Continuous Pricing with Advanced Discrete Choice Demand Modeling: A Spatial Branch and Benders Decomposition Algorithm

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Outline

The Continuous Pricing Problem (CPP)

CPP

- Supplier offers S products for sale. Goal: determine optimal **price** for each product to maximize total **profit**.
- Demand for each product is modeled using a **discrete choice model** (DCM).

DCM

- For every **costumer** n and **product** i a stochastic **utility** U_{in} is defined, which depends on **socio-economic** characteristics of the individual and **attributes** of the products (e.g. the price).

The Continuous Pricing Problem (CPP)

Utility

- **Utility** of alternative i for costumer n:

$$U_{in} = \sum_{k \neq p} \beta_k x_{ink} + \beta_p p_i + \varepsilon_{in}$$

- β_k : parameters (exogenous)
- x_{ink} : attributes (exogenous)
- p_i : price of alternative i
- ε_{in} : stochastic error term

The Continuous Pricing Problem (CPP)

Probability

- **Probability** that costumer n chooses alternative i:

$$P_n(i) \quad = \quad \mathbb{P}(U_{in} \geq U_{jn} \; \forall j \in J)$$

- **Logit** ($\varepsilon_{in} \sim$ i.i.d. Gumbel(0, 1)):

$$P_n(i) \quad = \quad \frac{e^{V_{in}}}{\sum_{j \in C_n} e^{V_{jn}}}$$

- **Mixed Logit** (Logit + $\beta_k \sim F(\beta_k|\theta)$):

$$P_n(i) \quad = \quad \int \frac{e^{V_{in}(\beta_{kn})}}{\sum_{j \in C_n} e^{V_{jn}(\beta_{kn})}} f(\beta_k|\theta) d\beta_k$$

Monte Carlo Simulation

- **Simulate** R scenarios (draws), each with **deterministic** utilities U_{inr}:

$$U_{inr} = \sum_{k \neq p} \beta_k x_{ink} + \beta_p p_i + \varepsilon_{inr}$$

- **Choice** variables:

$$\omega_{inr} = \begin{cases} 1 \text{ if } U_{inr} = \max_j U_{jnr} \\ 0 \text{ else} \end{cases}$$

- Probability **estimator**:

$$\widehat{P}_n(i) = \frac{1}{R} \sum_r \omega_{inr}$$

MILP formulation [Paneque et al., 2021]

$$\max_{p,\omega,U,H} \frac{1}{R} \sum_r \sum_n \sum_{i \in S} p_i \omega_{inr}$$

s.t.

$$\begin{aligned}
\sum_i \omega_{inr} &= 1 && \forall n, r && (\mu_{nr}) \\
H_{nr} &= \sum_i U_{inr} \omega_{inr} && \forall n, r && (\zeta_{nr}) \\
H_{nr} &\geq U_{inr} && \forall i, n, r && (\alpha_{inr}) \\
U_{inr} &= \sum_{k \neq p} \beta_k x_{ink} + \beta_p p_i + \varepsilon_{inr} && \forall i, n, r && (\kappa_{inr}) \\
\omega &\in \{0, 1\}^{JNR} \\
p, U, H &\in \mathbb{R}^S, \mathbb{R}^{JNR}, \mathbb{R}^{NR}
\end{aligned}$$

Literature

[Li and Huh, 2011], [Gallego and Wang, 2014], ...

- Extensive research for **Logit** and **Nested Logit** (NL) integration.

[Li et al., 2019], [Marandi and Lurkin, 2020], [van de Geer and den Boer, 2022], ...

- Tackled **Mixed Logit** (ML) integration in various ways, e.g. **approximations**, assuming consumer **homogeneity**, or considering only **discrete** probability measures.

[Paneque et al., 2022]

- Apply a **Lagrangian** decomposition scheme to speed up the solution of the MILP.
- Limited success.

Outline

QCQP formulation

$$\max_{p,\omega,U,H} \frac{1}{R} \sum_r \sum_n \sum_{i \in S} p_i \omega_{inr}$$

s.t.

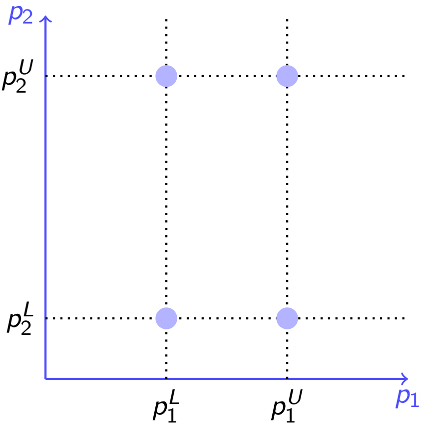
$$\begin{aligned}
\sum_i \omega_{inr} &= 1 && \forall n, r && (\mu_{nr}) \\
H_{nr} &= \sum_i U_{inr}\omega_{inr} && \forall n, r && (\zeta_{nr}) \\
H_{nr} &\geq U_{inr} && \forall i, n, r && (\alpha_{inr}) \\
U_{inr} &= \sum_{k \neq p} \beta_k x_{ink} + \beta_p p_i + \varepsilon_{inr} && \forall i, n, r && (\kappa_{inr}) \\
\omega &\in [0,1]^{JNR} \\
p, U, H &\in \mathbb{R}^S, \mathbb{R}^{JNR}, \mathbb{R}^{NR}
\end{aligned}$$

QCLP formulation

$$
\begin{aligned}
\max_{p,\omega,\eta,U,H} \quad & \frac{1}{R}\sum_{r}\sum_{n}\sum_{i\in S}\eta_{inr} & & \\
\text{s.t.} \quad & \sum_{i}\omega_{inr} = 1 & \forall n, r \quad & (\mu_{nr}) \\
& H_{nr} = \sum_{i}\Big(\sum_{k\neq p}\beta_k x_{ink} + \varepsilon_{inr}\Big)\omega_{inr} + \beta_p \eta_{inr} & \forall n, r \quad & (\zeta_{nr}) \\
& H_{nr} \geq U_{inr} & \forall i, n, r \quad & (\alpha_{inr}) \\
& U_{inr} = \sum_{k\neq p}\beta_k x_{ink} + \beta_p p_i + \varepsilon_{inr} & \forall i, n, r \quad & (\kappa_{inr}) \\
& \eta_{inr} = p_i \omega_{inr} & \forall i \in S, n, r \quad & (\lambda_{inr}) \\
& \omega \in [0,1]^{JNR} & & \\
& p, \eta, U, H \in \mathbb{R}^{S}, \mathbb{R}^{SNR}, \mathbb{R}^{JNR}, \mathbb{R}^{NR} & &
\end{aligned}
$$

Simplification

- Assume reasonable **bounds** on price, $p_i \in [p_i^L, p_i^U]$. This means some choices are fixed.



Simplification

Observations

- The number of controlled prices is generally **low** (usually, one or two).
- There are 2^J combinations of lower and upper bounds.

Procedure for each n and r

- For each combination, identify the **best alternative**.
- If alternative i is **never** the best, set $w_{inr} = 0$.
- If alternative i is **always** the best, set $w_{inr} = 1$.

Note

This happens often when bounds are **tight**.

Spatial Branch & Bound (B&B) Algorithm

Relaxation

- Relax the constraint $\eta_{inr} = p_i \omega_{inr}$ with a **McCormick** envelope:

$$\begin{aligned}
\eta_{inr} &\geq p_i^L \omega_{inr} && \forall i \in S, n, r && (\lambda^1_{inr}) \\
\eta_{inr} &\geq p_i^U \omega_{inr} + p_i - p_i{}^U && \forall i \in S, n, r && (\lambda^2_{inr}) \\
\eta_{inr} &\leq p_i^L \omega_{inr} + p_i - p_i{}^L && \forall i \in S, n, r && (\lambda^3_{inr}) \\
\eta_{inr} &\leq p_i^U \omega_{inr} && \forall i \in S, n, r && (\lambda^4_{inr})
\end{aligned}$$

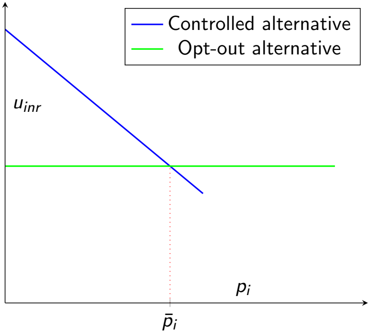
- Integrality is **preserved** for tight enough bounds.

TRANSP-OR

EPFL

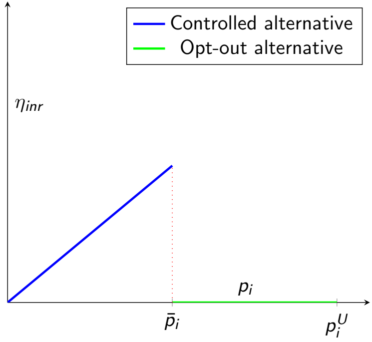
Break points

Competing with opt-out: utility



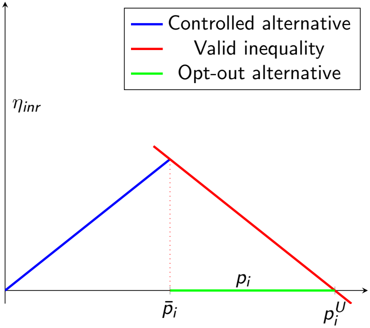
Break points

Competing with opt-out: revenue



Break points

Competing with opt-out: valid inequality



Valid inequalities based on break points

Competing with opt-out

$$\eta_{inr} \leq \frac{\bar{p}_i(p_i^U - p_i)}{p_i^U - \bar{p}_i}.$$

Competing with another controlled alternative

$$\eta_{inr} \leq \frac{\beta_j p_i^U p_j - c_i p_i^U + c_j p_i^U - p_i \left(\beta_j p_j^L - c_i + c_j\right)}{\beta_i p_i^U - \beta_j p_j^L + c_i - c_j}.$$

and

$$\eta_{inr} \leq \frac{\beta_j p_i^U p_j - c_i p_i^U + c_j p_i^U - p_i \left(\beta_j p_j^U - c_i + c_j\right)}{\beta_i p_i^U - \beta_j p_j^U + c_i - c_j}.$$

Spatial Branch & Bound (B&B) Algorithm

Convergence

- Every relaxation provides an **upper bound** on the maximal profit.
- Any solution value for the price gives an immediate feasible solution (**lower bound**) due to integrality.

Custom B&B vs. standard B&B

- We only branch on $\boldsymbol{S}$ continuous variables, instead of branching on $\boldsymbol{SNR}$ continuous (QCLP) or $\boldsymbol{JNR}$ binary (MILP) variables.

Benders Decomposition

- The McCormick relaxation (linear program) at each node is solved by the use of a **Benders decomposition**:

Benders Decomposition

- **Master problem (MP)**: compute *candidate solutions* for the price
- **Subproblem (SP)**: given a price, compute reduced costs to construct *optimality cut* to add to the MP
- SP is highly **separable**: utility maximization for each customer and scenario can be solved **independently**
- Make use of fully **disaggregated optimality cuts** (one cut per customer and scenario)
- Can add **valid inequalities** in the Master problem.

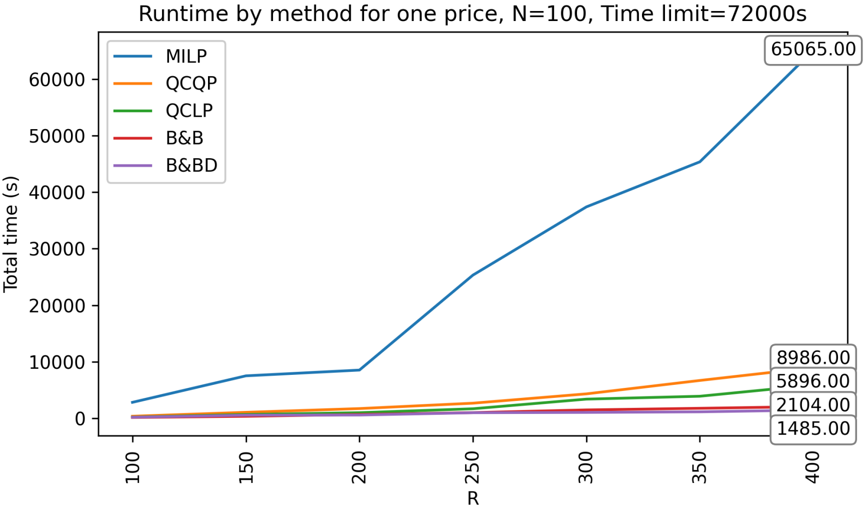
Outline

Case Study

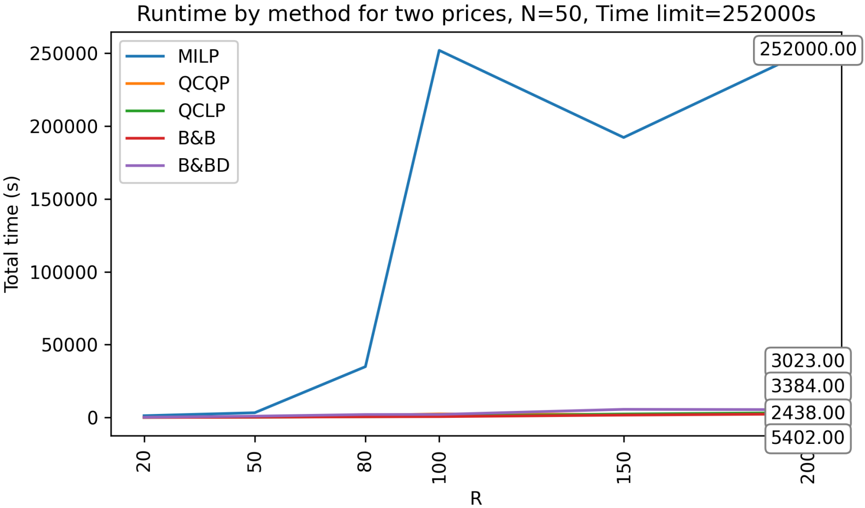
Parking space operator [Ibeas et al., 2014]

- **Alternatives**: Paid-Street-Parking (PSP), Paid-Underground-Parking (PUP) and Free-Street-Parking (FSP).
- Optimize prices for PSP and PUP, FSP is the **opt-out** alternative.
- **Socio-economic characteristics**: trip origin, vehicle age, driver income, residence area.
- **Product attributes**: access time to parking, access time to destination, and parking fee (price).
- Choice model is a **Mixed Logit**, $\beta_{\text{fee}}, \beta_{\text{time\_parking}} \sim \mathcal{N}(\mu, \sigma)$.

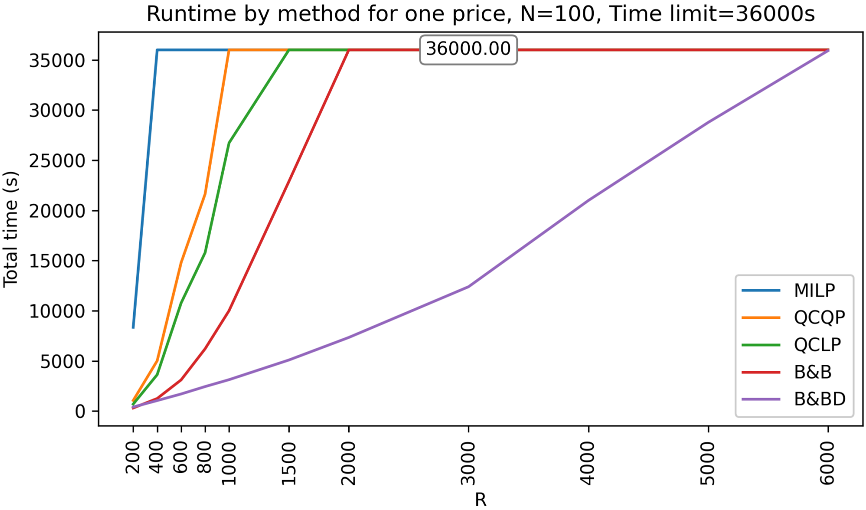
Computational results



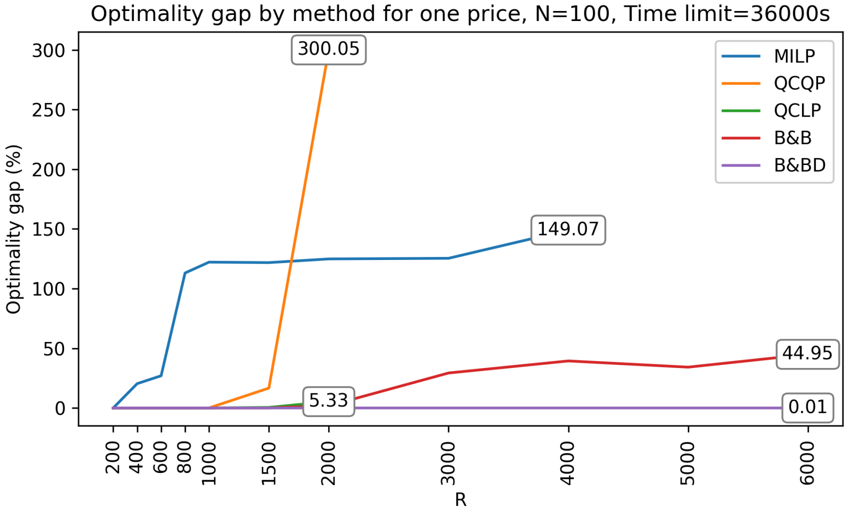
Computational results



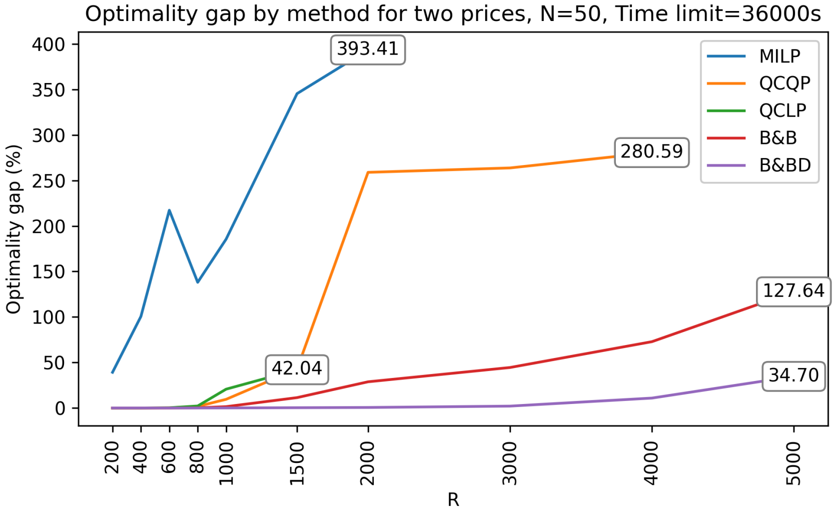
Computational results



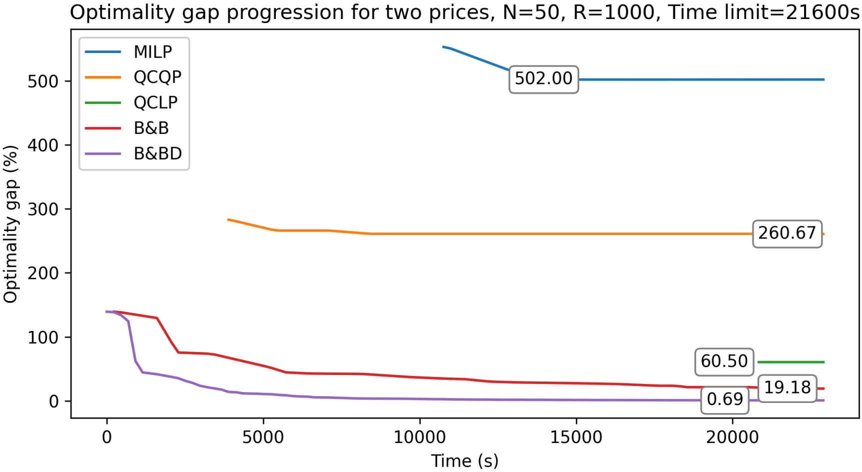
Computational results



Computational results



Computational results



Simplifications + valid inequalities

Table: One-price and two-price optimization runtime (seconds) when using simplifications (S) + valid inequalities (V1 and V2). Time limit = 36000s

N	R	QCLP	B&B	B&BD	B&BD+S	B&BD+S+V1	B&BD+S+V2
100	100	107	29	98	30	33	41
100	500	4739	625	851	252	673	519
100	1000	27586	10007	3387	1865	3329	2388
100	3000	-	25950	5606	3337	5019	3905

N	R	QCLP	B&B	B&BD	B&BD+S	B&BD+S+V1	B&BD+S+V2
50	100	840	660	1925	416	11253	18447
50	500	30600	16826	19904	4686	0.40%	1.01%
50	1000	20.68%	1.59%	0.07%	15066	1.87%	4.68%
50	3000	-	42.88%	2.07%	0.06%	3.54%	8.71%

Outline

Conclusions

- Introduced more efficient **formulation** of the CPP as a QCQP and QCLP.
- Developed methodology that is applicable to **any choice-based optimization problem** integrating **any advanced discrete choice model**.
- Showed that we can solve instances to optimality before GUROBI finds a first feasible solution.

Thank you for your attention!

Appendix

Table 1: Utility parameters reported in [Ibeas et al., 2014]

Parameter	Value
ASC_{FSP}	0.0
ASC_{PSP}	32.0
ASC_{PUP}	34.0
Fee (€)	$\sim \mathcal{N}(-32.328, 14.168)$
Fee PSP - low income (€)	-10.995
Fee PUP - low income (€)	-13.729
Fee PSP - resident (€)	-11.440
Fee PUP - resident (€)	-10.668
Access time to parking (min)	$\sim \mathcal{N}(-0.788, 1.06)$
Access time to destination (min)	-0.612
Age of vehicle (1/0)	4.037
Origin (1/0)	-5.762

Appendix

Table 2: Solve time (seconds) for single-price optimization (small-scale)

N	R	MILP	QCQP	QCLP	B&B	B&BD
100	100	2849	392	242	174	216
100	150	7534	1087	708	378	574
100	200	8549	1746	1018	701	603
100	250	25333	2698	1713	1032	1012
100	300	37396	4346	3416	1511	1066
100	350	45362	6715	3927	1795	1169
100	400	65065	8986	5896	2104	1485

Appendix

Table 3: Optimal profit and price for single-price optimization (small-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
100	100	54.134	[0.661]	54.134	[0.661]	54.134	[0.661]	54.133	[0.661]	54.133	[0.661]
100	150	54.233	[0.67]	54.233	[0.67]	54.233	[0.67]	54.233	[0.67]	54.232	[0.67]
100	200	54.599	[0.662]	54.599	[0.662]	54.599	[0.662]	54.598	[0.663]	54.596	[0.662]
100	250	54.622	[0.673]	54.622	[0.673]	54.622	[0.673]	54.619	[0.673]	54.618	[0.673]
100	300	54.48	[0.67]	54.48	[0.67]	54.479	[0.67]	54.479	[0.67]	54.478	[0.67]
100	350	54.449	[0.657]	54.448	[0.657]	54.449	[0.657]	54.448	[0.657]	54.447	[0.657]
100	400	54.389	[0.664]	54.389	[0.664]	54.389	[0.664]	54.389	[0.669]	54.388	[0.664]

Appendix

Table 4: Solve time (seconds) for two-price optimization (small-scale)

		MILP		QCQP	QCLP	B&B	B&BD
N	R	Time	Gap (%)	Time	Time	Time	Time
50	20	1238	0.01	60	32	32	184
50	50	3275	0.01	487	199	201	933
50	80	34907	0.01	1516	564	488	2051
50	100	251466	0.01	2475	843	614	2099
50	150	192213	0.01	2105	2404	1651	5614
50	200	252000	23.92	3023	3384	2438	5402

Appendix

Table 5: Optimal profit and price for two-price optimization (small-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
50	20	27.417	[0.609, 0.653]	27.417	[0.609, 0.653]	27.417	[0.609, 0.653]	27.416	[0.609, 0.653]	27.414	[0.609, 0.653]
50	50	26.71	[0.556, 0.654]	26.71	[0.556, 0.654]	26.71	[0.556, 0.654]	26.71	[0.556, 0.654]	26.707	[0.556, 0.654]
50	80	27.413	[0.57, 0.648]	27.413	[0.57, 0.648]	27.413	[0.57, 0.648]	27.412	[0.57, 0.648]	27.41	[0.57, 0.648]
50	100	27.546	[0.608, 0.704]	27.546	[0.608, 0.704]	27.546	[0.608, 0.704]	27.544	[0.608, 0.704]	27.544	[0.608, 0.704]
50	150	27.29	[0.562, 0.668]	27.289	[0.562, 0.668]	27.29	[0.562, 0.668]	27.29	[0.562, 0.668]	27.288	[0.562, 0.667]
50	200	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.995	[0.546, 0.679]	26.996	[0.546, 0.679]

Appendix

Table 6: Solve time (seconds) for single-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)
100	200	8348	0.01	1059	0.01	698	0.01	310	0.00	409	0.01
100	400	36000	20.39	5013	0.01	3629	0.01	1255	0.01	1050	0.01
100	600	36000	27.0	14796	0.01	10775	0.01	3110	0.01	1707	0.01
100	800	36000	113.12	21626	0.01	15784	0.01	6206	0.01	2444	0.01
100	1000	36000	122.21	36000	0.04	26727	0.01	10007	0.01	3131	0.01
100	1500	36000	121.82	36000	16.69	36000	0.49	22892	0.01	5093	0.01
100	2000	36000	124.91	36000	300.05	36000	5.33	36000	1.88	7341	0.01
100	3000	36000	125.44	36000	-	36000	-	36000	29.33	12396	0.01
100	4000	36000	149.07	36000	-	36000	-	36000	39.42	20990	0.01
100	5000	36000	-	36000	-	36000	-	36000	34.22	28768	0.01
100	6000	36000	-	36000	-	36000	-	36000	44.95	35917	0.01
100	7000	36000	-	36000	-	36000	-	36000	44.88	36000	0.16

Appendix

Table 7: Optimal profit and price for single-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
100	200	54.599	[0.662]	54.599	[0.662]	54.599	[0.662]	54.598	[0.663]	54.596	[0.662]
100	400	54.385	[0.664]	54.389	[0.664]	54.389	[0.664]	54.389	[0.669]	54.388	[0.664]
100	600	54.019	[0.625]	54.295	[0.667]	54.295	[0.667]	54.295	[0.667]	54.294	[0.667]
100	800	54.319	[0.662]	54.327	[0.653]	54.326	[0.653]	54.325	[0.653]	54.326	[0.653]
100	1000	54.421	[0.663]	54.429	[0.661]	54.429	[0.661]	54.429	[0.661]	54.429	[0.661]
100	1500	54.488	[0.67]	49.33	[0.971]	54.514	[0.654]	54.53	[0.659]	54.529	[0.659]
100	2000	54.511	[0.656]	22.469	[1.379]	53.966	[0.613]	54.54	[0.667]	54.541	[0.666]
100	3000	54.439	[0.664]	-	-	-	-	52.387	[0.801]	54.448	[0.661]
100	4000	54.422	[0.668]	-	-	-	-	51.175	[0.856]	54.428	[0.669]
100	5000	-	-	-	-	-	-	53.144	[0.764]	54.394	[0.661]
100	6000	-	-	-	-	-	-	49.207	[0.971]	54.399	[0.663]
100	7000	-	-	-	-	-	-	49.229	[0.97]	54.41	[0.669]

Appendix

Table 8: Solve time (seconds) for two-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)
50	200	36000	39.22	3098	0.01	3338	0.01	2426	0.01	5498	0.01
50	400	36000	100.58	17774	0.01	23325	0.01	11746	0.01	21838	0.01
50	600	36000	217.26	36000	0.18	36000	0.26	26662	0.01	35367	0.01
50	800	36000	138.07	36000	1.75	36000	2.21	36000	0.16	35938	0.01
50	1000	36000	185.45	36000	9.52	36000	20.68	36000	1.48	36000	0.07
50	1500	36000	345.36	36000	42.8	36000	42.04	36000	11.41	36000	0.32
50	2000	36000	393.41	36000	258.89	36000	-	36000	28.79	36000	0.58
50	3000	36000	-	36000	263.73	36000	-	36000	44.48	36000	2.08
50	4000	36000	-	36000	280.59	36000	-	36000	72.86	36000	10.90
50	5000	36000	-	36000	-	36000	-	36000	127.64	36000	34.70
50	6000	36000	-	36000	-	36000	-	36000	128.44	36000	41.96
50	7000	36000	-	36000	-	36000	-	36000	138.01	36000	51.96

Appendix

Table 9: Optimal profit and price for two-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
50	200	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.995	[0.546, 0.679]	26.996	[0.546, 0.679]
50	400	21.689	[0.789, 0.956]	27.174	[0.556, 0.665]	27.174	[0.556, 0.665]	27.172	[0.556, 0.665]	27.172	[0.556, 0.665]
50	600	13.801	[1.087, 1.281]	27.243	[0.561, 0.682]	27.245	[0.563, 0.671]	27.246	[0.562, 0.683]	27.246	[0.562, 0.683]
50	800	16.993	[0.877, 1.203]	27.072	[0.578, 0.668]	27.06	[0.559, 0.659]	27.082	[0.573, 0.667]	27.089	[0.574, 0.667]
50	1000	13.987	[1.2, 1.212]	26.968	[0.59, 0.684]	26.208	[0.584, 0.797]	27.012	[0.571, 0.667]	27.031	[0.573, 0.67]
50	1500	10.144	[1.415, 1.485]	26.319	[0.584, 0.799]	26.322	[0.584, 0.799]	26.982	[0.582, 0.698]	27.052	[0.569, 0.667]
50	2000	9.255	[1.239, 1.866]	11.82	[1.199, 1.395]	-	-	26.718	[0.632, 0.712]	27.094	[0.565, 0.661]
50	3000	-	-	11.849	[1.198, 1.397]	-	-	25.983	[0.5, 0.756]	27.144	[0.571, 0.677]
50	4000	-	-	11.844	[1.199, 1.396]	-	-	24.707	[1.242, 0.766]	27.078	[0.582, 0.699]
50	5000	-	-	-	-	-	-	18.988	[1.0, 1.0]	26.012	[0.5, 0.755]
50	6000	-	-	-	-	-	-	18.915	[1.0, 1.0]	25.973	[0.5, 0.757]
50	7000	-	-	-	-	-	-	18.926	[1.0, 1.0]	24.681	[1.231, 0.766]

Appendix: Callback implementation

```
def mycallback(model, where):
    if where == GRB.Callback.MIPNODE:
        status = model.cbGet(GRB.Callback.MIPNODE_STATUS)
        if status == GRB.OPTIMAL:
            sol = model.cbGetNodeRel(model._vars)
            omega, eta = compute_cpp_from_p_parking(sol[0], sol[1])
            mysol = [sol[0], sol[1]] + list(eta.values()) + list(omega.values())
            model.cbSetSolution(model._vars, mysol)
```

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