# A differentiable dynamic network loading model that yields queue length distributions and accounts for spillback

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# Motivation

- dynamic network loading with stochastic queueing model
  - captures uncertainty about queue lengths
  - specified in terms of differentiable equations
- convenient structure for, e.g., state estimation
- this talk describes the model, not its estimation

# Outline

Queueing theory and the Lighthill-Whitham-Richards (LWR) model

Dynamic queue modeling

Dynamic road modeling

Results: transient model behavior

Results: fundamental diagram

Summary, conclusion, outlook

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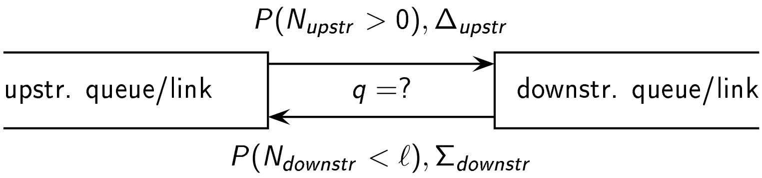
# Queueing theory and LWR model – symmetries


| queueing theory | LWR model (Lebacque, '96) |
|---:|---|
| arrival rate $\lambda$ | upstream demand $\Delta$ |
| service rate $\mu$ | downstream supply $\Sigma$ |
| number of jobs $N$ | traffic density $\varrho$ |
| max. number of jobs $\ell$ | maximum density $\hat{\varrho}$ |

# Boundary conditions – symmetries




| queueing theory | LWR model |
|---:|---|
| waiting prob. $P(N_{upstr} > 0)$ | upstr. demand $\Delta_{upstr}$ |
| non-blocking prob. $P(N_{downstr} < \ell)$ | downstr. supply $\Sigma_{downstr}$ |
| $q \propto P(N_{upstr} > 0) P(N_{downstr} < \ell)$ | $q = \min\{\Delta_{upstr}, \Sigma_{downstr}\}$ |

# Boundary conditions – differences

- invariance principle (IP) for LWR model (Lebacque, '05)
  - in congested conditions, increasing $\Delta_{upstr}$ has no effect
  - in uncongested conditions, increasing $\Sigma_{downstr}$ has no effect
- queueing theory does not satisfy the IP
  - in congested conditions, flow is $\propto P(N_{upstr} > 0)$
  - in uncongested conditions, flow is $\propto P(N_{downstr} < \ell)$
- speculation: stochastic LWR cannot satisfy the IP

$$E\{q\} = \int\int \min\{\Delta, \Sigma\} p(\Delta, \Sigma) d\Delta d\Sigma$$

... because demand and supply regime get mixed

# Outline

# Transient queue length distribution

$$
\begin{aligned}
p_n(t) &= s_n + \rho^{\frac{n}{2}} \sum_{j=1}^{\ell} C_j \left\{ \sin \frac{jn\pi}{\ell+1} - \sqrt{\rho} \sin \frac{j(n+1)\pi}{\ell+1} \right\} e^{\tau_j t} \\
\tau_j &= \lambda + \mu - 2\sqrt{\lambda\mu} \cos \frac{j\pi}{\ell+1}
\end{aligned}
$$

(Morse, '58)

- $p_n(t)$ probability of $n$ jobs at time $t$ in the queue
- $s_n$ stationary probability of $n$ jobs
- $\rho = \frac{\lambda}{\mu}$ traffic intensity at $t \geq 0$
- $C_j$ enforce consistency with queue length distribution at $t = 0$

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# Discrete time simulation of queue interactions

- update interactions in *discrete* time
- piecewise constant queue boundary conditions ($\lambda$,$\mu$)
  - updated at the beginning of every time step
  - kept constant for the duration of the time step
- continuous-time evolution of queue distributions
  - subject to jump-changes in boundary conditions
  - yield queue boundary conditions in next time step

# Outline



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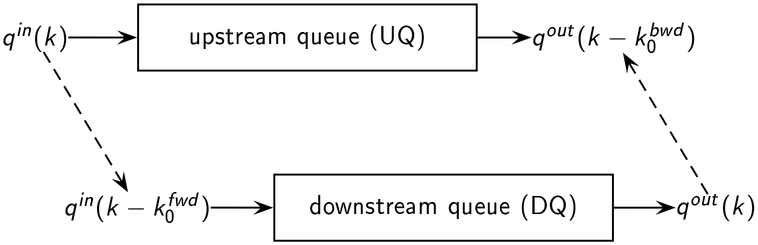
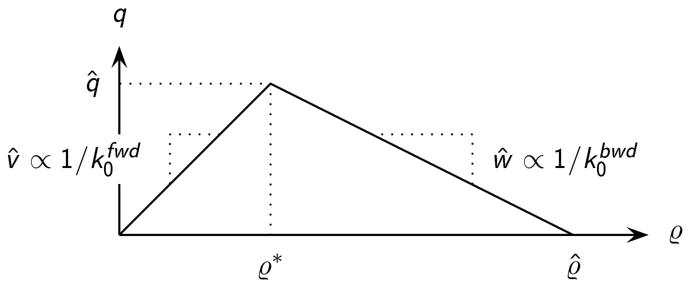
# One link consists of two queues




(Charypar et al., '07)

- $k$ is discrete time index
- forward lag $k_0^{fwd}$ generates lower bound on link travel time
- backward lag $k_0^{bwd}$ generates slow queue dissipation

# Fundamental diagram for deterministic queues



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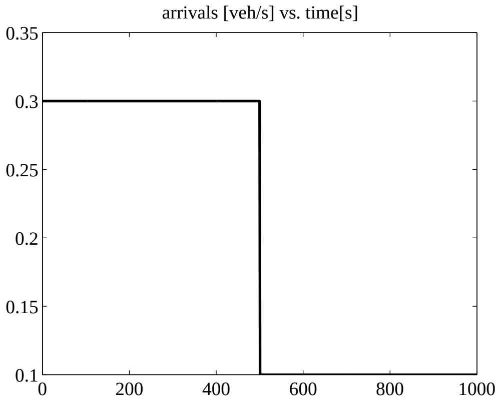
# Settings

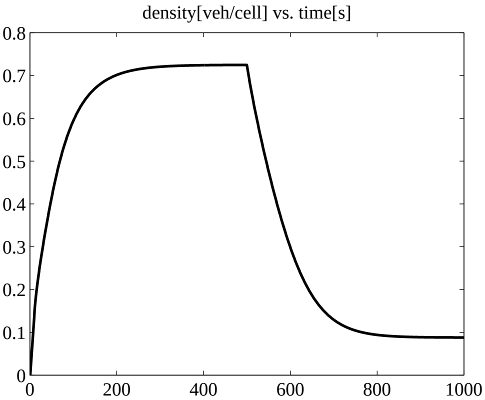
- a single, initially empty link under various boundary conditions
- link parameters

| parameter | value | normalized |
|---|---|---|
| vehicle length | 5 m | 1 place |
| link length | 100 m | 20 places |
| max. density $\hat{\varrho}$ | 200 veh/km | 1 veh/place |
| time step length | 1 s | 1 s |
| free flow velocity $\hat{v}$ | 36 km/h | 2 places/s |
| backward wave speed $\hat{w}$ | 18 km/h | 1 place/s |
| downstream bottleneck | 720 veh/h | 0.2 veh/s |

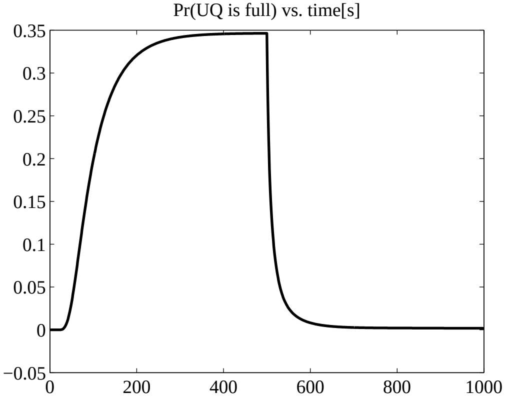
# Arrival profile – $\lambda(k)$



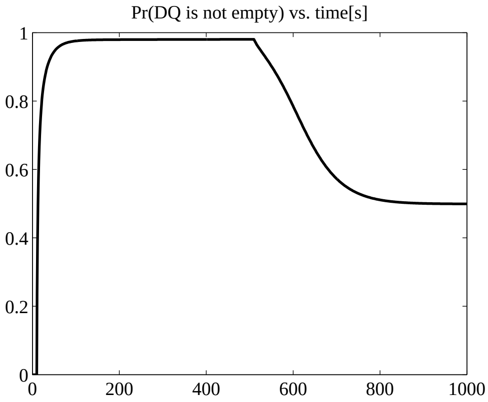
# Density profile – $\varrho(k)$



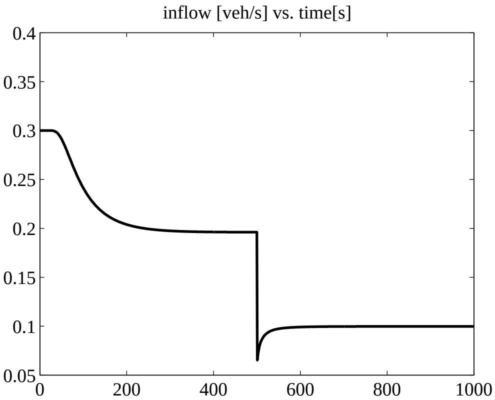
# Blocking probability – $P(N_{UQ} = \ell)$



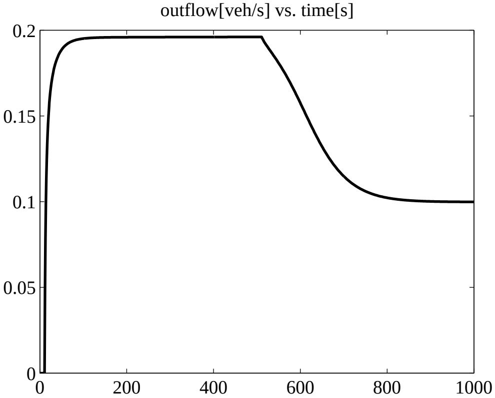
# Availability probability – $P(N_{DQ} > 0)$



# Inflow profile – $q^{in}(k)$



# Outflow profile – $q^{out}(k)$



# Transients: discussion

- build-up, spillback, and dissipation of queues is captured
- bottleneck shifted upstream in supply regime
- arrival rate shifted downstream in demand regime
- stochastic model captures
  - spillback probabilities
  - variances of queue length, travel time, ...
  - queueing in undersaturated conditions

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**Results: fundamental diagram**

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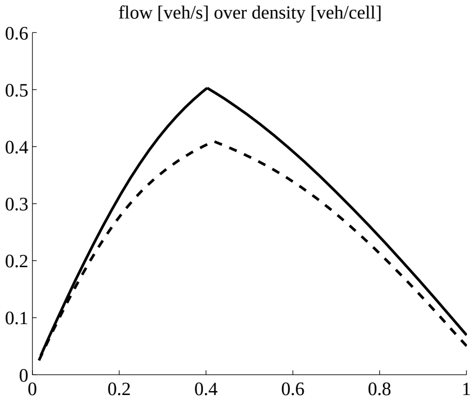
# Generation of fundamental diagram (FD)

- uncongested, left half (demand regime)
  - set bottleneck capacity $\mu$ to a large value
  - run until stationarity for one particular arrival rate $\lambda$
  - the resulting $(\varrho, q)$ tuple constitutes one point of the FD
  - repeat this for many $\lambda$ between zero and $\mu$
- congested, right half (supply regime)
  - set arrival rate $\lambda$ to a large value
  - run until stationarity for one particular bottleneck capacity $\mu$
  - the resulting $(\varrho, q)$ tuple constitutes one point of the FD
  - repeat this for many $\mu$ between zero and $\lambda$

# Two fundamental diagrams



# Fundamental diagrams: discussion

- slopes at the extreme points correspond to $\hat{v}$ and $\hat{w}$
- concave, plausible maximum value (link capacity)
- positive flow at jam density probably a numerical artifact
- the two FDs differ only in the "large value" ...
  - of the bottleneck when computing the demand regime
  - of the arrivals when computing the supply regime

$\Rightarrow$ downstream bottleneck takes effect in demand regime

$\Rightarrow$ upstream arrivals take effect in supply regime

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  - captures build-up and dissipation of queues
  - generates plausible fundamental diagram
- differentiability good for estimation, optimization, assignment
- ongoing work: more complex node models, network traffic

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**Thank you for your attention.**