Large-Scale Continuous Pricing With Discrete Choice Demand Modeling

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Outline

The Continuous Pricing Problem (CPP)

CPP

- Supplier offers J products for sale. Goal: determine optimal **price** for each product to maximize total **profit**.
- Demand for each product is modeled using a **discrete choice model** (DCM).
- Have to include an **opt-out** alternative (free).

DCM

- For every **costumer** n and **product** i a stochastic **utility** U_{in} is defined, which depends on **socio-economic** characteristics of the individual and **attributes** of the products (e.g. the price).

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The Continuous Pricing Problem (CPP)

Utility

- **Utility** of alternative i for costumer n:

$$U_{0n} = \sum_{k \neq p} \beta_k^{0n} x_{0nk} + \varepsilon_{0n}$$

$$U_{in} = \sum_{k \neq p} \beta_k^{in} x_{ink} + \beta_p^{in} p_i + \varepsilon_{in} \qquad \forall i \geq 1$$

- β_k^{in} : parameters (exogenous)
- x_{ink} : attributes (exogenous)
- p_i : price of alternative i
- ε_{in} : stochastic error term

The Continuous Pricing Problem (CPP)

Probability

- **Probability** that costumer n chooses alternative i:

$$P_n(i) = \mathbb{P}(U_{in} \geq U_{jn} \ \forall j \in \{0, \ldots, J\})$$

- **Logit** ($\varepsilon_{in} \sim$ i.i.d. Gumbel(0, 1)):

$$P_n(i) = \frac{e^{V_{in}}}{\sum_{j \in C_n} e^{V_{jn}}}$$

- **Mixed Logit** (Logit + $\beta_k \sim F(\beta_k|\theta)$):

$$P_n(i) = \int \frac{e^{V_{in}(\beta_{kn})}}{\sum_{j \in C_n} e^{V_{jn}(\beta_{kn})}} f(\beta_k|\theta) d\beta_k$$

Monte Carlo Simulation

- **Simulate** R scenarios (draws), each with **deterministic** utilities U_{inr}:

$$\begin{aligned} U_{inr} &= \sum_{k \neq p} \beta_k^{in} x_{ink} + \beta_p^{in} p_i + \varepsilon_{inr} \\ &= c_{inr} + \beta_p^{in} p_i \end{aligned}$$

- **Choice** variables:

$$\omega_{inr} = \begin{cases} 1 \text{ if } U_{inr} = \max_j U_{jnr} \\ 0 \text{ else} \end{cases}$$

- Probability **estimator**:

$$\widehat{P}_n(i) = \frac{1}{R} \sum_r \omega_{inr}$$

MILP formulation [Paneque et al., 2021]

$$\max_{p,\omega,U,H} \frac{1}{R} \sum_{r} \sum_{n} \sum_{i \geq 1} p_i \omega_{inr}$$

s.t.

$$\begin{aligned}
\sum_i \omega_{inr} &= 1 && \forall n, r && (\mu_{nr}) \\
H_{nr} &= \sum_i U_{inr}\omega_{inr} && \forall n, r && (\zeta_{nr}) \\
H_{nr} &\geq c_{0nr} && \forall n, r && (\alpha_{0nr}) \\
H_{nr} &\geq U_{inr} && \forall i \geq 1, n, r && (\alpha_{inr}) \\
U_{inr} &= c_{inr} + \beta_p^{in} p_i && \forall i \geq 1, n, r && (\kappa_{inr}) \\
\omega &\in \{0,1\}^{(J+1)NR} \\
p, U, H &\in \mathbb{R}^J, \mathbb{R}^{(J+1)NR}, \mathbb{R}^{NR}
\end{aligned}$$

Literature

[Li and Huh, 2011], [Gallego and Wang, 2014], ...

- Extensive research for **Logit** and **Nested Logit** (NL) integration.

[Li et al., 2019], [Marandi and Lurkin, 2020],
[van de Geer and den Boer, 2022], ...

- Tackled **Mixed Logit** (ML) integration in various ways, e.g. **approximations**, assuming consumer **homogeneity**, or considering only **discrete** probability measures.

[Paneque et al., 2022]

- Apply a **Lagrangian** decomposition scheme to speed up the solution of the MILP.
- Include capacity constraints.
- Solve instances with up to 100 draws.

Outline

QCQP formulation

$$\max_{p,\omega,U,H} \frac{1}{R} \sum_{r} \sum_{n} \sum_{i \geq 1} p_i \omega_{inr}$$

s.t.

$$\begin{aligned}
\sum_i \omega_{inr} &= 1 && \forall n, r && (\mu_{nr}) \\
H_{nr} &= \sum_i U_{inr}\omega_{inr} && \forall n, r && (\zeta_{nr}) \\
H_{nr} &\geq c_{0nr} && \forall n, r && (\alpha_{0nr}) \\
H_{nr} &\geq U_{inr} && \forall i \geq 1, n, r && (\alpha_{inr}) \\
U_{inr} &= c_{inr} + \beta_p^{in} p_i && \forall i \geq 1, n, r && (\kappa_{inr}) \\
\omega &\in [0,1]^{(J+1)NR} \\
p, U, H &\in \mathbb{R}^J, \mathbb{R}^{(J+1)NR}, \mathbb{R}^{NR}
\end{aligned}$$

QCLP formulation

$$\max_{p,\omega,\eta,U,H} \ \frac{1}{R}\sum_{r}\sum_{n}\sum_{i\geq 1}\eta_{inr}$$

s.t.

$$\begin{aligned}
\sum_{i}\omega_{inr} &= 1 && \forall n, r && (\mu_{nr})\\
H_{nr} &= \sum_{i} c_{inr}\omega_{inr} + \beta_p^{in}\eta_{inr} && \forall n, r && (\zeta_{nr})\\
H_{nr} &\geq c_{0nr} && \forall n, r && (\alpha_{0nr})\\
H_{nr} &\geq U_{inr} && \forall i \geq 1, n, r && (\alpha_{inr})\\
U_{inr} &= c_{inr} + \beta_p^{in} p_i && \forall i \geq 1, n, r && (\kappa_{inr})\\
\eta_{inr} &= p_i\omega_{inr} && \forall i \geq 1, n, r && (\lambda_{inr})\\
\omega &\in [0,1]^{(J+1)NR}\\
p, U, H &\in \mathbb{R}^{J}, \mathbb{R}^{(J+1)NR}, \mathbb{R}^{NR}
\end{aligned}$$

Spatial Branch & Bound (B&B) Algorithm

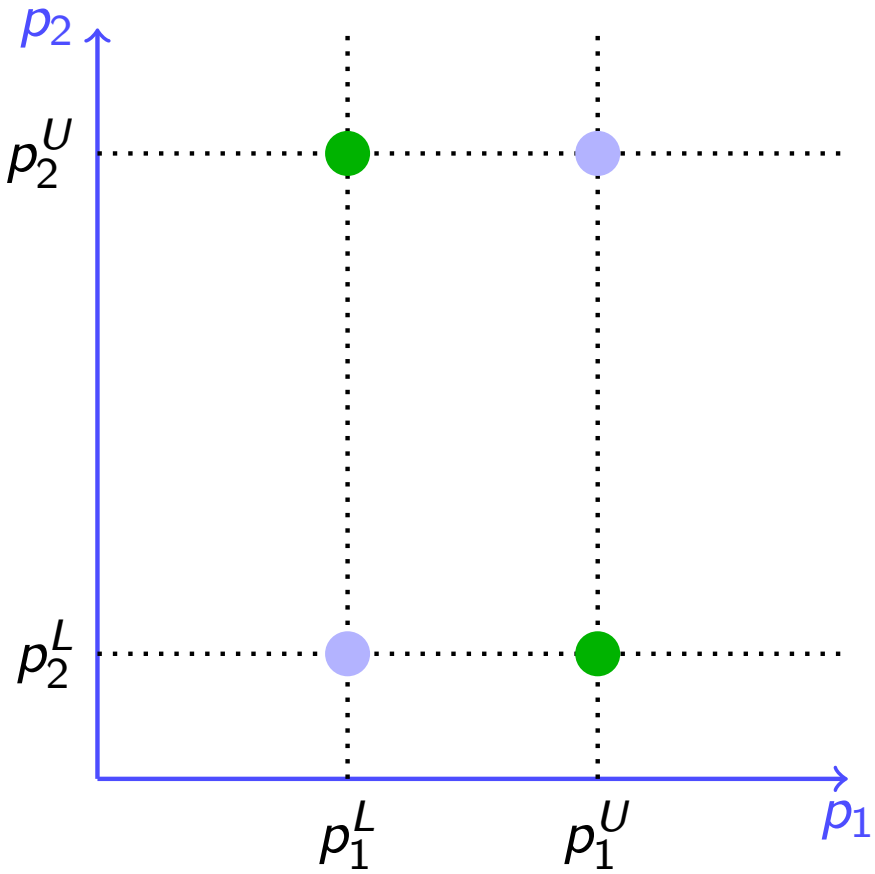
Relaxation

- Assume reasonable **bounds** on price, $p_i \in [p_i^L, p_i^U]$.
- Relax the constraint $\eta_{inr} = p_i \omega_{inr}$ with a **McCormick** envelope:

$$
\begin{aligned}
\eta_{inr} &\geq p_i^L \omega_{inr} && \forall i \geq 1, n, r && (\lambda^1_{inr}) \\
\eta_{inr} &\geq p_i^U \omega_{inr} + p_i - {p_i}^U && \forall i \geq 1, n, r && (\lambda^2_{inr}) \\
\eta_{inr} &\leq p_i^L \omega_{inr} + p_i - {p_i}^L && \forall i \geq 1, n, r && (\lambda^3_{inr}) \\
\eta_{inr} &\leq p_i^U \omega_{inr} && \forall i \geq 1, n, r && (\lambda^4_{inr})
\end{aligned}
$$

Simplification

- For a given set of bounds $p_i \in [p_i^L, p_i^U]$, some choices could be pre-determined.



Simplification

Procedure for each n and r

- For each alternative, consider its **best case** and **worst case** comparison to the others, i.e. set its price to the lower bound and all others to the upper bound, or inversely.
- If alternative i is **dominated** in its best case, set $w_{inr} = 0$.
- If alternative i is **dominating** in its worst case, set $w_{inr} = 1$.

Note

This happens often when bounds are **tight**.

Spatial Branch & Bound (B&B) Algorithm

Convergence

- Every relaxation provides an **upper bound** on the maximal profit.
- Any solution value for the price gives an immediate feasible solution (**lower bound**) due to integrality.

Custom B&B vs. standard B&B

- We only branch on $\boldsymbol{J}$ continuous variables, instead of branching on $(\boldsymbol{J}+\mathbf{1})\boldsymbol{NR}$ continuous (QCLP) or binary (MILP) variables.

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Spatial Branch & Bound (B&B) Algorithm

Priority queue

- We utilize a **best-first search**, as we can easily generate feasible solutions (upper bounds).

Branching strategy

- Branch on alternative where relaxation has **lowest average resolution.**

$$\texttt{branching\_index} \quad = \quad \text{argmax}\left\{ \sum_{nr} |\overline{\eta}_{inr} - \overline{p}_i \overline{\omega}_{inr}| \ , i \in \{1, \ldots, J\} \right\}$$

Enumeration strategy

- If two nodes have the same upper bound, we chose the one first that explores the **lower price range**, i.e. where the **guaranteed capture rate is largest**.

Benders Decomposition

- The McCormick relaxation (linear program) at each node is solved by the use of a **Benders decomposition**:

Benders Decomposition

- **Master problem (MP)**: compute *candidate solutions* for the price.
- **Subproblem (SP)**: given a price, compute reduced costs to construct *optimality cut* to add to the MP.
- SP is highly **separable**: utility maximization for each customer and scenario can be solved **independently**.
- Make use of fully **disaggregated optimality cuts** (one cut per customer and scenario).

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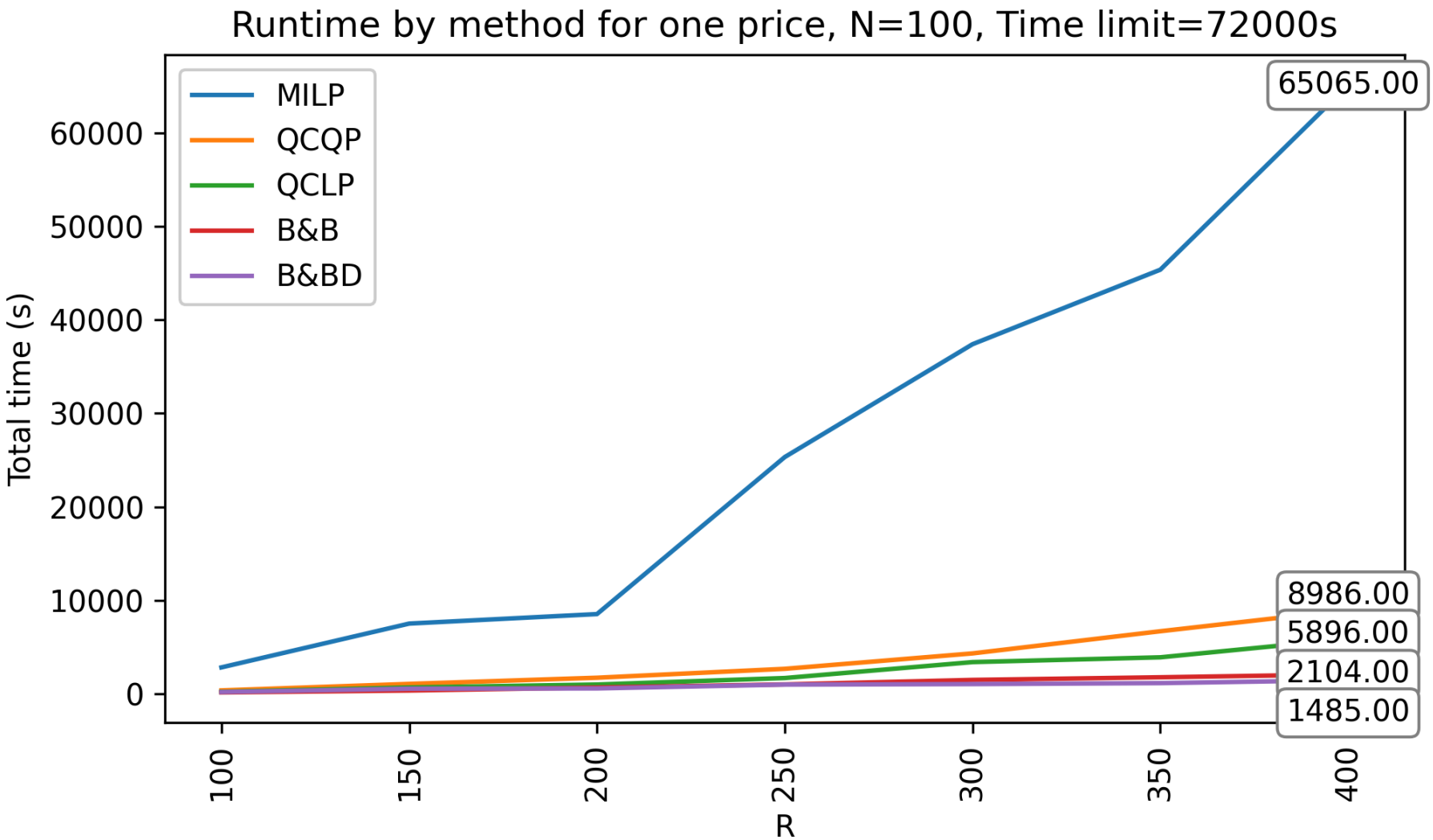
Case Study

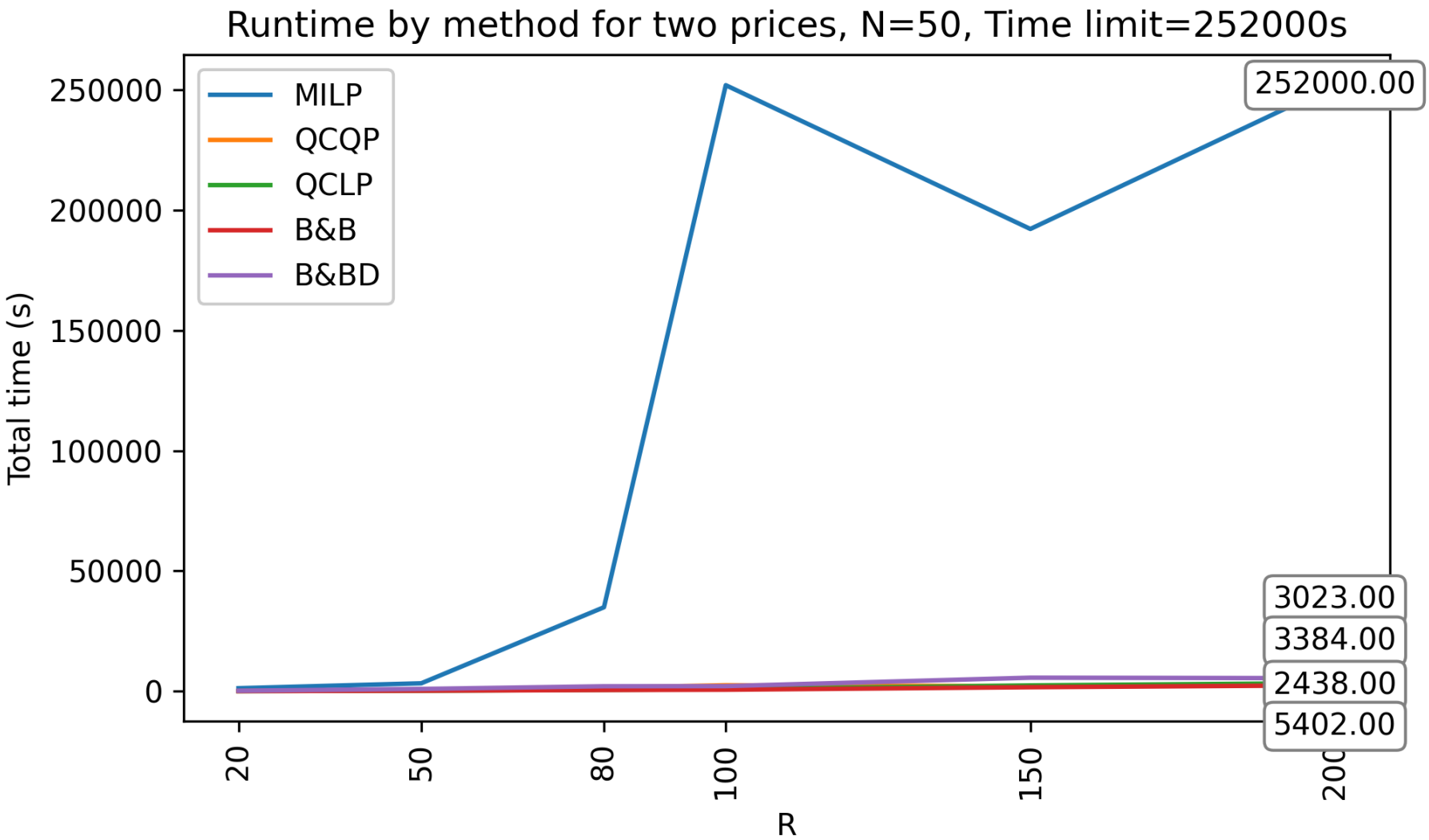
Parking space operator [Ibeas et al., 2014]

- **Alternatives**: Paid-Street-Parking (PSP), Paid-Underground-Parking (PUP) and Free-Street-Parking (FSP).
- Optimize prices for PSP and PUP, FSP is the **opt-out** alternative.
- **Socio-economic characteristics**: trip origin, vehicle age, driver income, residence area.
- **Product attributes**: access time to parking, access time to destination, and parking fee (price).
- Choice model is a **Mixed Logit**, $\beta_{\text{fee}}, \beta_{\text{time\_parking}} \sim \mathcal{N}(\mu, \sigma)$.

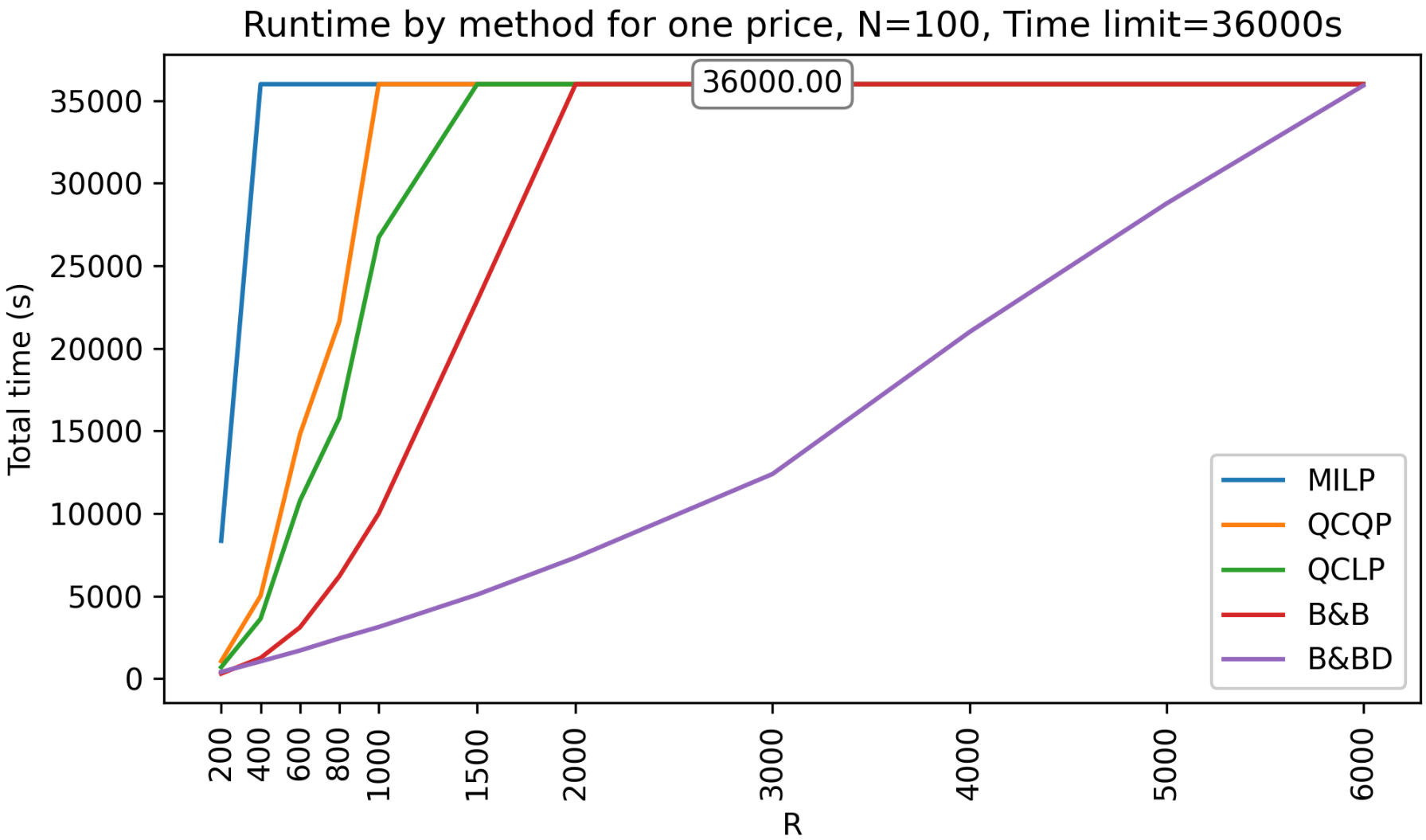
Computational results



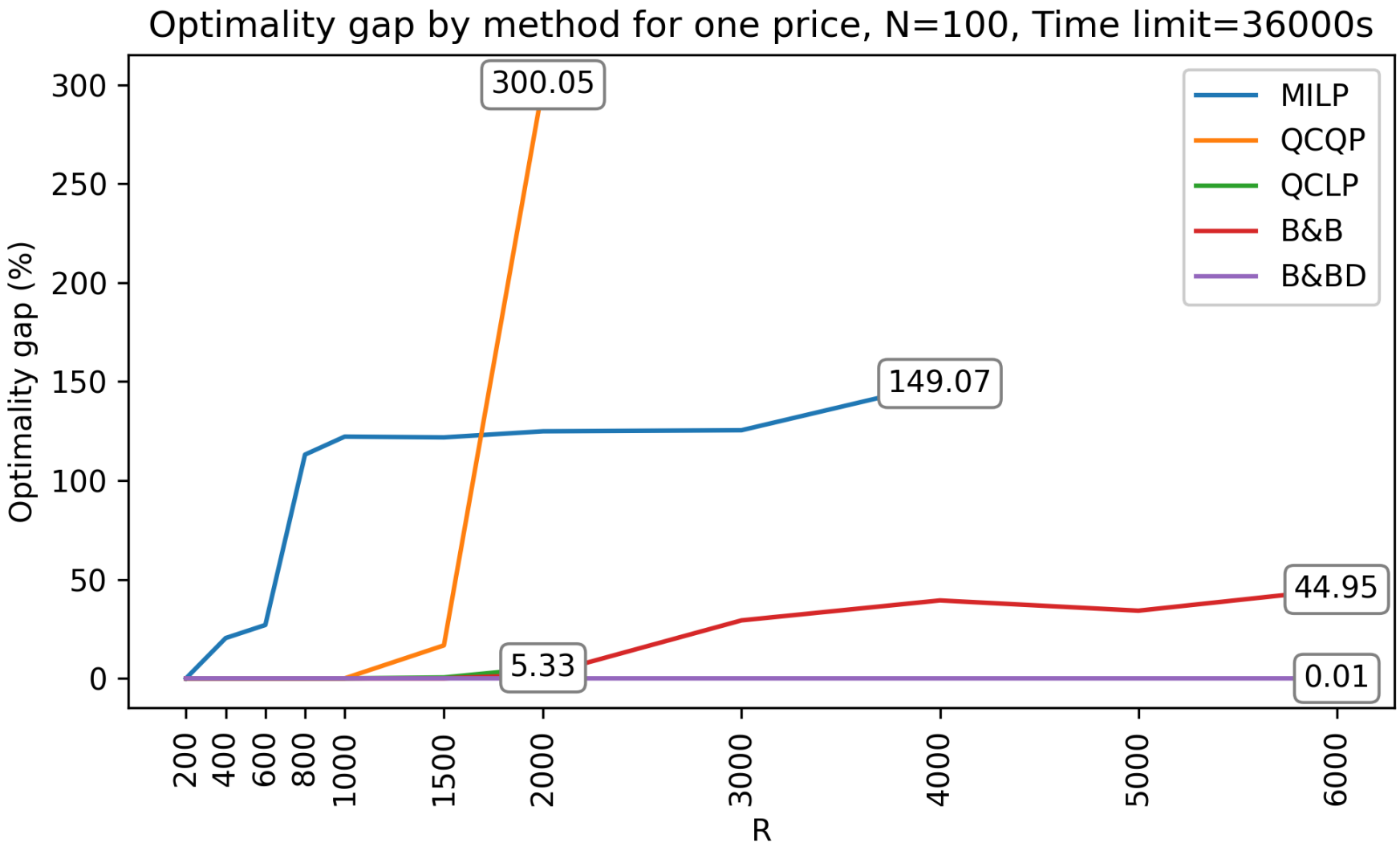
Computational results



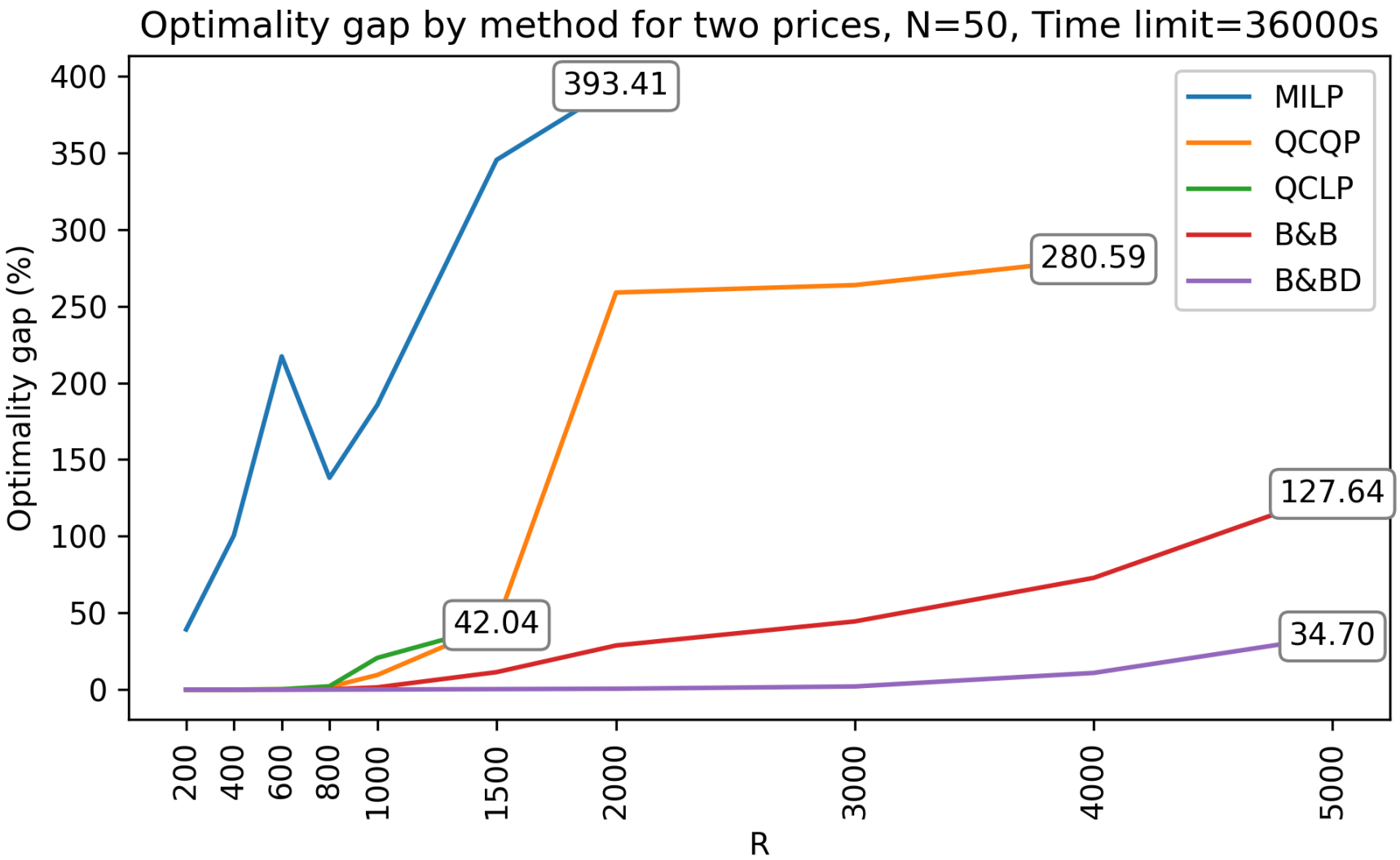
Computational results



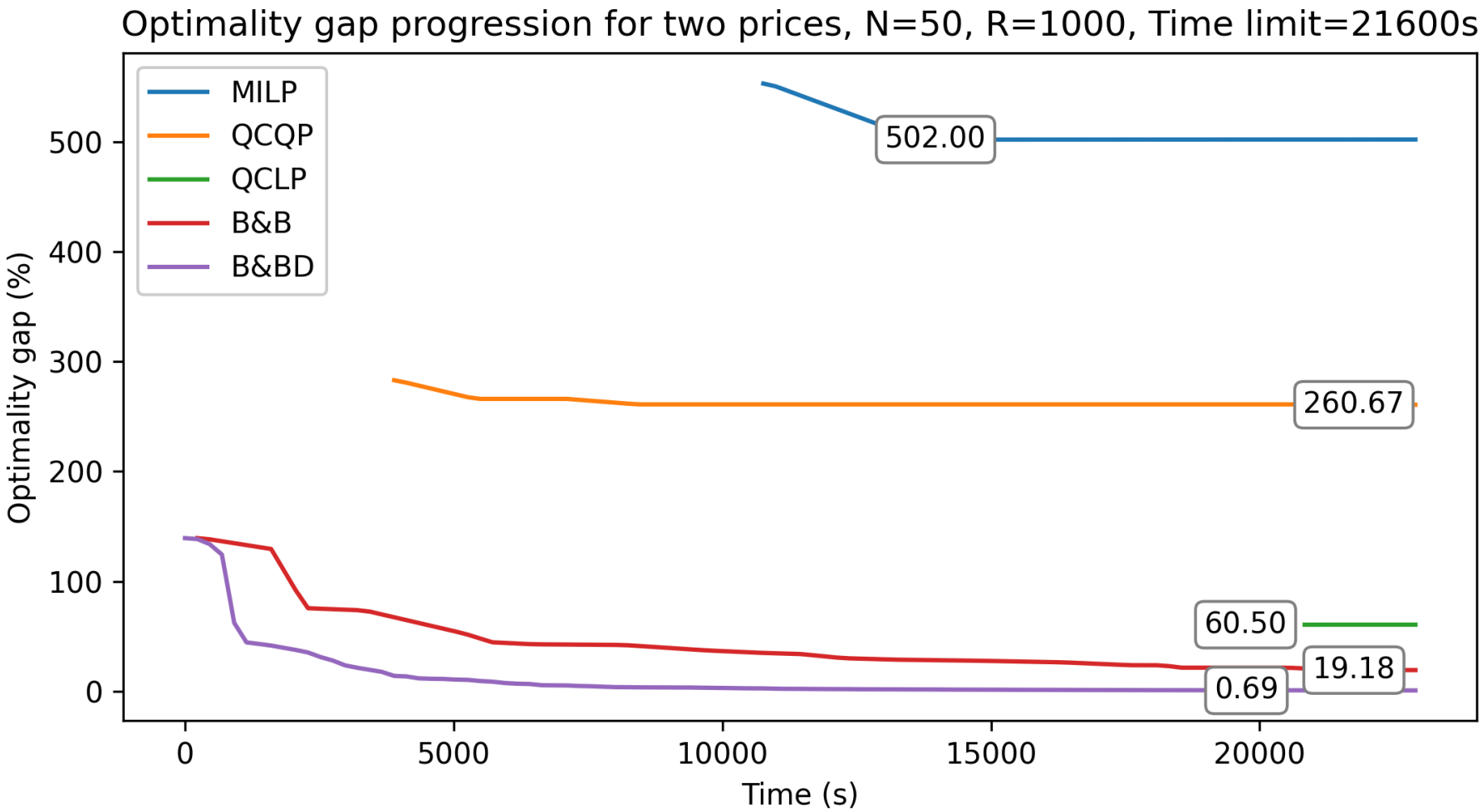
Computational results



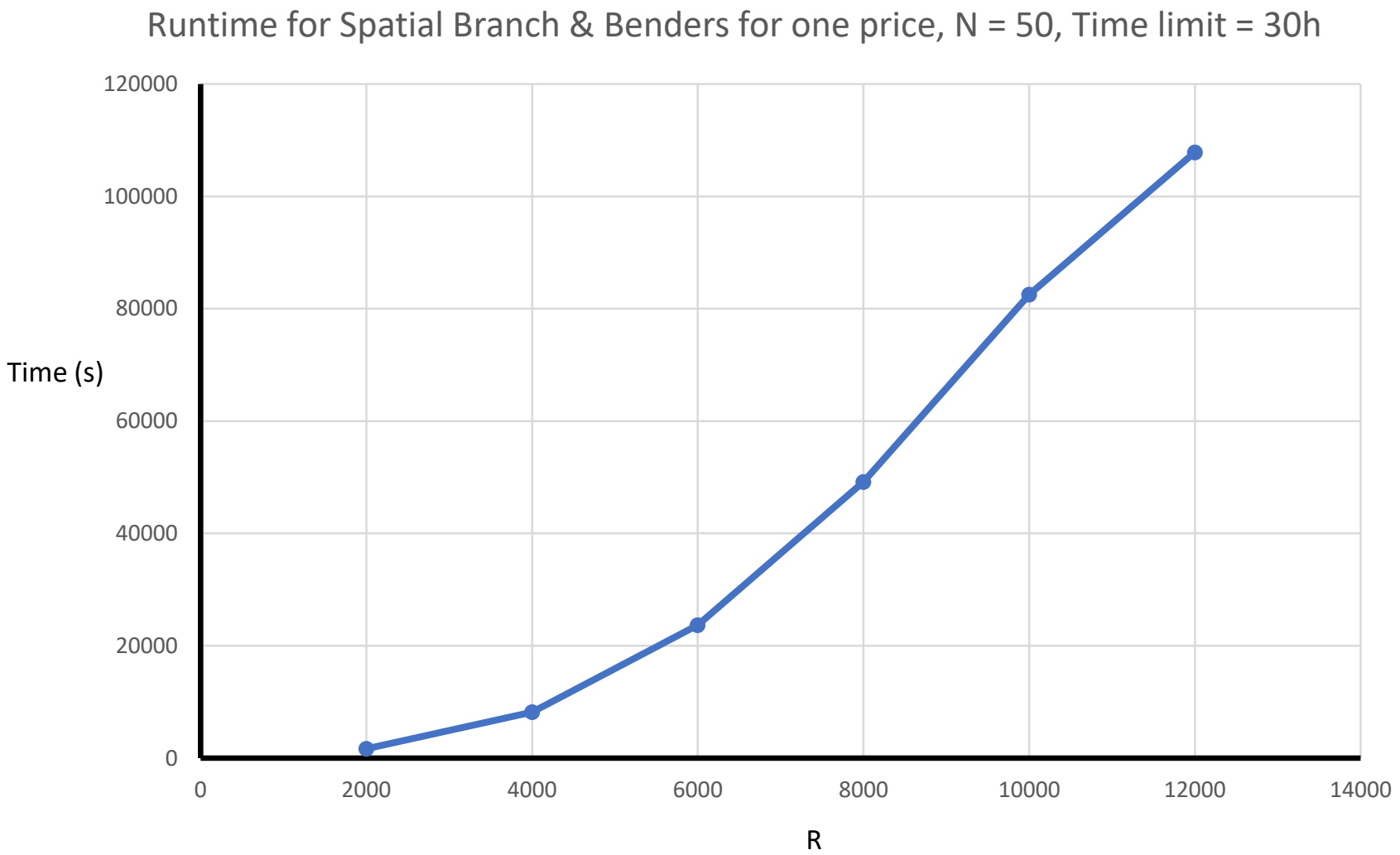
Computational results



Computational results



Computational results



- Seems to at least not be exponential.

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Conclusions

- Introduced more efficient **formulation** of the CPP as a QCQP and QCLP.
- Developed methodology to solve the QCLP efficiently
- Approach is applicable to **any choice-based optimization problem** integrating **any advanced discrete choice model**.
- Showed that we can solve instances to optimality before GUROBI finds a first feasible solution.

Thank you for your attention!

Appendix

Table 1: Utility parameters reported in [Ibeas et al., 2014]

Parameter	Value
ASC_{FSP}	0.0
ASC_{PSP}	32.0
ASC_{PUP}	34.0
Fee (€)	$\sim \mathcal{N}(-32.328, 14.168)$
Fee PSP - low income (€)	-10.995
Fee PUP - low income (€)	-13.729
Fee PSP - resident (€)	-11.440
Fee PUP - resident (€)	-10.668
Access time to parking (min)	$\sim \mathcal{N}(-0.788, 1.06)$
Access time to destination (min)	-0.612
Age of vehicle (1/0)	4.037
Origin (1/0)	-5.762

Appendix

Table 2: Solve time (seconds) for single-price optimization (small-scale)

N	R	MILP	QCQP	QCLP	B&B	B&BD
100	100	2849	392	242	174	216
100	150	7534	1087	708	378	574
100	200	8549	1746	1018	701	603
100	250	25333	2698	1713	1032	1012
100	300	37396	4346	3416	1511	1066
100	350	45362	6715	3927	1795	1169
100	400	65065	8986	5896	2104	1485

Appendix

Table 3: Optimal profit and price for single-price optimization (small-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
100	100	54.134	[0.661]	54.134	[0.661]	54.134	[0.661]	54.133	[0.661]	54.133	[0.661]
100	150	54.233	[0.67]	54.233	[0.67]	54.233	[0.67]	54.233	[0.67]	54.232	[0.67]
100	200	54.599	[0.662]	54.599	[0.662]	54.599	[0.662]	54.598	[0.663]	54.596	[0.662]
100	250	54.622	[0.673]	54.622	[0.673]	54.622	[0.673]	54.619	[0.673]	54.618	[0.673]
100	300	54.48	[0.67]	54.48	[0.67]	54.479	[0.67]	54.479	[0.67]	54.478	[0.67]
100	350	54.449	[0.657]	54.448	[0.657]	54.449	[0.657]	54.448	[0.657]	54.447	[0.657]
100	400	54.389	[0.664]	54.389	[0.664]	54.389	[0.664]	54.389	[0.669]	54.388	[0.664]

Appendix

Table 4: Solve time (seconds) for two-price optimization (small-scale)

		MILP		QCQP	QCLP	B&B	B&BD
N	R	Time	Gap (%)	Time	Time	Time	Time
50	20	1238	0.01	60	32	32	184
50	50	3275	0.01	487	199	201	933
50	80	34907	0.01	1516	564	488	2051
50	100	251466	0.01	2475	843	614	2099
50	150	192213	0.01	2105	2404	1651	5614
50	200	252000	23.92	3023	3384	2438	5402

Appendix

Table 5: Optimal profit and price for two-price optimization (small-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
50	20	27.417	[0.609, 0.653]	27.417	[0.609, 0.653]	27.417	[0.609, 0.653]	27.416	[0.609, 0.653]	27.414	[0.609, 0.653]
50	50	26.71	[0.556, 0.654]	26.71	[0.556, 0.654]	26.71	[0.556, 0.654]	26.71	[0.556, 0.654]	26.707	[0.556, 0.654]
50	80	27.413	[0.57, 0.648]	27.413	[0.57, 0.648]	27.413	[0.57, 0.648]	27.412	[0.57, 0.648]	27.41	[0.57, 0.648]
50	100	27.546	[0.608, 0.704]	27.546	[0.608, 0.704]	27.546	[0.608, 0.704]	27.544	[0.608, 0.704]	27.544	[0.608, 0.704]
50	150	27.29	[0.562, 0.668]	27.289	[0.562, 0.668]	27.29	[0.562, 0.668]	27.29	[0.562, 0.668]	27.288	[0.562, 0.667]
50	200	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.995	[0.546, 0.679]	26.996	[0.546, 0.679]

Appendix

Table 6: Solve time (seconds) for single-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)
100	200	8348	0.01	1059	0.01	698	0.01	310	0.00	409	0.01
100	400	36000	20.39	5013	0.01	3629	0.01	1255	0.01	1050	0.01
100	600	36000	27.0	14796	0.01	10775	0.01	3110	0.01	1707	0.01
100	800	36000	113.12	21626	0.01	15784	0.01	6206	0.01	2444	0.01
100	1000	36000	122.21	36000	0.04	26727	0.01	10007	0.01	3131	0.01
100	1500	36000	121.82	36000	16.69	36000	0.49	22892	0.01	5093	0.01
100	2000	36000	124.91	36000	300.05	36000	5.33	36000	1.88	7341	0.01
100	3000	36000	125.44	36000	-	36000	-	36000	29.33	12396	0.01
100	4000	36000	149.07	36000	-	36000	-	36000	39.42	20990	0.01
100	5000	36000	-	36000	-	36000	-	36000	34.22	28768	0.01
100	6000	36000	-	36000	-	36000	-	36000	44.95	35917	0.01
100	7000	36000	-	36000	-	36000	-	36000	44.88	36000	0.16

Appendix

Table 7: Optimal profit and price for single-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
100	200	54.599	[0.662]	54.599	[0.662]	54.599	[0.662]	54.598	[0.663]	54.596	[0.662]
100	400	54.385	[0.664]	54.389	[0.664]	54.389	[0.664]	54.389	[0.669]	54.388	[0.664]
100	600	54.019	[0.625]	54.295	[0.667]	54.295	[0.667]	54.295	[0.667]	54.294	[0.667]
100	800	54.319	[0.662]	54.327	[0.653]	54.326	[0.653]	54.325	[0.653]	54.326	[0.653]
100	1000	54.421	[0.663]	54.429	[0.661]	54.429	[0.661]	54.429	[0.661]	54.429	[0.661]
100	1500	54.488	[0.67]	49.33	[0.971]	54.514	[0.654]	54.53	[0.659]	54.529	[0.659]
100	2000	54.511	[0.656]	22.469	[1.379]	53.966	[0.613]	54.54	[0.667]	54.541	[0.666]
100	3000	54.439	[0.664]	-	-	-	-	52.387	[0.801]	54.448	[0.661]
100	4000	54.422	[0.668]	-	-	-	-	51.175	[0.856]	54.428	[0.669]
100	5000	-	-	-	-	-	-	53.144	[0.764]	54.394	[0.661]
100	6000	-	-	-	-	-	-	49.207	[0.971]	54.399	[0.663]
100	7000	-	-	-	-	-	-	49.229	[0.97]	54.41	[0.669]

Appendix

Table 8: Solve time (seconds) for two-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)
50	200	36000	39.22	3098	0.01	3338	0.01	2426	0.01	5498	0.01
50	400	36000	100.58	17774	0.01	23325	0.01	11746	0.01	21838	0.01
50	600	36000	217.26	36000	0.18	36000	0.26	26662	0.01	35367	0.01
50	800	36000	138.07	36000	1.75	36000	2.21	36000	0.16	35938	0.01
50	1000	36000	185.45	36000	9.52	36000	20.68	36000	1.48	36000	0.07
50	1500	36000	345.36	36000	42.8	36000	42.04	36000	11.41	36000	0.32
50	2000	36000	393.41	36000	258.89	36000	-	36000	28.79	36000	0.58
50	3000	36000	-	36000	263.73	36000	-	36000	44.48	36000	2.08
50	4000	36000	-	36000	280.59	36000	-	36000	72.86	36000	10.90
50	5000	36000	-	36000	-	36000	-	36000	127.64	36000	34.70
50	6000	36000	-	36000	-	36000	-	36000	128.44	36000	41.96
50	7000	36000	-	36000	-	36000	-	36000	138.01	36000	51.96

Appendix

Table 9: Optimal profit and price for two-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
50	200	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.995	[0.546, 0.679]	26.996	[0.546, 0.679]
50	400	21.689	[0.789, 0.956]	27.174	[0.556, 0.665]	27.174	[0.556, 0.665]	27.172	[0.556, 0.665]	27.172	[0.556, 0.665]
50	600	13.801	[1.087, 1.281]	27.243	[0.561, 0.682]	27.245	[0.563, 0.671]	27.246	[0.562, 0.683]	27.246	[0.562, 0.683]
50	800	16.993	[0.877, 1.203]	27.072	[0.578, 0.668]	27.06	[0.559, 0.659]	27.082	[0.573, 0.667]	27.089	[0.574, 0.667]
50	1000	13.987	[1.2, 1.212]	26.968	[0.59, 0.684]	26.208	[0.584, 0.797]	27.012	[0.571, 0.667]	27.031	[0.573, 0.67]
50	1500	10.144	[1.415, 1.485]	26.319	[0.584, 0.799]	26.322	[0.584, 0.799]	26.982	[0.582, 0.698]	27.052	[0.569, 0.667]
50	2000	9.255	[1.239, 1.866]	11.82	[1.199, 1.395]	-	-	26.718	[0.632, 0.712]	27.094	[0.565, 0.661]
50	3000	-	-	11.849	[1.198, 1.397]	-	-	25.983	[0.5, 0.756]	27.144	[0.571, 0.677]
50	4000	-	-	11.844	[1.199, 1.396]	-	-	24.707	[1.242, 0.766]	27.078	[0.582, 0.699]
50	5000	-	-	-	-	-	-	18.988	[1.0, 1.0]	26.012	[0.5, 0.755]
50	6000	-	-	-	-	-	-	18.915	[1.0, 1.0]	25.973	[0.5, 0.757]
50	7000	-	-	-	-	-	-	18.926	[1.0, 1.0]	24.681	[1.231, 0.766]

Appendix: Callback implementation

```
def mycallback(model, where):
    if where == GRB.Callback.MIPNODE:
        status = model.cbGet(GRB.Callback.MIPNODE_STATUS)
        if status == GRB.OPTIMAL:
            sol = model.cbGetNodeRel(model._vars)
            omega, eta = compute_cpp_from_p_parking(sol[0], sol[1])
            mysol = [sol[0], sol[1]] + list(eta.values()) + list(omega.values())
            model.cbSetSolution(model._vars, mysol)
```

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