Specification of the cross nested logit model with sampling of alternatives for route choice models

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Outline

Motivation

Route choice model

- Given an origin and a destination
- what is the preferred itinerary of a given traveler?

Main difficulties

- Very large choice set
- Structural correlation among alternatives

Very large choice set

Issue

Number of paths grows exponentially with the number of nodes

Literature

- link elimination Azevedo et al. (1993)
- link penalty de la Barra et al. (1993)
- labeled paths Ben-Akiva et al. (1984)
- SP on random costs Ramming (2002), Bovy and Fiorenzo-Catalano (2006)
- Sampling Frejinger et al. (2009)

Structural correlation

Issue

Significant physical overlap

Literature

- C-logit Cascetta et al. (1996)
- Path-size Ben-Akiva and Bierlaire (1999)
- Link-based cross-nested logit Prashker and Bekhor (1999)
- Error components Ramming (2002); Frejinger and Bierlaire (2007)

In this paper...

Methodology

- Cross Nested logit
- Sampling of alternatives

Builds on...

- McFadden (1978)
- Vovsha and Bekhor (1998)
- Bierlaire et al. (2008)
- Frejinger et al. (2009)
- Guevara and Ben-Akiva (2013)
- Flötteröd and Bierlaire (2013)

Outline

Logit model

$$P(i|\mathcal{C}) = \frac{e^{V_i}}{\sum_{j \in \mathcal{C}} e^{V_j}}$$

McFadden (1978)

Sampling protocol

- Sample subset $\mathcal{D} \subseteq \mathcal{C}$
- Sampling probability $q(\mathcal{D}|j)$
- Positive conditioning property

$$q(\mathcal{D}|i) > 0 \implies q(\mathcal{D}|j) > 0 \; \forall j \in \mathcal{D}.$$

Logit model

$$P(i|\mathcal{C}) \approx P(i|\mathcal{D}) = \frac{e^{V_i + \ln q(\mathcal{D}|i)}}{\sum_{j\in\mathcal{D}} e^{V_j + \ln q(\mathcal{D}|j)}}$$

Simple random sampling

- $q(\mathcal{D}|i) = q(\mathcal{D}|j)\ \forall i, j \in \mathcal{C}$
- Correction terms cancel out
- Irrelevant, circuitous paths
- How to draw?

Importance sampling

- $\ln q(\mathcal{D}|i)$ are confounded with ASC
- In route choice, usually no ASC
- How to draw?

How to draw?

Shortest path-based procedures

- link elimination: deterministic
- link penalty: deterministic
- labeled paths: deterministic
- SP on random costs:
  - some paths have 0 probability to be drawn
  - how to compute the sampling probability?

Metropolis-Hastings algorithm

Flötteröd and Bierlaire (2013)

Features

- Designed to draw from complex distributions
- Does not require the exact pmf/pdf
- Only a quantity proportional to it.
- For instance, to draw a path p with probability

$$\frac{b_p}{\sum_{q \in \mathcal{C}} b_q}$$

only b_p are needed.

Metropolis-Hastings algorithm

Methodology

- Design a Markov chain Q visiting the states/paths
- Accept/reject method
- Accept probability depends on
  - target (unnormalized) probabilities
  - transition probabilities of the Markov chain:

$$P(\text{accept}) = \min\left(\frac{b_q Q_{qp}}{b_p Q_{pq}}, 1\right)$$

Example

$$b = (20, 8, 3, 1) \quad \pi = (\frac{5}{8}, \frac{1}{4}, \frac{3}{32}, \frac{1}{32})$$

$$Q = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Run MH for 10000 iterations. Collect statistics after 1000

- Accept: [2488, 1532, 801, 283]
- Reject: [0, 952, 1705, 2239]
- Simulated: [0.627, 0.250, 0.095, 0.028]
- Target: [0.625, 0.250, 0.09375, 0.03125]

Sampling of paths

Difficulties

Design Q such that

- Every path can be generated with nonzero probability
- Both Q_{pq} and Q_{qp} are known

Flötteröd and Bierlaire (2013)

- Proof of concept on synthetic data
- Application to Tel Aviv (17K links, 8K nodes)

Outline

MEV models

Generic model

$$P(i|\mathcal{C}) = \frac{\exp(V_i + \ln G_i(\mathcal{C}))}{\sum_{j\in\mathcal{C}} \exp(V_j + \ln G_j(\mathcal{C}))}$$

where $G_i(\mathcal{C}) = G_i(e^{V_1}, \ldots, e^{V_J})$ is the derivative of the CPGF wrt e^{V_i}.

Cross nested logit

$$G_i(\mathcal{C}) = \sum_{m=1}^{M} \left[\mu\alpha_{im} e^{V_i(\mu_m - 1)} \left(\sum_{j\in\mathcal{C}} \alpha_{jm} e^{\mu_m V_j} \right)^{\frac{\mu-\mu_m}{\mu_m}} \right],$$

MEV models

Generic model

$$P(i|\mathcal{C}) = \frac{\exp(V_i + \ln G_i(\mathcal{C}))}{\sum_{j\in\mathcal{C}} \exp(V_j + \ln G_j(\mathcal{C}))}$$

where $G_i(\mathcal{C}) = G_i(e^{V_{1n}}, \ldots, e^{V_j})$ is the derivative of the CPGF wrt e^{V_i}.

Cross nested logit

$$G_i(\mathcal{C}) = \sum_{m=1}^{M} \left[\mu\alpha_{im} e^{V_i(\mu_m - 1)} \left(\sum_{j\in\mathcal{C}} \alpha_{jm} e^{\mu_m V_j} \right)^{\frac{\mu-\mu_m}{\mu_m}} \right],$$

Sampling and MEV

$$P(i|\mathcal{C}) = \frac{\exp(V_i + \ln G_i(\mathcal{C}))}{\sum_{j\in\mathcal{C}} \exp(V_j + \ln G_j(\mathcal{C}))}$$

Sampling correction

Bierlaire et al. (2008)

- If $\ln G_j(\mathcal{C})$ is known, same idea as for logit

$$\Pr(i|\mathcal{D}) = \frac{\exp(V_i + \ln G_i(\mathcal{C}) + \ln \Pr(\mathcal{D}|i))}{\sum_{j\in\mathcal{D}} \exp(V_j + \ln G_j(\mathcal{C}) + \ln \Pr(\mathcal{D}|j))}.$$

- Not confounded with the constants anymore.

Sampling and MEV

Correction term

$$\Pr(\mathcal{D}|p) \propto \frac{k_p}{q(p)}$$

where

- k_p is the number of times path p has been generated
- $q(p)$ is the sampling probability of path p
- $q(p) \propto b_p$

Model I

$$\Pr(i|\mathcal{D}) = \frac{\exp(V_i + \ln G_i(\mathcal{C}) + \ln \frac{k_i}{b_i})}{\sum_{j\in\mathcal{D}} \exp(V_j + \ln G_j(\mathcal{C}) + \ln \frac{k_j}{b_j})},$$

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Approximation of $\ln G_i(\mathcal{C})$

Guevara and Ben-Akiva (2013)

$$G_i(\mathcal{C}) \approx \widehat{G}_i(D, w) = \sum_{m=1}^{M} \left[\mu \alpha_{im} e^{V_i(\mu_m - 1)} \left(\sum_{j \in \mathcal{D}} w_j \alpha_{jm} e^{\mu_m V_j} \right)^{\frac{\mu - \mu_m}{\mu_m}} \right]$$

where w_j expansion factor to be defined.

Expansion factors: Guevara and Ben-Akiva (2013)

Realized / expected

$$w_j^G = \frac{k_j}{\mathrm{E}[k_j]} = \frac{k_j}{q(j)R} = \frac{k_j B}{b(j)R}$$

where

- R is the number of draws used to generate $\mathcal{D}$
- $B = \sum_{j \in \mathcal{C}} b(j)$ [Requires enumeration of $\mathcal{C}$]

Approximate B

$$B = \sum_{j \in \mathcal{C}} b(j) = |\mathcal{C}| \frac{\sum_{i \in \mathcal{C}} b(i)}{|\mathcal{C}|} = |\mathcal{C}| \bar{b},$$

and

$$\bar{b} = \frac{\sum_{i \in \mathcal{C}} b(i)}{|\mathcal{C}|} \approx \frac{\sum_{i \in \mathcal{D}} b(i)}{|\mathcal{D}|}.$$

Expansion factors: Guevara and Ben-Akiva (2013)

Approximation

$$w_j^G = \frac{k_j}{b(j)R} \frac{|\mathcal{C}|}{|\mathcal{D}|} \sum_{i \in \mathcal{D}} b(i)$$

which require $|\mathcal{C}|$

Approximate $|\mathcal{C}|$

Roberts and Kroese (2007)

N random walks in the network

$$|\mathcal{C}| \approx \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\ell(i)}.$$

$\ell^{(i)}$: likelihood of the path generated by the algorithm during run i

Expansion factors: Frejinger et al. (2009)

Account for the upper bound

$$w_j^F = \begin{cases} 1 & \text{if } b(j)R > B, \\ \frac{B}{b(j)R} & \text{otherwise.} \end{cases}$$

Same approximation of B

$$B \approx \frac{|\mathcal{C}|}{|\mathcal{D}|} \sum_{i \in \mathcal{D}} b(i)$$

Again, requires $|\mathcal{C}|$

Expansion factors: Lai and Bierlaire (2014)

Avoiding $|\mathcal{C}|$

- Let s be the path which has been sampled the most in $\mathcal{D}$
- $k_s \geq k_p$, for each $p \in \mathcal{D}$.
- If sample is large enough, $k_s \approx q(s)R$

$$w_j^G = \frac{k_j}{q(j)R} \approx w_j^L = \frac{k_j}{q(j)R}\frac{q(s)R}{k_s} = \frac{k_j}{b(j)}\frac{b(s)}{k_s}$$

which does not require B or $|\mathcal{C}|$.

Expansion factors

- Guevara and Ben-Akiva (2013)

$$w_j^G = \frac{k_j}{b(j)R} B \text{ with } B \approx \frac{|\mathcal{C}|}{|\mathcal{D}|} \sum_{i \in \mathcal{D}} b(i)$$

- Frejinger et al. (2009)

$$w_j^F = \begin{cases} 1 & \text{if } b(j)R > B, \\ \frac{B}{b(j)R} & \text{otherwise.} \end{cases} \quad \text{with } B \approx \frac{|\mathcal{C}|}{|\mathcal{D}|} \sum_{i \in \mathcal{D}} b(i).$$

- Lai and Bierlaire (2014)

$$w_j^L = \frac{k_j}{b(j)} \frac{b(s)}{k_s}$$

Models to be compared

- Model I: true G_i (impossible in practice)

$$\Pr(i|\mathcal{D}) = \frac{\exp(V_i + \ln G_i(\mathcal{C}) + \ln \frac{k_i}{b(i)})}{\sum_{j\in\mathcal{D}} \exp(V_j + \ln G_j(\mathcal{C}) + \ln \frac{k_j}{b(j)})}$$

- Model II: the proposed model

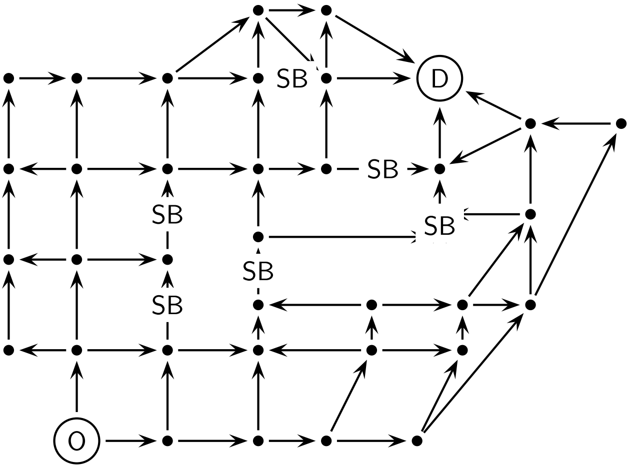
$$\Pr(i|\mathcal{D},\mathcal{D}',w) = \frac{\exp(V_i + \ln G_i(\mathcal{D}',w)) + \ln \frac{k_i}{b(i)})}{\sum_{j\in\mathcal{D}} \exp(V_j + \ln G_j(\mathcal{D}',w) + \ln \frac{k_j}{b(j)})}.$$

- Model III: no expansion factor, no sampling correction (benchmark)

$$\Pr(i|\mathcal{D},\mathcal{D}') = \frac{\exp(V_i + \ln G_i(\mathcal{D}',1))}{\sum_{j\in\mathcal{D}} \exp(V_j + \ln G_j(\mathcal{D}',1))},$$

Outline

The network: 170 paths (Frejinger (2008))



The true model: cross-nested logit

Utility

$$V_i = \beta_L L_i + \beta_{\mathrm{SB}} \mathrm{SB}_i,$$

“True” parameters

- $\beta_L = -0.5$ and $\beta_{\mathrm{SB}} = -0.1$
- $\mu_m = 1.5$ for each link m
- $\alpha_{im} = \ell_m / L_i$

Data

3000 synthetic choices

Re-estimate the parameters of the true model

Full choice set

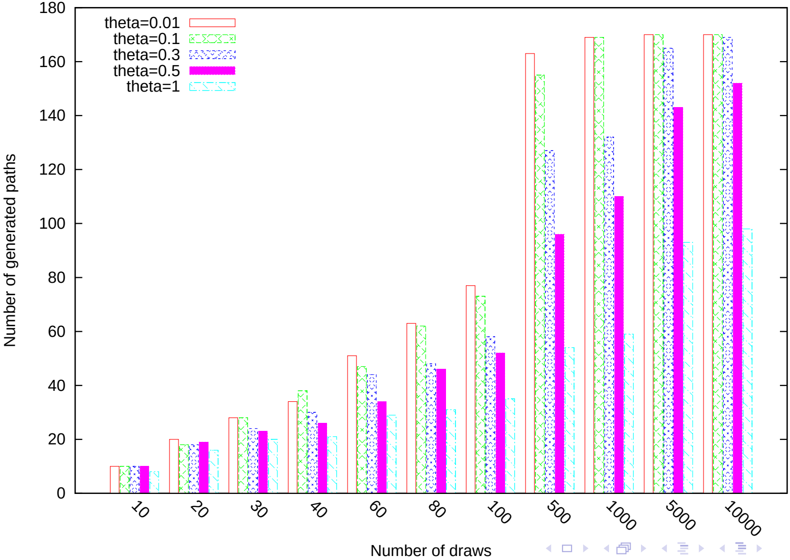
Parameters	Est.	Std err.	t-test (0)	t-test (true)
β_L	-0.501	0.0118	43.1	0.678
β_{SB}	-0.0910	0.0240	3.19	0.375
μ_m	1.49	0.0269	55.2	0.0535

Sampling paths

Metropolis-Hastings

$$b(i) = \exp(-\theta L_i), \quad \theta \geq 0$$

Number of generated paths



Model I: true G_i — MH $\theta = 0.5$

10 draws	Est.	Std err.	t-test(0)	t-test(true)
β_L (-0.5)	-0.443	0.0163	27.3	3.48
β_{SB} (-0.1)	-0.0647	0.0427	1.51	0.826
μ_m (1.5)	1.56	0.0340	45.8	1.72
Estimation time: 1362 seconds				
40 draws	Est.	Std err.	t-test(0)	t-test(true)
β_L (-0.5)	-0.479	0.0156	30.8	1.34
β_{SB} (-0.1)	-0.0720	0.0393	1.83	0.713
μ_m (1.5)	1.51	0.0322	47.0	0.367
Estimation time: 4648 seconds				

Model I: true G_i — MH $\theta = 0.01$

10 draws	Est.	Std err.	t-test(0)	t-test(true)
β_L (-0.5)	-0.535	0.0174	30.8	2.01
β_{SB} (-0.1)	-0.132	0.0545	2.42	0.580
μ_m (1.5)	1.41	0.0355	39.8	2.47
Estimation time: 1612 seconds				
40 draws	**Est.**	**Std err.**	**t-test(0)**	**t-test(true)**
β_L (-0.5)	-0.544	0.0160	33.9	2.76
β_{SB} (-0.1)	-0.130	0.0410	3.16	0.726
μ_m (1.5)	1.41	0.0322	43.8	2.85
Estimation time: 4914 seconds				

Model I: comments

- Trade-off between dispersion (low θ) and number of draws
- Lower value of θ requires more draws
- $\theta = 0.5$, 40 draws: parameters are correctly estimated
- First sampling scheme is validated
- No specific guideline for θ and R

Approximating $\bar{b}$ and $|\mathcal{C}|$

Protocol

- For $\bar{b}$: generate $\mathcal{D}$ using MH with 100 draws and $\theta = 0.01$
- For $|\mathcal{C}|$: generate 10000 paths using random walk
- Repeat 100 times
- Compute the empirical mean and standard error

Results

	True	Mean	Std err	t-test(true)
$\bar{b}$	0.688	0.684	0.0023	1.62
$\|\mathcal{C}\|$	170	169.8	2.52	0.0722

Model II

Protocol

- Denominator: $\mathcal{D}$ generated with MH (40 draws, $\theta = 0.5$)
- Expansion factor: $\mathcal{D}'$ MH with various values

Model II: 100 draws (t-test vs true value)

Sampling protocol for $\mathcal{D}'$: $\theta = 0.5$					
	Mod. II				Mod. III
	w^G	w^F	w^L	$w = 1$	
β_L	2.48	4.34	**1.25**	3.59	19.4
β_{SB}	**0.910**	**0.867**	**0.722**	**0.179**	**0.221**
μ_m	2.02	3.09	**0.437**	2.98	**1.06**
Sampling protocol for $\mathcal{D}'$: $\theta = 0.01$					
	Mod. II				Mod. III
	w^G	w^F	w^L	$w = 1$	
β_L	4.61	4.23	4.48	4.30	18.9
β_{SB}	**0.303**	**0.297**	**0.254**	**0.467**	**0.634**
μ_m	4.70	4.71	5.38	4.55	3.63

Model II: 200 draws (t-test vs true value)

Sampling protocol for $\mathcal{D}'$: $\theta = 0.5$					
	Mod. II				Mod. III
	w^G	w^F	w^L	$w = 1$	
β_L	**0.578**	10.5	**0.0374**	3.38	18.9
β_{SB}	**0.513**	**0.194**	**0.440**	**0.259**	**0.269**
μ_m	**1.36**	5.02	**1.34**	3.07	**0.965**
Sampling protocol for $\mathcal{D}'$: $\theta = 0.01$					
	Mod. II				Mod. III
	w^G	w^F	w^L	$w = 1$	
β_L	3.51	3.84	2.86	4.37	18.5
β_{SB}	**0.173**	**0.119**	**0.298**	**0.409**	**0.571**
μ_m	9.11	8.65	7.19	5.41	3.72

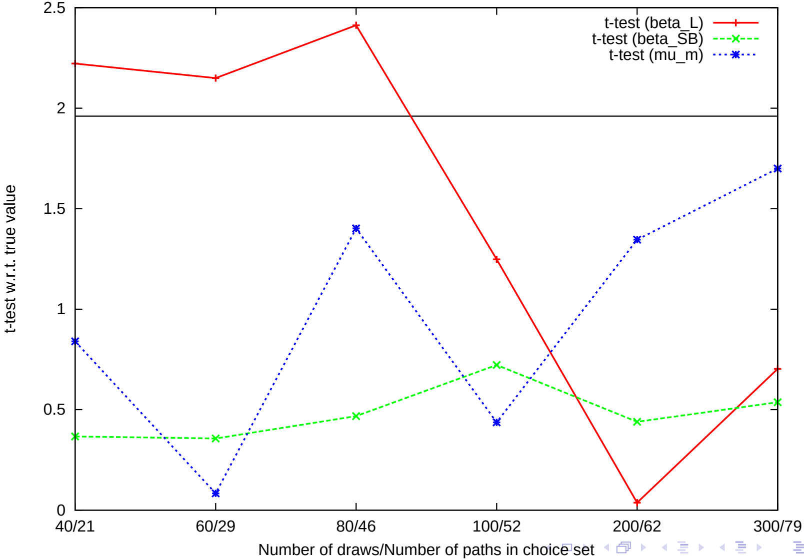
Model II: 300 draws (t-test vs true value)

Sampling protocol for $\mathcal{D}'$: $\theta = 0.5$					
	Mod. II				Mod. III
	w^G	w^F	w^L	$w = 1$	
β_L	**0.981**	3.62	**0.703**	**0.981**	19.3
β_{SB}	**0.428**	**1.34**	**0.537**	**0.428**	**0.0052**
μ_m	2.28	3.12	**1.70**	2.28	**1.66**
Sampling protocol for $\mathcal{D}'$: $\theta = 0.01$					
	Mod. II				Mod. III
	w^G	w^F	w^L	$w = 1$	
β_L	**0.809**	**0.0271**	**1.02**	5.05	18.5
β_{SB}	**0.565**	**0.780**	**0.480**	**0.564**	**0.654**
μ_m	**1.66**	**0.650**	**1.84**	5.19	3.01

Comments

- $\theta = 0.5$ seems again the most appropriate
- Model II outperforms Model III (no correction, no expansion factor)
- New expansion factor is the most appropriate (already good results with 100 draws)
- μ_m seems to be the most sensitive parameters

t-tests with w^L and $\theta = 0.5$



Outline

Tianhe region (CBD) of Guangzhou (China)

Data

Network

- 208 nodes
- 662 links
- 24 major roads
- 34 arterial streets
- 32 minor streets
- 57 signalized intersections

GPS traces from taxis

- 7 ODs
- 740 trips

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Model

Utility

$$V_i = \beta_L \text{Length}_i + \beta_{\text{ARR}} \text{ArteryRoadRatio}_i + \beta_S \text{Signal}_i.$$

Cross-nested logit

- Two nests: μ: non-artery roads, μ_{mA}: artery roads
- $\alpha_{im} = \ell_m / L_i$

MH sampling

θ	$\lvert\mathcal{D}\rvert$	θ	$\lvert\mathcal{D}\rvert$
0.005	29	0.0025	3813
0.004	54	0.0023	5624
0.003	201	0.002	7766
0.0028	2036	0.001	9836

Estimation results (with Matlab, Intel i5 with 4GB RAM, one processor)

$\theta = 0.003$			
	Model II		
	Est.	Std. err.	t-test (0)
β_L	-1.58	0.0566	27.9
β_{ARR}	8.09	0.636	12.7
β_S	-0.513	0.267	1.91
μ_m	3.90	0.117	33.3
μ_{mA}	2.22	0.257	8.62
Number of observations	740 trips from 7 OD		
Null log likelihood	-3.4078e+03		
Final log likelihood	-1.9206e+03		
Estimation time	22.32 hours		

Conclusion

Contributions

- Application of sampling of alternative for MEV and route choice
- New expansion factor
- Validity check: synthetic data
- Feasibility check: real data
- Heavy, but tractable

Future work

- Investigate other nesting structures
- Different ways to approximate G_i
- Estimation of α_{im} (?)

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