A Lagrangian relaxation technique for the demand-based benefit maximization problem

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Outline

1. Introduction
2. Demand-based benefit maximization problem
3. Decomposition by operator and customers
4. Decomposition by customers and draws
5. Conclusions and future work

1 Introduction

2 Demand-based benefit maximization problem

3 Decomposition by operator and customers

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5 Conclusions and future work

Discrete choice models and optimization

- Behavioral realism
- Unrealistic assumptions
- Complex formulations

- Tractability
- Linearity and/or convexity
- MILP models

Bridging the gap

- General framework integrating a choice model with an MILP model
- Simulation to linearize the discrete choice model
- Demand-based benefit maximization problem

Lagrangian relaxation

MILP

$$
\begin{aligned}
Z = \quad \max \; & cx + fy \\
\text{s.t.} \; & Ax \le b \\
& Bx + Dy \le e \\
& Gy \le h \\
& x, y \ge 0 \\
& y \text{ integer}
\end{aligned}
$$

Lagrangian subproblem

$$
\begin{aligned}
Z_D(\lambda) = \quad \max \; & cx + fy + \lambda(e - Bx - Dy) \\
\text{s.t.} \; & Bx + Dy \le e \\
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$Z_D(\lambda) \ge Z$

Lagrangian dual

$$Z_D = \min_{\lambda} Z_D(\lambda)$$

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Linearization of the choice model

$$U_{in} = V_{in} + \varepsilon_{inr} \xrightarrow{\text{draw distribution}} U_{inr} = \overbrace{\sum_k \beta_k x^e_{ink} + g^d_{in}(x^d_{in})}^{V_{in}} + \xi_{inr}$$

x^e endogenous variable

x^d exogenous variable (choice)

$$w_{inr} = \begin{cases} 1 & \text{if } U_{inr} \geq U_{jnr} \forall j \in \mathscr{C}_n, j \neq i \\ 0 & \text{otherwise} \end{cases} \longrightarrow D_i = \frac{1}{R} \sum_{r=1}^{R} \sum_{n=1}^{N} w_{inr}$$

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Benefit maximization problem (1)

- Set of services $\mathscr{C}$ $(i > 0)$
- Population N $(n \geq 1)$
- Price $a_i \leq p_{in} \leq b_{in}$
- Opt-out option $i = 0$
- Capacity $c_i = \sum_{q=1}^{Q} c_{iq} y_{iq}$
- Fixed f_{iq} and variable v_{iq} costs

Benefit maximization problem (2)

Expected gain:

$$G_i = \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \frac{1}{R} \overbrace{p_{in} w_{iqnr}}^{\eta_{iqnr}}$$

Operating costs:

$$C_i = \sum_{q=1}^{Q} (f_{iq} + v_{iq} c_{iq}) y_{iq}$$

Benefit maximization problem (3)

obj. fun.	$\sum_{i>0} G_i - C_i$
availability	operator level and scenario level
disc. utility	variable capturing availability and utility
choice	linearization of the highest utility
price	linearization of the variable η_{iqnr}
capacity	relation with the availability at scenario level

Computational results

- Parking choices: mixtures of logit model
- $R = 50$ draws and $N = 50$ customers
- $|\mathscr{C}| = 3$: PSP, PUP and FSP (opt-out)
- Experiment testing two approaches:
  1. operator can decide to offer a service
  2. operator is forced to offer all services

		Capacity		Demand			Prices		
Approach	**Solution time** (h)	PSP	PUP	PSP	PUP	FSP	PSP	PUP	**Benefit**
1	18.7	20	-	19.4	-	30.6	0.76	-	6.27
2	33.7	15	5	14.8	4.56	30.7	0.76	1.32	4.99

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Motivation

obj. fun.

availability

price

availability

disc. utility

choice

price

capacity

Lagrangian decomposition

Duplicate w_{iqnr} and relax the copy constraint → Chain of constraints to define new relations → Price specification in the customer subproblem

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+

Operator subproblem

Capacitated Facility Location Problem (CFLP)

$$\max \quad Z_o(\theta) = \sum_{i\in\mathscr{C}} \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \theta_{iqnr} w'_{iqnr} - \sum_{i>0} \sum_{q=1}^{Q} (f_{iq} + v_{iq} c_{iq}) y_{iq} \tag{1}$$

$$\text{subject to} \quad \sum_{q=1}^{Q} y_{iq} \leq 1 \qquad \forall i > 0 \tag{2}$$

$$\sum_{i\in\mathscr{C}} \sum_{q=1}^{Q} w'_{iqnr} = 1 \qquad \forall n, r \tag{3}$$

$$w'_{iqnr} \leq y_{iq} \qquad \forall i > 0, q, n, r \tag{4}$$

$$w'_{iqnr} = 0 \qquad \forall i \notin \mathscr{C}_n, q, n, r \tag{5}$$

$$\sum_{n=1}^{N} w'_{iqnr} \leq c_{iq} y_{iq} \qquad \forall i > 0, q, r \tag{6}$$

Customer subproblem

obj. fun.	$\sum_{i>0} G_i$ and the relaxed term
availability	operator level and scenario level
disc. utility	variable capturing availability and utility
choice	linearization of the highest utility
price	linearization of the variable η_{iqnr}
capacity	relation with the availability at scenario level

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Motivation

- Maximization of own utility
- Objective function and capacity constraints

- Behavioral scenario
- Objective function

Uncapacitated case and revenue maximization

obj. fun.	$\sum_{i>0} G_i$
availability	
utility	no need for discounted utility (no availability)
choice	linearization of the highest utility
price	linearization of the variable η_{iqnr}
capacity	

Lagrangian decomposition

⚠ Price p_{in} is the same across draws $\Rightarrow$ **no** decomposition by n and r

$$p_{in(r-1)} = p_{inr}$$

$$p_{in1} = p_{in2} = \cdots = p_{inR} = p_{in1}$$

Relaxation with Lagrangian multipliers $\lambda_{inr} \Rightarrow$ decomposition by n and r

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Relaxation with Lagrangian multipliers $\lambda_{inr} \Rightarrow$ decomposition by n and r

Subproblem for each n and r

$$\max \quad \sum_{i>0} \left[\frac{1}{R}\eta_{inr} + (\lambda_{inr} - \lambda_{in(r-1)})p_{inr} \right] \tag{7}$$

subject to

$$U_{inr} = \beta_{in}p_{inr} + g^d_{in}(x^d_{in}) + \xi_{inr} \qquad \forall i \in \mathscr{C}_n \tag{8}$$

$$U_{inr} \leq U_{nr} \qquad \forall i \tag{9}$$

$$U_{nr} \leq U_{inr} + M_{nr}(1 - w_{inr}) \qquad \forall i \tag{10}$$

$$\sum_{i \in \mathscr{C}} w_{inr} = 1 \tag{11}$$

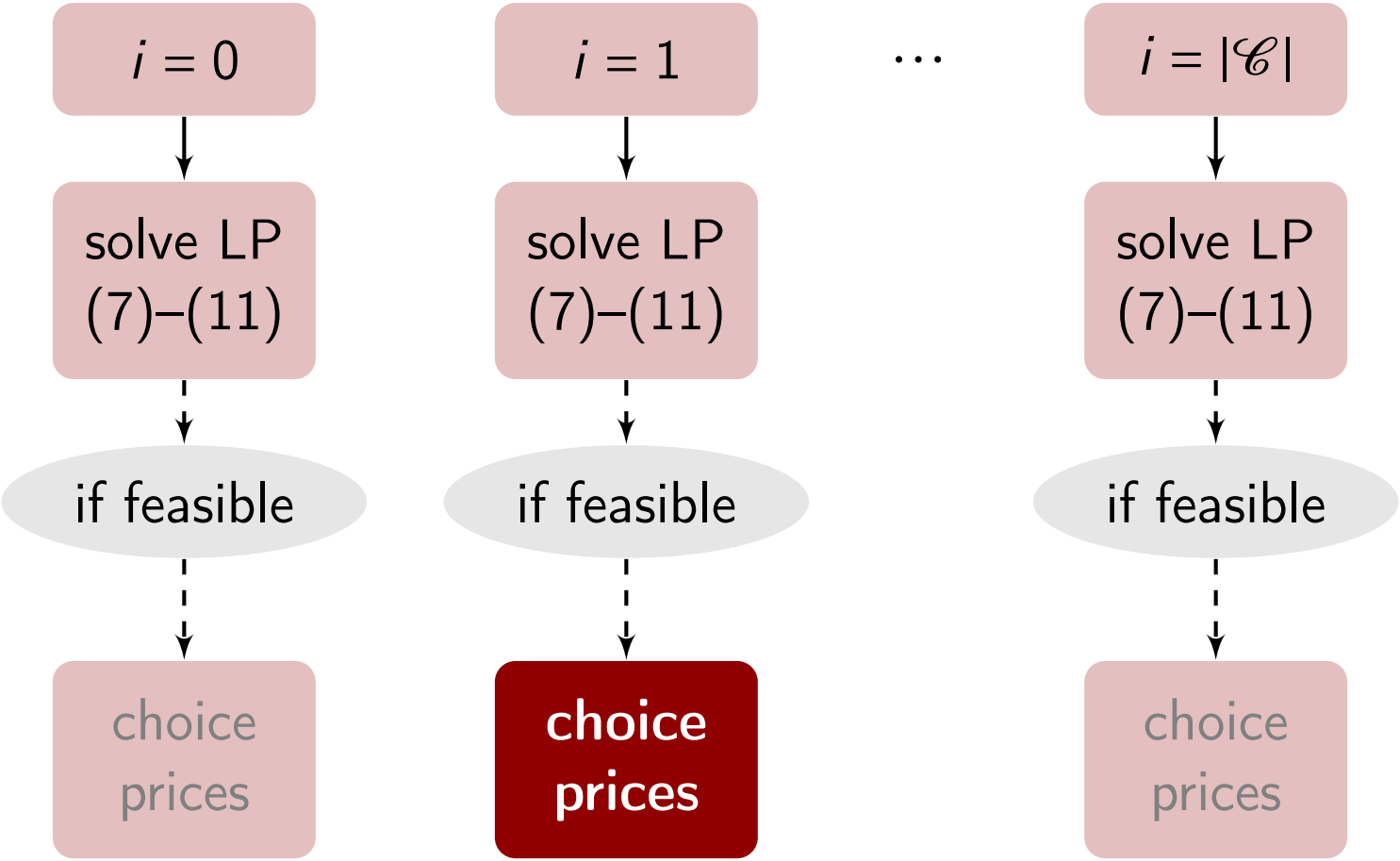
$$a_{in}w_{inr} \leq \eta_{inr} \qquad \forall i > 0 \tag{12}$$

$$\eta_{inr} \leq b_{in}w_{inr} \qquad \forall i > 0 \tag{13}$$

$$p_{inr} - (1 - w_{inr})b_{in} \leq \eta_{inr} \qquad \forall i > 0 \tag{14}$$

$$\eta_{inr} \leq p_{inr} - (1 - w_{inr})a_{in} \qquad \forall i > 0 \tag{15}$$

Algorithm


$i = 0$
solve LP (7)–(11)
if feasible
choice prices
$i = 1$
solve LP (7)–(11)
if feasible
choice prices
$\cdots$
$i = |\mathscr{C}|$
solve LP (7)–(11)
if feasible
choice prices

Subgradient method

Initialization

Number iterations K

Initialize $k = 0$

Starting values $\lambda^0_{inr} = 0$

Subgradients

$s^k_{inr} = p^k_{in(r-1)} - p^k_{inr}$

$s^k_{inr} = 0 \; \forall i > 0, n, r \Rightarrow$ **STOP**

Lagrangian multipliers

Compute step size γ^k

$\lambda^{k+1}_{inr} = \lambda^k_{inr} + \gamma^k s^k_{inr}$

Stopping criterion

Increment k

If $k = K$, then **STOP**,

else go to *Subgradients*

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Conclusions and future work

- Assess the validity of the LR for the uncapacitated case
- Characterize a LR for the capacitated case
- Implement the LR scheme for the customer subproblem
- Test the complete LR with a large number of N and R

Questions?

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