Discrete choice models with multiplicative error terms

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Introduction

- Random utility models:

$$
\begin{aligned}
P(i|\mathcal{C}) &= \Pr(U_i \geq U_j \ \forall j \in \mathcal{C}) \\
&= \Pr(\mu V_i + \varepsilon_i \geq \mu V_j + \varepsilon_j \ \forall j \in \mathcal{C})
\end{aligned}
$$

- ε_i i.i.d. across individuals, so the scale is normalized.
- As a consequence, the scale is confounded with the parameters of V_i.
- The scale is directly linked with the variance of U_i

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Introduction

- The scale may vary from one individual to the next
- The scale may vary from one choice context to the next
  - SP/RP data
- Linear-in-parameter: $V_i = \mu\beta' x_i$
- Even if β is fixed, $\mu\beta$ is distributed

Introduction

Proposed solutions:

- Deterministically identify groups and estimate different scale parameters (introduces non linearities)
- Assume a distribution for μ: Bhat (1997); Swait and Adamowicz (2001); De Shazo and Fermo (2002); Caussade et al. (2005); Koppelman and Sethi (2005); Train and Weeks (2005)

Multiplicative error

Our proposal:

- RUM with multiplicative error

$$U_i = \mu V_i \varepsilon_i.$$

where

- μ is an independent individual specific scale parameter,
- $V_i < 0$ is the systematic part of the utility function, and
- $\varepsilon_i > 0$ is a random variable, independent of V_i and μ.

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Multiplicative error

- ε_i are i.i.d. across individuals
- Potential heteroscedasticity is captured by the individual specific scale μ.
- Sign restriction on V_i: natural if, for instance, generalized cost

Choice probability

The scale disappears

$$\begin{aligned} P(i|\mathcal{C}) &= \Pr(U_i \geq U_j, j \in \mathcal{C}) \\ &= \Pr(\mu V_i \varepsilon_i \geq \mu V_j \varepsilon_j, j \in \mathcal{C}) \\ &= \Pr(V_i \varepsilon_i \geq V_j \varepsilon_j, j \in \mathcal{C}), \end{aligned}$$

Taking logs

$$\begin{aligned} P(i|\mathcal{C}) &= \Pr(V_i \varepsilon_i \geq V_j \varepsilon_j, j \in \mathcal{C}) \\ &= \Pr(-V_i \varepsilon_i \leq -V_j \varepsilon_j, j \in \mathcal{C}) \\ &= \Pr(\ln(-V_i) + \ln(\varepsilon_i) \leq \ln(-V_j) + \ln(\varepsilon_j), j \in \mathcal{C}) \\ &= \Pr(-\ln(-V_i) - \ln(\varepsilon_i) \geq -\ln(-V_j) - \ln(\varepsilon_j), j \in \mathcal{C}). \end{aligned}$$

Choice probability

We define

$$-\ln(\varepsilon_i) = (c_i + \xi_i)/\lambda,$$

where

- c_i is the intercept,
- λ is the scale, constant across the population, as a consequence of the i.i.d. assumption on ε_i
- ξ_i are random variables with a fixed mean and scale

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Choice probability

- $P(i|\mathcal{C}) =$

$$\Pr(-\lambda \ln(-V_i) + c_i + \xi_i \geq -\lambda \ln(-V_j) + c_j + \xi_j, j \in \mathcal{C}),$$

  which is now a classical RUM with additive error.
- Important: contrarily to μ, the scale λ is constant across the population
- V_i must be normalized for the model to be identified. Indeed, for any $\alpha > 0$,

$$-\lambda \ln(-\alpha V_i) + c_i = -\lambda \ln(-V_i) - \lambda \ln(\alpha) + c_i$$

Choice probability

- When V_i is linear-in-parameters, it is sufficient to fix one parameter to either 1 or -1.
- e.g. normalize the cost coefficient to 1. Others become willingness-to-pay indicators.

Choice probability: MNL

$$P(i|\mathcal{C}) = \frac{e^{-\lambda \ln(-V_i)+c_i}}{\sum_{j\in\mathcal{C}} e^{-\lambda \ln(-V_j)+c_j}} = \frac{e^{c_i}(-V_i)^{-\lambda}}{\sum_{j\in\mathcal{C}} e^{c_j}(-V_j)^{-\lambda}},$$

where

- e^{c_j} are constants to be estimated

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Properties: distribution

If ξ_i is extreme value distributed, the CDF of ε_i is a generalization of an exponential distribution

$$F_{\varepsilon_i}(x) = 1 - e^{-x^{\lambda} e^{c_i}}.$$

Properties: elasticities

Define

$$\bar{V}_i = -\lambda \ln(-V_i) + c_i,$$

Then

$$e_i = \frac{\partial P(i)}{\partial \bar{V}_i} \frac{\partial \bar{V}_i}{\partial V_i} \frac{\partial V_i}{\partial x_{ik}} \frac{x_{ik}}{P(i)} = -\frac{\lambda}{V_i} \frac{\partial P(i)}{\partial \bar{V}_i} \frac{\partial V_i}{\partial x_{ik}} \frac{x_{ik}}{P(i)}$$

where $\partial P(i)/\partial \bar{V}_i$ is derived from the corresponding additive model. For MNL:

$$\frac{\partial P(i)}{\partial \bar{V}_i} = P(i)(1 - P(i)),$$

and

$$e_i = -\frac{\lambda}{V_i}(1 - P(i)) \frac{\partial V_i}{\partial x_{ik}} x_{ik}.$$

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Properties

In the paper (see `transp-or.epfl.ch`)

- Trade-offs: the same
- Expected Maximum Utility: derivation for MEV models
- Marshallian compensating variation when $-V_i$ is a generalized cost

$$-\int_a^b P(i) dV_i.$$

  - not as simple as the logsum
  - integral with no closed form

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Discussion

- Fairly general specification
- Free to make assumptions about ξ_i
- Parameters inside V_i can be random
- We may obtain MNL, GEV and mixtures of GEV models.
- c_i may depend on covariates, such that it is also possible to incorporate both observed and unobserved heterogeneity both inside and outside the log (examples in the paper).

Discussion

- If random parameters are involved, one must ensure that $P(V_i \geq 0) = 0$.
- How? The sign of a parameter can be restricted using, e.g., an exponential.
- For deterministic parameters: bounds constraints
- Maximum likelihood estimation is complicated in the general case.
- Taking logs provides an equivalent specification with additive independent error terms

Discussion

- Classical softwares can be used
- However, even when the Vs are linear in the parameters, the equivalent additive specification is nonlinear.
- OK with Biogeme

Case study: value of time in Denmark

- Danish value-of-time study
- SP data
- involves several attributes in addition to travel time and cost

Case study: value of time in Denmark

Model 1: Additive specification

$$V_i = \lambda(- \text{cost} + \beta_1 \text{ ae} + \beta_2 \text{ changes} + \beta_3 \text{ headway} + \beta_4 \text{ inVehTime} + \beta_5 \text{ waiting}),$$

Model 1: Multiplicative specification

$$V_i = -\lambda \log(\text{cost} - \beta_1 \text{ ae} - \beta_2 \text{ changes} - \beta_3 \text{ headway} - \beta_4 \text{ inVehTime} - \beta_5 \text{ waiting})$$

Model 1: additive

Variable number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	ae	-2.00	0.211	-9.46	0.00
2	changes	-36.1	6.89	-5.23	0.00
3	headway	-0.656	0.0754	-8.71	0.00
4	in-veh. time	-1.55	0.159	-9.76	0.00
5	waiting time	-1.68	0.770	-2.18	0.03
6	λ	0.0141	0.00144	9.82	0.00

Number of observations = 3455

$$\mathcal{L}(0) = -2394.824$$
$$\mathcal{L}(\hat{\beta}) = -1970.846$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 847.954$$
$$\rho^2 = 0.177$$
$$\bar{\rho}^2 = 0.175$$

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Model 1: multiplicative

Variable number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	ae	-0.672	0.0605	-11.11	0.00
2	changes	-5.22	1.54	-3.40	0.00
3	headway	-0.224	0.0213	-10.53	0.00
4	in-veh. time	-0.782	0.0706	-11.07	0.00
5	waiting time	-1.06	0.206	-5.14	0.00
6	λ	5.37	0.236	22.74	0.00

Number of observations = 3455

$$\mathcal{L}(0) = -2394.824$$
$$\mathcal{L}(\hat{\beta}) = -1799.086$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 1191.476$$
$$\rho^2 = 0.249$$
$$\bar{\rho}^2 = 0.246$$

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Model 1: result

- Same number of parameters
- Significant improvement of the fit: 171.76, from -1970.846 to -1799.086

Model 2: taste heterogeneity

- Additive specification:

$$V_i = \lambda(-\text{cost} - e^{\beta_5+\beta_6\xi}Y_i)$$

where

  - $Y_i =$

$$\text{inVehTime} + e^{\beta_1}\ \text{ae} + e^{\beta_2}\ \text{changes} + e^{\beta_3}\ \text{headway} + e^{\beta_4}\ \text{waiting}$$

  - $\xi \sim N(0,1)$

- Multiplicative specification

$$V_i = -\lambda \log(\text{cost} + e^{\beta_5+\beta_6\xi}Y_i),$$

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Model 2: additive

Variable number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	ae	0.0639	0.357	0.18	0.86
2	changes	2.88	0.373	7.73	0.00
3	headway	-0.999	0.193	-5.17	0.00
4	waiting time	-0.274	0.433	-0.63	0.53
5	scale (mean)	0.331	0.178	1.86	0.06
6	scale (stderr)	0.934	0.130	7.19	0.00
7	λ	0.0187	0.00301	6.20	0.00

Number of observations = 3455

Number of individuals = 523

Number of draws for SMLE = 1000

$$\mathcal{L}(0) = -2394.824$$

$$\mathcal{L}(\hat{\beta}) = -1925.467$$

$$\bar{\rho}^2 = 0.193$$

Model 2: multiplicative

Variable number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	ae	0.0424	0.0946	0.45	0.65
2	changes	2.24	0.239	9.38	0.00
3	headway	-1.03	0.0983	-10.48	0.00
4	waiting time	0.355	0.207	1.72	0.09
5	scale (mean)	-0.252	0.106	-2.38	0.02
6	scale (stderr)	1.49	0.123	12.04	0.00
7	λ	7.04	0.370	19.02	0.00

Number of observations = 3455

Number of individuals = 523

Number of draws for SMLE = 1000

$$\mathcal{L}(0) = -2394.824$$

$$\mathcal{L}(\hat{\beta}) = -1700.060$$

$$\bar{\rho}^2 = 0.287$$

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Model 2: result

- Same number of parameters
- Significant improvement of the fit: 225.764, from -1925.824 to -1700.060

Observed and unobs. heterogeneity

- Additive specification

$$V_i = \lambda(-\text{cost} - e^{W_i} Y_i)$$

where

- Y_i is defined as before
- $W_i =$

$$\beta_5 \text{ highInc} + \beta_6 \log(\text{inc}) + \beta_7 \text{ lowInc}$$
$$+\beta_8 \text{ missingInc} + \beta_9 + \beta_{10}\xi$$

- $\xi \sim N(0,1)$.

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Observed and unobs. heterogeneity

- Multiplicative specification:

$$V_i = -\lambda \log(\text{cost} + e^{W_i} Y_i).$$

Results:

- Again large improvement of the fit with the same number of parameters
- Additive: -1914.180
- Multiplicative: -1675.412
- Difference: 238.777

Summary: train data set

		Number of observations	3455
		Number of individuals	523
Model	Additive	Multiplicative	Difference
1	-1970.85	-1799.09	171.76
2	-1925.824	-1700.06	225.764
3	-1914.12	-1674.67	239.45

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Summary: bus data set

	Number of observations:		7751
	Number of individuals:		1148
Model	Additive	Multiplicative	Difference
1	-4255.55	-3958.35	297.2
2	-4134.56	-3817.49	317.07
3	-4124.21	-3804.9	319.31

Summary: car data set

		Number of observations:	8589
		Number of individuals:	1585
Model	Additive	Multiplicative	Difference
1	-5070.42	-4304.01	766.41
2	-4667.05	-3808.22	858.83
3	-4620.56	-3761.57	858.99

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Swiss value of time (SP)

- No improvement with fixed parameters
- Small improvement for random parameters

	Additive	Multiplicative	Diff.
Fixed param.	-1668.070	-1676.032	-7.96
Random param.	-1595.092	-1568.607	26.49

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Swissmetro (SP)

- Nested logit
- 16 variants of the model
  - Alternative Specific Socio-economic Characteristics (ASSEC)
  - Error component (EC)
  - Segmented travel time coefficient (STTC)
  - Random coefficient (RC): the coefficients for travel time and headway are distributed, with a lognormal distribution.

	RC	EC	STTC	ASSEC	Additive	Multiplicative	Difference
1	0	0	0	0	-5188.6	-4988.6	200.0
2	0	0	0	1	-4839.5	-4796.6	42.9
3	0	0	1	0	-4761.8	-4745.8	16.0
4	0	1	0	0	-3851.6	-3599.8	251.8
5	1	0	0	0	-3627.2	-3614.4	12.8
6	0	0	1	1	-4700.1	-4715.5	-15.4
7	0	1	0	1	-3688.5	-3532.6	155.9
8	0	1	1	0	-3574.8	-3872.1	-297.3
9	1	0	0	1	-3543.0	-3532.4	10.6
10	1	0	1	0	-3513.3	-3528.8	-15.5
11	1	1	0	0	-3617.4	-3590.0	27.3
12	0	1	1	1	-3545.4	-3508.1	37.2
13	1	0	1	1	-3497.2	-3519.6	-22.5
14	1	1	0	1	-3515.1	-3514.0	1.1
15	1	1	1	0	-3488.2	-3514.5	-26.2
16	1	1	1	1	-3465.9	-3497.2	-31.3

Concluding remarks

- Error term does not have to be additive
- With multiplicative errors, an equivalent additive formulation can be derived by taking logs
- Multiplicative is not systematically superior
- In our experiments, it outperforms additive spec. in the majority of the cases
- In quite a few cases, the improvement is very large, sometimes even larger than the improvement gained from allowing for unobserved heterogeneity.

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Concluding remarks

- Model with multiplicative error terms should be part of the toolbox of discrete choice analysts

Thank you!