behavior: towards a linear formulation

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May 16, 2023

Outline

1. Choice models
2. Beyond rationality
3. Optimization

Decision rule

Homo economicus

Rational and narrowly self-interested economic actor who is optimizing her outcome

Utility

$$U_n : \mathcal{C}_n \longrightarrow \mathbb{R} : a \rightsquigarrow U_n(a)$$

- captures the attractiveness of an alternative
- measure that the decision maker wants to optimize

Behavioral assumption

- the decision maker associates a utility with each alternative
- the decision maker is a perfect optimizer
- the alternative with the highest utility is chosen

Random utility model

Random utility

$$U_{in} = V_{in} + \varepsilon_{in} = \beta^T X_{in} + \varepsilon_{in}.$$

Choice model

$$P(i|\mathcal{C}_n) = \Pr(U_{in} \geq U_{jn}, \forall j \in \mathcal{C}_n),$$

Logit model

Assumptions

ε_{in} are i.i.d. EV(0, μ).

Choice model

$$P_n(i|\mathcal{C}_n) = \frac{y_{in} e^{\mu V_{in}}}{\sum_{j=1}^{J} y_{jn} e^{\mu V_{jn}}}.$$

Outline

Beyond rationality

Motivation

- There is evidence that human beings are not necessarily rational in the way assumed by random utility models.
- We first review some experiments that illustrate that (apparent) irrationality.

Example: pain lovers

[Kahneman et al., 1993]

- Short trial: immerse one hand in water at 14° for 60 sec.
- Long trial: immerse the other hand at 14° for 60 sec, then keep the hand in the water 30 sec. longer as the temperature of the water is gradually raised to 15°.
- Outcome: most people prefer the long trial.
- Explanation: duration plays a small role, the peak and the final moments matter.

Example: The Economist

[Ariely, 2008]

Subscription to The Economist

Web only	@ $59
Print only	@ $125
Print and web	@ $125

Example: The Economist

[Ariely, 2008]

Subscription to The Economist

Experiment 1	Experiment 2
Web only @ $59	Web only @ $59
Print only @ $125	
Print and web @ $125	Print and web @ $125

Example: The Economist

[Ariely, 2008]

Subscription to The Economist

	Experiment 1	Experiment 2	
16	Web only @ $59	Web only @ $59	68
0	Print only @ $125		
84	Print and web @ $125	Print and web @ $125	32

The Economist: explanations

- Dominated alternative.
- According to utility maximization, should not affect the choice.
- But it affects the perception, which affects the choice.

Decoy effect

Decoy

High-price, low-value product compared to other items in the choice set.

Behavior

Consumers shift their choice to more expensive items.

Applications

- Travel and tourism. [Josiam and Hobson, 1995]
- Wine lists in restaurants. [Kimes et al., 2012]
- Tobacco treatment. [Rogers et al., 2020]
- Online diamond retail. [Wu and Cosguner, ta]

Example: good or bad wine?

Choose a bottle of wine...

	Experiment 1	Experiment 2
1	McFadden red at $10	McFadden red at $10
2	Nappa red at $12	Nappa red at $12
3		McFadden special reserve pinot noir at $60
	Most would choose 2	Most would choose 1

- Context plays a role on perceptions.
- Here, perceived quality is increased.

Example: live and let die

[Kahneman and Tversky, 1986]

Population of 600 is threatened by a disease.

Two alternative treatments to combat the disease have been proposed.

	Experiment 1 # resp. = 152	Experiment 2 # resp. = 155	
72%	Treatment A: 200 people saved	Treatment C: 400 people die	22%
28%	Treatment B: 600 saved with prob. 1/3 0 saved with prob. 2/3	Treatment D: 0 die with prob. 1/3 600 die with prob. 2/3	78%

Example: to be free

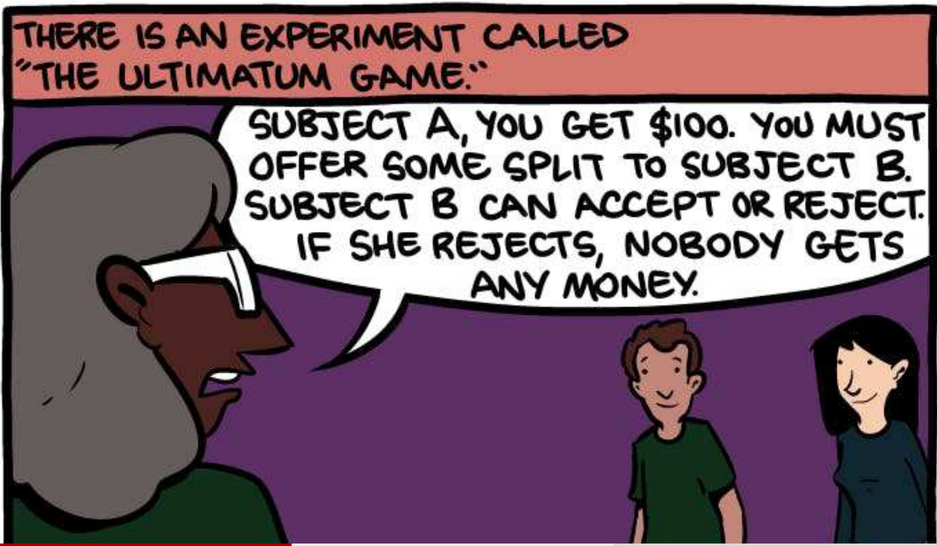
[Ariely, 2008]

Choice between a fine and a regular chocolate

	Experiment 1	Experiment 2
Lindt	\$0.15	\$0.14
Hershey	\$0.01	\$0.00
Lindt chosen	73%	31%
Hershey chosen	27%	69%

Discontinuity at 0

Ultimatum game



Ultimatum game

Ultimatum game

Ultimatum game

Optimal solution

Subject B should accept any offer.

In practice

Offers of less than 30% are often rejected.

Modeling latent concepts

Motivation

- Some observed behavior may appear irrational, and inconsistent with random utility.
- It is only apparent, as these behaviors can be explained by more complex formulations of the concept of utility.
- In particular, this may involve subjective and latent concepts such as perceptions and attitudes.
- Latent concepts can be introduced in choice models.

Indirect measurements of latent concepts

Attitude towards the environment

For each question, response on a scale: strongly agree, agree, neutral, disagree, strongly disagree, no idea.

- The price of oil should be increased to reduce congestion and pollution.
- More public transportation is necessary, even if it means additional taxes.
- Ecology is a threat to minorities and small companies.
- People and employment are more important than the environment.
- I feel concerned by the global warming.
- Decisions must be taken to reduce the greenhouse gas emission.

Indirect measurements of latent concepts

Psychometric indicators

- Usually easy to respond.
- Arbitrary units.
- Important to minimize framing.

Data

For each individual, we have

- Vector of independent variables: x.
- Choice: i.
- vector of psychometric indicators: I.

Prediction model

Latent variable

- Captures perceptions, attitudes, anchors, etc.
- Not observed.
- Modeled as a function of observed variables:

$$X^* = \text{EnvironmentalAttidude} = \text{f}(\text{Age, Education, etc.}; \theta) + \xi.$$

Random utility model

- Utility is also unobserved.
- Modeled as a function of observed variables, as well as the latent variable(s):

$$\text{Utility(PublicTransport)} = \text{f}(\text{Price, Time, Frequency, EnvironmentalAttitude}; \theta) + \varepsilon$$

Prediction model

Choice model: mixture of logit models

$$P_n(i|x_n, X_n^*, \mathcal{C}_n) = \frac{y_{in} e^{\mu V_{in}(x_n, X_n^*)}}{\sum_{j=1}^{J} y_{jn} e^{\mu V_{jn}(x_n, X_n^*)}}.$$

$$\begin{aligned} P_n(i|x_n, \mathcal{C}_n) &= \int_t P_n(i|x_n, t, \mathcal{C}_n) f_{X_n^*}(t) dt \\ &= \int_t \frac{y_{in} e^{\mu V_{in}(x_n, t)}}{\sum_{j=1}^{J} y_{jn} e^{\mu V_{jn}(x_n, t)}} f_{X_n^*}(t) dt. \end{aligned}$$

Outline

Demand-based optimization

Context

- An operator providing goods or services.
- Potentially, competing operators.
- Customers who freely decide which service/good to choose.

Objective

Help the operator with strategic, tactical or operational decisions.

Comments

- This is the core business of operations research.
- But the decisions of customers are often assumed to be given, exogenous.
- Challenge: use choice models to capture the demand, the decisions of customers.

Demand-based optimization

Examples

- Pricing, toll setting.
- Revenue management.
- Facility location.
- Assortment optimization.
- Passenger-centric railway timetabling.
- ...

Main issue

Demand representation

- $d_i(x)$: number of customers who select service/good i, under decision x.
- Using a choice model:

$$d_i(x) = \sum_n P_n(i|\mathcal{C}_n) = \sum_n \int_t \frac{y_{in}(x)e^{\mu V_{in}(x,t)}}{\sum_{j=1}^J y_{jn}(x)e^{\mu V_{jn}(x,t)}} f_{X^*}(t)dt.$$

Issue

- Most optimization models in OR rely on convenient relaxations of the original problem.
- Usually, "convenient" means linear or convex.
- But mixtures of logit models are far from being convex.

Exogenous and endogenous variables

Endogenous variables

- Decision variables of the operator that influence the choice of customers.
- Examples: price, quality of service, properties of goods, etc.

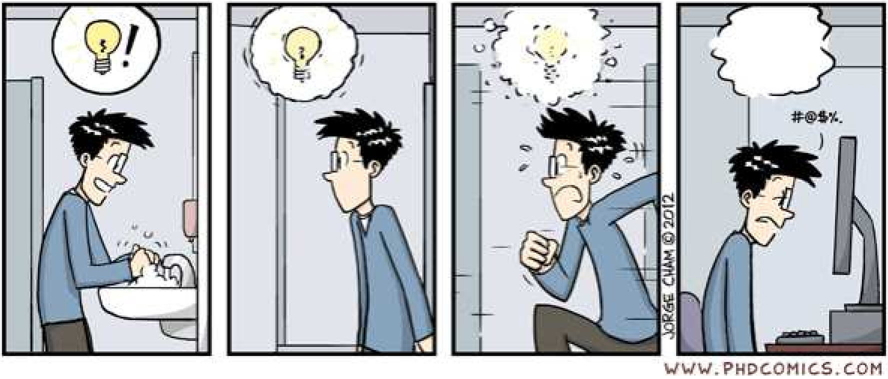
Exogenous variables

- Variables influencing the choice of customers, but not decided by the operator.
- Examples: decisions of the competing operators, attitudes, perceptions, etc.

Mathematical requirement

We need linearity (or convexity) in the endogenous variables.

The main idea



The main idea

Linearization

- Hopeless to linearize the logit formula (we tried...)
- Anyway, we want to go beyond logit.

Idea

Work with the utility and not the probability.

A linear formulation

Latent variable

$$X_n^* = f_X(z_{\text{endo}}, z_{\text{exo}}) + \xi_n, \text{where } f_X \text{ is linear (or convex) in } z_{\text{endo}}.$$

Simulation

- Assume a distribution for ξ_n
- E.g. normal distribution.
- Draw R realizations ξ_{nr}, $r = 1, \ldots, R$

A linear formulation

Utility function

$$U_{in} = V_{in}(x_{\text{endo}}, x_{\text{exo}}, X_n^*) + \varepsilon_{in},$$

where V_{in} is linear (or convex) in x_{endo} and X_n^* (and so, in z_{endo}).

Simulation

- Assume a distribution for ε_{in}
- E.g. logit: i.i.d. extreme value
- Draw R realizations ε_{inr}, $r = 1, \ldots, R$

Scenarios

Draws

- Draw R realizations ξ_{inr}, $r = 1, \ldots, R$
- We obtain R scenarios

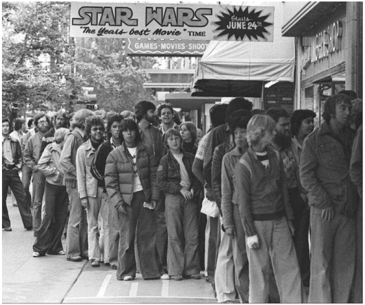
$$X^*_{nr} = \sum_k \theta_k z_{\text{endo}} + f(z_{\text{exo}}) + \xi_{inr}.$$

$$U_{inr} = \sum_k \beta_k x_{\text{endo}} + f(x_{\text{exo}}) + \varepsilon_{inr}.$$

- For each scenario r, we can identify the largest utility.
- It corresponds to the chosen alternative.

Capacities

- Demand may exceed supply
- Each alternative i can be chosen by maximum c_i individuals.
- An exogenous priority list is available.
- Can be randomly generated, or according to some rules.
- The numbering of individuals is consistent with their priority.



Choice set

Variables

$y_i \in \{0,1\}$ operator decision

$y^d_{in} \in \{0,1\}$ customer decision (data)

$y_{in} \in \{0,1\}$ product of decisions

$y_{inr} \in \{0,1\}$ capacity restrictions

Constraints

$$y_{in} = y^d_{in} y_i \quad \forall i, n$$
$$y_{inr} \leq y_{in} \quad \forall i, n, r$$

Utility

Variables

U_{inr}	utility
$z_{inr} = \begin{cases} U_{inr} & \text{if } y_{inr} = 1 \\ \ell_{nr} & \text{if } y_{inr} = 0 \end{cases}$	discounted utility
(ℓ_{nr} smallest lower bound)	

Constraint: utility

$$U_{inr} = \overbrace{\sum_k \beta_k x_{kn,\text{endo}} + f(x_{n,\text{exo}})}^{V_{in}} + \varepsilon_{inr} \; \forall i, n, r$$

Utility (ctd)

Constraints: discounted utility

$$\begin{aligned} \ell_{nr} &\leq z_{inr} && \forall i, n, r \\ z_{inr} &\leq \ell_{nr} + M_{inr} y_{inr} && \forall i, n, r \\ U_{inr} - M_{inr}(1 - y_{inr}) &\leq z_{inr} && \forall i, n, r \\ z_{inr} &\leq U_{inr} && \forall i, n, r \end{aligned}$$

Choice

Variables

$$U_{nr} = \max_{i \in \mathcal{C}} z_{inr}$$

$$w_{inr} = \begin{cases} 1 & \text{if } z_{inr} = U_{nr} \\ 0 & \text{otherwise} \end{cases} \qquad \text{choice}$$

Constraints

$$z_{inr} \leq U_{nr} \qquad \forall i, n, r$$

$$U_{nr} \leq z_{inr} + M_{nr}(1 - w_{inr}) \qquad \forall i, n, r$$

$$\sum_i w_{inr} = 1 \qquad \forall n, r$$

$$w_{inr} \leq y_{inr} \qquad \forall i, n, r$$

Capacity

If capacity reached $\Rightarrow y_{inr} = 0$

$$\sum_{m=1}^{n-1} w_{imr} \leq (c_i - 1)y_{inr} + (n-1)(1 - y_{inr}) \ \forall i > 0, n > c_i, r$$

If capacity not reached $\Rightarrow y_{inr} = 1$

$$c_i(y_{in} - y_{inr}) \leq \sum_{m=1}^{n-1} w_{imr}, \ \forall i > 0, n, r$$

Family of models

Constraints

- Set of linear constraints characterizing choice behavior
- Can be included in any relevant optimization problem.

Examples

- Profit maximization
- Facility location

Difficulties

- big M constraints
- large dimensions

Profit maximization

Profit

If p_{in} is the price paid by individual to purchase option i, the revenue generated by this option is

$$\frac{1}{R}\sum_{r=1}^{R}\sum_{n=1}^{N} p_{in} w_{inr}.$$

Linearization

If $a_{in} \leq p_{in} \leq b_{in}$, we define $\eta_{inr} = p_{in} w_{inr}$, and the following constraints:

$$\begin{aligned} a_{in} w_{inr} &\leq \eta_{inr} \\ \eta_{inr} &\leq b_{in} w_{inr} \\ p_{in} - (1 - w_{inr}) b_{in} &\leq \eta_{inr} \\ \eta_{inr} &\leq p_{in} - (1 - w_{inr}) a_{in} \end{aligned}$$

A case study

Challenge

- Take a choice model from the literature.
- It must be a mixture of logit.
- It must involve heterogeneity.
- Show that it can be integrated in a relevant MILP.

A case study

Challenge

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- It must involve heterogeneity.
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Parking choice

- [Ibeas et al., 2014]

Parking choices [Ibeas et al., 2014]

Alternatives

- Paid on-street parking
- Paid underground parking
- Free street parking

Model

- $N = 50$ customers
- $\mathcal{C} = \{\text{PSP}, \text{PUP}, \text{FSP}\}$
- $\mathcal{C}_n = \mathcal{C} \quad \forall n$
- $p_{in} = p_i \quad \forall n$
- Capacity of 20 spots
- Mixture of logit models

General experiments

Uncapacitated vs Capacitated case

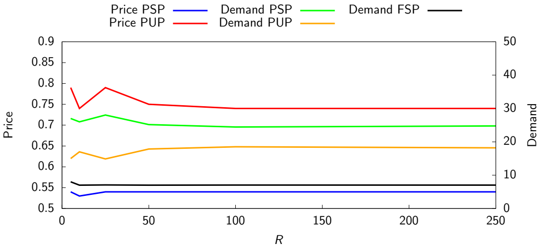
- Maximization of revenue
- Unlimited capacity
- Capacity of 20 spots for PSP and PUP

Price differentiation by population segmentation

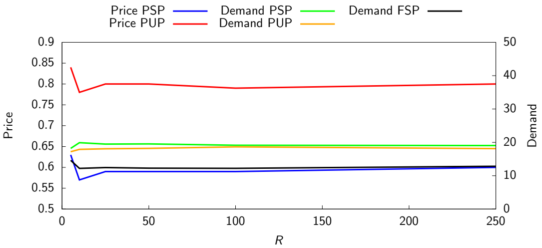
- Reduced price for residents
- Two scenarios
  1. Subsidy offered by the municipality
  2. Operator is forced to offer a reduced price

Uncapacitated vs Capacitated case

Uncapacitated



Capacitated



Computational time

	Uncapacitated case				Capacitated case			
R	Sol time	PSP	PUP	Rev	Sol time	PSP	PUP	Rev
5	2.58 s	0.54	0.79	26.43	12.0 s	0.63	0.84	25.91
10	3.98 s	0.53	0.74	26.36	54.5 s	0.57	0.78	25.31
25	29.2 s	0.54	0.79	26.90	13.8 min	0.59	0.80	25.96
50	4.08 min	0.54	0.75	26.97	50.2 min	0.59	0.80	26.10
100	20.7 min	0.54	0.74	26.90	6.60 h	0.59	0.79	26.03
250	2.51 h	0.54	0.74	26.85	1.74 days	0.60	0.80	25.93

Facility location

Data

- U_{in}: exogenous,
- C_i: fixed cost to open a facility,
- c_i: operational cost per customer to run the facility.

Objective function

$$\min \sum_{i \in \mathcal{C}_k} C_i y_i + \frac{1}{R} \sum_r \sum_i \sum_n c_i w_{inr}$$

Benders decomposition

$$\min \sum_{i \in \mathcal{C}_k} C_i y_i + \frac{1}{R} \sum_r \sum_i \sum_n c_i w_{inr}$$

subject to

$$\max_w U_{nr} = \sum_i U_{inr} w_{inr}$$

$$\sum_i w_{inr} \leq 1$$

$$w_{inr} \leq y_i$$

$$w_{inr} \geq 0$$

$$w_{inr}, y_i \in \{0, 1\}.$$

Benders decomposition

Customer subproblem: fix y_i^*

$$\max_w U_{nr} = \sum_i U_{inr} w_{inr}$$

subject to

$$\sum_i w_{inr} = 1$$

$$w_{inr} \leq y_i^*$$

$$w_{inr} \geq 0.$$

Property

Totally unimodular: no integrality constraint is required.

Benders decomposition

Primal

$$\min_{w} U = -\sum_i U_i w_i$$

subject to

$$\sum_i w_i = 1$$

$$w_i \leq y_i^* \qquad \forall i$$

$$w_i \geq 0.$$

Dual

$$\max_{\lambda,\mu} \lambda + \sum_i \mu_i y_i^*$$

subject to

$$\lambda + \mu_i \leq -U_i \qquad \forall i$$

$$\mu_i \leq 0 \qquad \forall i$$

Bender decomposition

Ongoing work

- Exploit the duality results to generate cuts for the master problem.
- Investigate the use of Benders for other problems.
    - profit maximization,
    - maximum likelihood estimation of the parameters.

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