Centralized Versus Decentralized Control - A Solvable Stylized Model in Transportation

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Smart Parts Dynamics - A Fashionable Trend in Logistics

Highly complex decision issues ⇒ tendency to decentralize the management

- Huge number of control parameters
- Feedback (*i.e.* non-linearity) in the underlying dynamics
- Ubiquitous presence of randomness in the dynamics
- ...

⇓

Decisions based on limited rationality ⇒ Rigid pre-planning offers poor performance

mutual interactions ⇓ self-organization

Autonomous agents might better perform than an effective central controller

⇓ goal of today's presentation

Exhibit a solvable model showing performance of decentralized control

A Simple Model for Competitive Dynamics

$$\dot{X}_k(t) = \underbrace{v_k(t)}_{\text{velocity}} + \gamma_k \underbrace{\mathbb{I}_k(\vec{X}(t), X_k(t))}_{\text{multi-agent interactions}} + \underbrace{q_k(v_k(t))dB_{k,t}}_{\text{noise sources}}, \qquad k = 1, 2, \ldots, N.$$

Multi-agent interactions:

$$\mathbb{I}_k(\vec{X}(t), X_k(t)) = \frac{1}{\mathcal{N}_k} \sum_{j \neq k}^{\mathcal{N}_k} \mathcal{I}_k(X_j(t)), \qquad \mathcal{N}_k := \text{neighbourhood of agent } k,$$

$$\mathcal{I}_k(X_j(t)) = \begin{cases} 0 & \text{if} \quad 0 \leq X_j(t) < X_k(t), \quad (\underline{\text{velocity unchanged}}), \\ 1 & \text{if} \quad X_k(t) \leq X_j(t) < X_k(t) + U, \quad (U > 0), \quad (\underline{\text{accelerate}}), \\ 0 & \text{if} \quad X_j(t) > X_k(t) + U, \quad (\underline{\text{velocity unchanged}}). \end{cases}$$

(U := *"mutual influence"* interval)

A Simple Model for Competitive Dynamics - Applications

Logistics

Economy

Human Mimetism

...

Homogeneous Population of Agents

$$dX_k(t) = \underbrace{\left[v(t) + \gamma \mathbb{I}(\vec{X}(t), X_k(t))\right]}_{:= \text{ drift field } \mathcal{D}_{k,v}(x,t)} dt \quad + \quad \underbrace{q \; dB_{k,t}}_{\text{indep. White Gaussian Noise}}.$$

$\Downarrow$ diffusion process

Fokker - Planck diffusion equation:

$$\frac{\partial}{\partial t} P(\vec{x}, t) = -\sum_k \frac{\partial}{\partial x_k} \left[\mathcal{D}_{k,v(\vec{x},t)} P(\vec{x}, t)\right] + \frac{1}{2} q^2 \sum_k \frac{\partial^2}{\partial x_k^2} \left[P(\vec{x}, t)\right],$$

$P(\vec{x}, t)$:= conditional probability density

Mean-Field Dynamics for Homogeneous Agents

$\mathcal{N}_k \equiv \mathcal{N} \to \infty \quad \Rightarrow$ Mean-Field Dynamics (MFD)

$\Downarrow$ dynamics for a representative effective agent

trajectories point of view

$$\underbrace{\frac{1}{\mathcal{N}} \sum_{j \neq k}^{\mathcal{N}} \mathcal{I}(X_j(t))}_{\text{proportion of } \underline{\text{velocity}-\text{active agents}} \text{ acting on } k} \approx \underbrace{\int_x^{x+U} P(x,t)\, dx}_{\text{proportion of } \underline{\text{representative agents}} \text{ located in } [x, x+U]}$$

probabilistic point of view

$\Downarrow$

Effective Fokker-Planck equation:

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} \underbrace{\left\{ \left[v(t) + \gamma \left(\int_x^{x+U} P(x,t) dx \right) \right] P(x,t) \right\}}_{non-linear \ and \ non-local \ field \ equation} + \frac{1}{2} q^2 \frac{\partial^2}{\partial x^2} [P(x,t)] \, ,$$

Small Influence Region - Burgers' Equation Dynamics

Small values of U $\Rightarrow$ Taylor expand up to 1st order in U

$$\Downarrow \quad \int_x^{x+U} P(x,t)dx \simeq U\, P(x,t)$$

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\underbrace{\{[v(t) + \gamma\, U\, P(x,t)]\, P(x,t)\}}_{non-linear\ \ but\ \ local\ \ drift\ \ field} + \frac{1}{2}q^2\frac{\partial^2}{\partial x^2}[P(x,t)]$$

$$t \mapsto \tau = \gamma t \quad \Downarrow \quad x \mapsto z = \frac{x - \int_0^t v(s)\, ds}{2U}$$

Burgers' Equation (to be solved with initial condition $P(z,t) = \delta(z)\Theta(z)$)

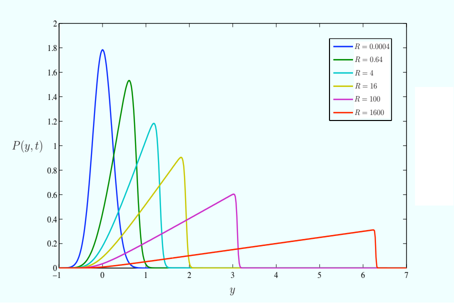
$$\boxed{\dot{P}(z,t) = \frac{1}{2}\frac{\partial}{\partial z}\left[P(z,t)^2\right] + \left[\frac{q^2}{8U^2\gamma}\right]\frac{\partial^2}{\partial z^2}[P(z,t)]}$$

Burgers' Eq. $\Leftarrow$ logarithmic transformation (Hopf - Cole) $\Rightarrow$ Heat Eq.

$\Downarrow$ exact integration

$$P(y,t) = -\frac{q^2}{4\gamma U^2}\frac{\partial}{\partial y}\ln\left[1+\frac{(e^R-1)}{2}\mathrm{Erfc}\left(\frac{y}{q\sqrt{t}}\right)\right] =$$

$$= \frac{1}{R}\left[\frac{(e^R-1)\frac{1}{\sqrt{\pi q^2 t}}e^{-\frac{y^2}{q^2 t}}}{1+\frac{(e^R-1)}{2}\mathrm{Erfc}\left(\frac{y}{q\sqrt{t}}\right)}\right] := \frac{1}{R}\frac{(e^R-1)\mathbb{G}(y,t)}{\mathbb{E}(y,t)}$$

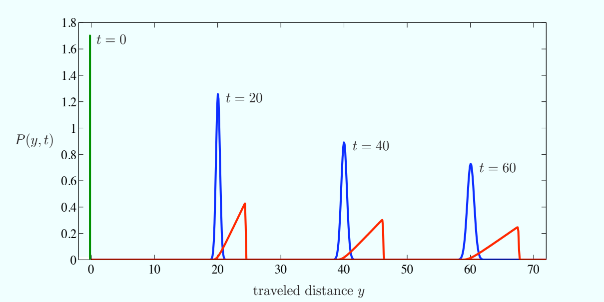


Typical shape of $P(y,t)$ for various $R := \frac{4U^2\gamma}{q^2}$ factors

(viewed from the relative moving frame)

Normalization and positivity are visually manifest !!

Benefit of Competition - Noise Induced Transport Enhancement



Position probability distribution: without interaction, with interactions

- Additional traveled distance when $R = \frac{4\gamma U^2}{q^2} \rightarrow \infty$: $\langle X(t) \rangle_{t\rightarrow\infty} \simeq \frac{4U}{3}\sqrt{\gamma t}$,
- Additional traveled distance when $R = \frac{4\gamma U^2}{q^2} \rightarrow 0$: $\langle X(t) \rangle_{t\rightarrow\infty} \simeq 0$.

Optimal Effective Centralized Control

Controlled diffusion process:

$$dY_t = \underbrace{c(Y,t)\,dt}_{\text{effective central controller}} + q\,dB_t, \qquad \underbrace{Y_0 = 0}_{\text{initial condition}}, \qquad (0 \leq t \leq T),$$

$\Downarrow$ (Fokker-Planck equation)

$$\frac{\partial}{\partial t} P_c(y,t) = -\frac{\partial}{\partial y}\left[c(y,t) P_c(y,t)\right] + \frac{q^2}{2}\frac{\partial^2}{\partial y^2} P_c(y,t)$$

Construct a drift controller $c(Y,t)$ which, for time T, fulfills

$$\underbrace{P_c(y,T)}_{\text{Prob. density with central controller}} = \underbrace{P(y,T)}_{\text{Prob. density due to agent interactions}}$$

Burgers' exact solution

Optimal Effective Centralized Control (continued)

Introduce a utility function $J_{\text{central},T}\,[c(y,t;T)]$ defined as:

$$J_{\text{central},T}\,[c(y,t;T)] = \Big\langle \int_0^T \underbrace{\frac{c^2(y,s;T)}{2q^2}}_{\text{cost rate } \rho(y,s)} ds \Big\rangle,$$

($\langle \cdot \rangle$:= average over the realization of underlying stochastic process)

Optimal Control Problem

Construct an $\underbrace{\text{optimal drift}}_{\text{i.e. yielding minimal cost}}$ $c^*(y,t;T)$ such that:

$$J_{\text{central},T}\,[c^*(y,t;T)] \le J_{\text{central},T}\,[c(y,t;T)]$$

The Dai Pra Solution of the Optimal Control Problem

Optimal drift controller:

$$c^*(y,t;T) = \frac{\partial}{\partial y} \ln \left[h(y,t) \right],$$

$$h(y,t) = \int_{\mathbb{R}} \mathbb{G}\left[(z-y), (T-t) \right] \frac{P(z,T)}{\mathbb{G}(z,t)} dz.$$

Paolo Dai Pra, *"A Stochastic Control Approach to Reciprocal Diffusion Processes"*, Appl. Math. Optim. **23**, (1991), 313-329.

Minimal cost:

$$J_{\text{central},T}\left[c^*(y,t;T) \right] = \underbrace{N}_{\sharp \text{ population}} \cdot \underbrace{\mathcal{D}(P|\mathbb{G})}_{\text{Kullback}-\text{Leibler}} = \begin{cases} 0 & \text{for} \quad t = 0, \\ N\frac{R}{2} + N \ln\left[\frac{(e^R - 1)}{R} \right] & \text{for} \quad t > 0. \end{cases}$$

Decentralized Agent Control - Cost Estimation

Cost $J_{\text{agents},T}$ for decentralized evolution during time horizon T:

$$J_{\text{agents},T} := \underbrace{N}_{\sharp\ \text{population}} \cdot \rho \cdot \int_0^T ds \underbrace{\Phi(s)}_{\text{interacting agents}},$$

- $\rho = \dfrac{\overbrace{\gamma^2 U^2/2}^{\text{kinetic energy}}}{\underbrace{q^2}_{\text{diffusion rate}}} :=$ individual cost rate function,
- $\Phi(t) \in [0,1] :=$ proportion of interacting agents at time t.

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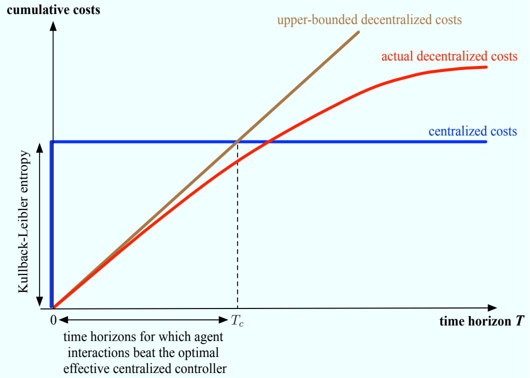
Cost upper-bound, reached when $\Phi(t) \equiv 1$

$$\Downarrow$$

$$\boxed{J_{\text{agents},T} \leq N\rho T}$$

Costs Comparison - Centralized vs Decentralized



To Summarize and to Somehow "Philosophically" Conclude

The stylized model cartoons basic and somehow "universal" features:

- Agents' mimetic interactions produce an emergent structure - (here a "shock"- like wave),
- Competition enhances global transport flow - (here a $\sqrt{t}$-increase of the traveled distance),
- Self-organization via autonomous agents interactions can reduce costs.