Introduction to disaggregate demand models

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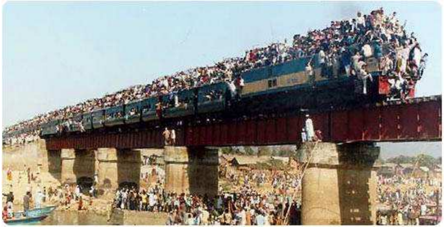
Outline

Demand models

- Supply = infrastructure
- Demand = behavior, choices
- Congestion = mismatch

Demand models



- Usually in OR:
- optimization of the supply
- for a given (fixed) demand

Aggregate demand

- Homogeneous population
- Identical behavior
- Price (P) and quantity (Q)
- Demand functions: $P = f(Q)$
- Inverse demand: $Q = f^{-1}(P)$

Disaggregate demand

- Heterogeneous population
- Different behaviors
- Many variables:
  - Attributes: price, travel time, reliability, frequency, etc.
  - Characteristics: age, income, education, etc.
- Complex demand/inverse demand functions.

Choice

Choice: outcome of a sequential decision-making process

- Definition of the choice problem: How do I get to work?
- Generation of alternatives: car as driver, car as passenger, train
- Evaluation of the attributes of the alternatives: price, time, flexibility, comfort
- Choice: decision rule
- Implementation: travel

Choice theory

A choice theory defines

1. decision maker
2. alternatives
3. attributes of alternatives
4. decision rule

Decision-maker

The decision maker is

- an individual or a group of persons.
- If group of persons, we ignore internal interactions.
- Important to capture difference in tastes and decision-making processes.
- Socio-economic characteristics: age, gender, income, education, etc.

Alternatives

Choice set

- Environment: *universal choice set* ($\mathcal{U}$)
- Individual n: *choice set* ($\mathcal{C}_n$)

Choice set generation

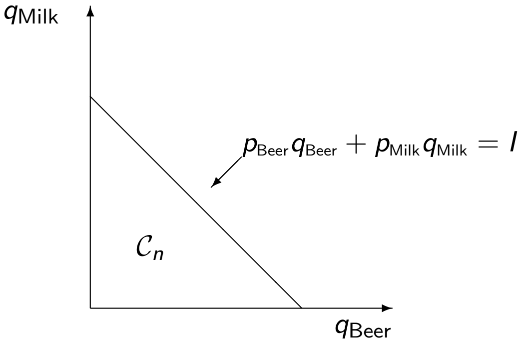
- Availability
- Awareness

Choice set type

- Continuous
- Discrete

Continuous vs. discrete

Continuous choice set



Discrete choice set

$$\mathcal{C}_n = \{\text{ Car, Bus, Bike}\}$$

Attributes

Describe the item:

- cost
- travel time
- walking time
- comfort
- bus frequency
- etc.

Decision rules

Neoclassical economic theory

Preference-indifference operator $\gtrsim$

1. reflexivity

$$a \gtrsim a \quad \forall a \in \mathcal{C}_n$$

2. transitivity

$$a \gtrsim b \text{ and } b \gtrsim c \Rightarrow a \gtrsim c \quad \forall a, b, c \in \mathcal{C}_n$$

3. comparability

$$a \gtrsim b \text{ or } b \gtrsim a \quad \forall a, b \in \mathcal{C}_n$$

Decision rules

Utility

$$\exists\ U_n : \mathcal{C}_n \longrightarrow \mathbb{R} : a \rightsquigarrow U_n(a) \text{ such that}$$

$$a \gtrsim b \Leftrightarrow U_n(a) \geq U_n(b) \quad \forall a, b \in \mathcal{C}_n$$

Remarks

- Utility is a latent concept
- It cannot be directly observed

Utility and continuous choice set

Context for the decision maker

- $Q = \{q_1, \ldots, q_L\}$ consumption bundle
- q_i is the quantity of product i consumed
- Utility of the bundle:
$$U(q_1, \ldots, q_L)$$
- $Q_a \gtrsim Q_b$ iff $U(q_1^a, \ldots, q_L^a) \geq U(q_1^b, \ldots, q_L^b)$
- Budget constraint:
$$\sum_{i=1}^{L} p_i q_i \leq I.$$

Utility and continuous choice set

Decision-maker solves the optimization problem

$$\max_{q \in \mathbb{R}^L} U(q_1, \ldots, q_L)$$

subject to

$$\sum_{i=1}^{L} p_i q_i = I.$$

Example with two products

$$\max_{q_1, q_2} U = \beta_0 q_1^{\beta_1} q_2^{\beta_2}$$

subject to

$$p_1 q_1 + p_2 q_2 = I.$$

Utility and continuous choice set

Example with two products

$$\max_{q_1,q_2} U = \beta_0 q_1^{\beta_1} q_2^{\beta_2}$$

subject to

$$p_1 q_1 + p_2 q_2 = I.$$

Lagrangian of the problem

$$L(q_1, q_2, \lambda) = \beta_0 q_1^{\beta_1} q_2^{\beta_2} + \lambda(I - p_1 q_1 - p_2 q_2).$$

Necessary optimality condition (KKT)

$$\nabla L(q_1, q_2, \lambda) = 0$$

Utility and continuous choice set

KKT

$$\begin{array}{rcrcl} \beta_0\beta_1 q_1^{\beta_1-1} q_2^{\beta_2} & - & \lambda p_1 & = & 0 \\ \beta_0\beta_2 q_1^{\beta_1} q_2^{\beta_2-1} & - & \lambda p_2 & = & 0 \\ p_1 q_1 + p_2 q_2 & - & I & = & 0. \end{array}$$

We have

$$\begin{array}{rcrcl} \beta_0\beta_1 q_1^{\beta_1} q_2^{\beta_2} & - & \lambda p_1 q_1 & = & 0 \\ \beta_0\beta_2 q_1^{\beta_1} q_2^{\beta_2} & - & \lambda p_2 q_2 & = & 0 \end{array}$$

so that

$$\lambda I = \beta_0 q_1^{\beta_1} q_2^{\beta_2} (\beta_1 + \beta_2)$$

KKT (ctd.)

Therefore

$$\beta_0 q_1^{\beta_1} q_2^{\beta_2} = \frac{\lambda I}{(\beta_1 + \beta_2)}$$

As $\beta_0\beta_2 q_1^{\beta_1} q_2^{\beta_2} = \lambda p_2 q_2$, we obtain (assuming $\lambda \neq 0$)

$$q_2 = \frac{I\beta_2}{p_2(\beta_1 + \beta_2)}$$

Similarly, we obtain

$$q_1 = \frac{I\beta_1}{p_1(\beta_1 + \beta_2)}$$

Utility and continuous choice set

Demand functions

$$q_1 = \frac{I\beta_1}{p_1(\beta_1+\beta_2)}$$

$$q_2 = \frac{I\beta_2}{p_2(\beta_1+\beta_2)}$$

Utility and discrete choice set

Binary optimization

$$\max_{q \in \{0,1\}^L} U = U(q_1, \ldots, q_L)$$

with

$$q_i = \begin{cases} 1 & \text{if product } i \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_i q_i = 1.$$

No optimality condition. Calculus cannot be used anymore.

Methodology

Utility functions

- Do not work with demand functions anymore
- Work with utility functions
- U is the "global" utility
- Define U_i the utility associated with product i.
- It is a function of the attributes of the product (price, quality, etc.)
- We say that product i is chosen if

$$U_i \geq U_j \quad \forall j.$$

Example

Two transportation modes

$$\begin{aligned} U_1 &= -\beta t_1 - \gamma c_1 \\ U_2 &= -\beta t_2 - \gamma c_2 \end{aligned}$$

with β, $\gamma > 0$

$$U_1 \geq U_2 \text{ iff } -\beta t_1 - \gamma c_1 \geq -\beta t_2 - \gamma c_2$$

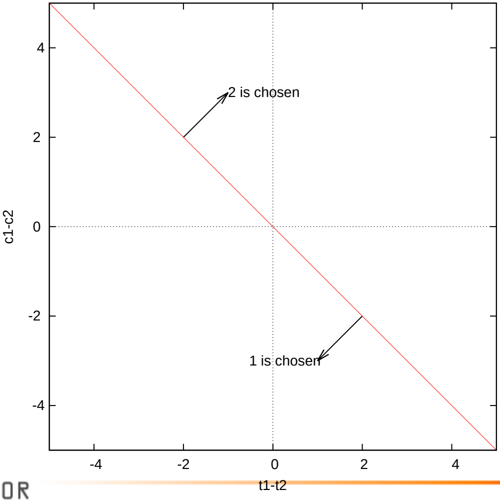
that is

$$-\frac{\beta}{\gamma} t_1 - c_1 \geq -\frac{\beta}{\gamma} t_2 - c_2$$

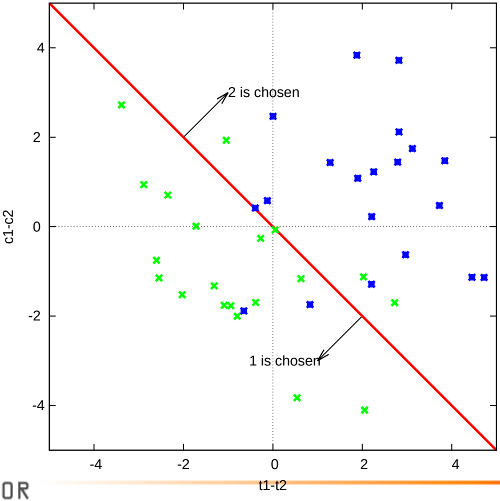
or

$$c_1 - c_2 \leq -\frac{\beta}{\gamma}(t_1 - t_2)$$

Example



Example



Assumptions

Decision-maker

- perfect discriminating capability
- full rationality
- permanent consistency

Analyst

- knowledge of all attributes
- perfect knowledge of $\gtrsim$ (or $U_n(\cdot)$)
- no measurement error

Must deal with uncertainty

- Random utility models
- For each individual n and alternative i

$$U_{in} = V_{in} + \varepsilon_{in}$$

and

$$P(i|\mathcal{C}_n) = P[U_{in} = \max_{j \in \mathcal{C}_n} U_{jn}] = P(U_{in} \geq U_{jn} \; \forall j \in \mathcal{C}_n)$$

Daniel L. McFadden

- UC Berkeley 1963, MIT 1977, UC Berkeley 1991
- Laureate of *The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel 2000*
- Owns a farm and vineyard in Napa Valley
- "Farm work clears the mind, and the vineyard is a great place to prove theorems"

Assumptions on V_{in}

$$V_{in} = V(x_{in}) = V(z_{in}, S_n)$$

Variables

- z_{in}: vector of attributes of alternative i for individual n
- S_n: vector of socio-economic characteristics of n
- $x_{in} = (z_{in}, S_n)$

Functional form

- Common assumption: Linear-in-parameter

$$V_{in} = \sum_p \beta_p (x_{in})_p$$

- β: vector of unknown parameters to be estimated
- Possibility for nonlinear specifications.

Assumptions on ε_{in}

- Mean: parameter to be estimated.
- Variance: parameter that cannot be estimated, must be normalized.
- Distribution: two common assumptions

Probit model

- Normal distribution
- Motivation: central limit theorem

Logit model

- Extreme value distribution
- i.i.d. across i and n
- Motivation: Gumbel's theorem

Logit model

Utility

$$U_{in} = V_{in} + \varepsilon_{in}$$

Choice probability

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}}}{\sum_{j\in\mathcal{C}_n} e^{V_{jn}}}.$$

- Decision-maker n
- Alternative $i \in \mathcal{C}_n$

Model estimation

- Design a sampling protocol
- For each individual n in the sample, and for each alternative i,
  - observe the explanatory variables x_{in}
  - observe the dependent variable, that is the choice i_n.
  - Compute the likelihood as a function of the parameters β:

$$\Lambda_n(\beta) = P_n(i_n|C_n) = \frac{e^{V_{i_n n}}}{\sum_{j \in \mathcal{C}_n} e^{V_{jn}}}.$$

- Compute the likelihood of the whole sample:

$$\mathcal{L}^*(\beta) = \prod_n \Lambda_n(\beta)$$

- Solve the maximum likelihood problem:

$$\widehat{\beta} = \operatorname{argmax}_\beta \log \mathcal{L}^*(\beta) = \operatorname{argmax}_\beta \sum_m \log \Lambda_n(\beta).$$

Aggregation

Reality

- Population composed of N individuals
- For each alternative i, and each individual n, define

$$y_{in} = \begin{cases} 1 & \text{if individual } n \text{ chooses alternative } i, \\ 0 & \text{otherwise.} \end{cases}$$

- Total number of individuals selecting item i in the population:

$$N(i) = \sum_{n=1}^{N} y_{in}.$$

- Market share for item i:

$$W(i) = N(i)/N = \sum_{n=1}^{N} y_{in}/N.$$

Forecasting

- Replace y_{in} by $P_n(i|x_n, C_n)$
- Total number of individuals selecting item i in the population:

$$\widehat{N}(i) = \sum_{n=1}^{N} P_n(i|x_n, C_n)$$

- Market share for item i:

$$\widehat{W}(i) = \widehat{N}(i)/N = \sum_{n=1}^{N} P_n(i|x_n, C_n)/N.$$

- Issue: no way to access x_n for the entire population.
- Solution: use a representative sample

Demand models

- $\widehat{N}(i)$ or $\widehat{W}(i)$ are the demand models required for OR
- They can be computed for subgroups of the population
- They account for the heterogeneity of
  - behavior,
  - taste,
  - choice contexts.

Value of time

Objective

- Monetary value of travel time?
- Cost - benefit analysis
- Costs : CHF
- Benefits : travel time savings

Definition

Price that travelers are willing to pay to decrease the travel time.

Motivation

Total time budget is limited, saved time can be used for other activities and, therefore, has value.

Example

Utility functions

$$\begin{aligned} U_1 &= -\beta t_1 - \gamma c_1 \\ U_2 &= -\beta t_2 - \gamma c_2 \end{aligned}$$

with β, $\gamma > 0$

Choice assumption

$$U_1 \geq U_2 \text{ if } \underbrace{c_1 - c_2}_{\text{CHF}} \leq - \underbrace{\frac{\beta}{\gamma}}_{\frac{\text{CHF}}{\text{hours}}} \underbrace{(t_1 - t_2)}_{\text{hours}}$$

Value of time

- If utility function is linear
- the value of time is the ratio between
  - the coefficient of the “time” variable, and
  - the coefficient of the “cost” variable
- Warning: utility is not always linear
- Value of time varies with
  - trip purpose
  - transportation mode
  - trip length
  - income

Example: model choice in Nijmegen

$$
\begin{array}{rcllll}
V_{\text{car}} & = & -0.798 & -0.110 \cdot \text{cost}_{\text{car}} & -1.33 \cdot \text{time}_{\text{car}} \\
V_{\text{train}} & = & & -0.110 \cdot \text{cost}_{\text{train}} & -1.33 \cdot \text{time}_{\text{train}}
\end{array}
$$

Value of time = -1.33 / -0.110 $\approx$ 12 euros / h $\approx$ 0.20 euros / min

	Case 1	Case 2
Time	2 h	1.5 h
Cost	7 €	13 €
Utility of train	-3.43	-3.43

Other willingness to pay indicators

- Headway (i.e. time between two buses)
- Number of transfers
- Reliability
- etc.

Same methodology:

- The model must involve the corresponding variable
- Willingness-to-pay = ratio between the coefficient of the variable and the cost coefficient

$$U = -\beta t - \gamma c - \alpha n$$

Willingness-to-pay to have one less transfer: α/γ

Value of time in Switzerland

Reference

Axhausen, K., Hess, S., Koenig, A., Abay, G., Bates, J., and Bierlaire, M. (2008). Income and distance elasticities of values of travel time savings: new Swiss results, *Transport Policy* **15**(3):173-185.

Data collection

- Source for recruitment: survey "Kontinuierliche Erhebung zum Personenverkehr" (KEP) by SBB/CFF
- Stated preferences
- Questionnaire designed based on a real reference trip
- Three parts:
  - SP mode choice (car / bus or rail)
  - SP route choice (current mode or alternative mode)
  - Socio-demographics and information about the reference trip

SP survey

Mode choice car – rail (main study version)

Travel costs:	18 Fr.	Travel costs:	23 Fr.
Total travel time:	40 minutes	Travel time:	30 minutes
... congested:	10 minutes	Headway:	30 minutes
... uncongested:	30 minutes	No. of changes:	0 times

☐ ← **Your choice** → ☐

Route choice rail (main study version)

Travel costs:	20 Fr.	Travel costs:	23 Fr.
Travel time:	40 minutes	Travel time:	30 minutes
Headway:	15 minutes	Headway:	30 minutes
No. of changes:	1 times	No. of changes:	0 times

☐ ← **Your choice** → ☐

Data

Number of observations (1225 individuals)

	Busin.	Comm.	Leis.	Shop.	Total
Mode : car/bus	6	162	186	126	480
Mode : car/rail	426	1716	2538	1104	5784
Route : bus for bus users	9	405	450	342	1206
Route : car for car users	156	846	1176	660	2838
Route : rail for car users	126	594	837	504	2061
Route : rail for rail users	324	1008	1881	288	3501
Total	1047	4731	7068	3024	15870

In average, $15870/1225 \approx 13$ valid responses out of 16 per respondent.

Model specification

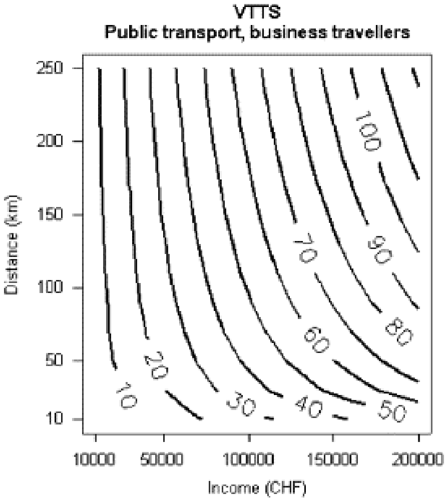
Variables

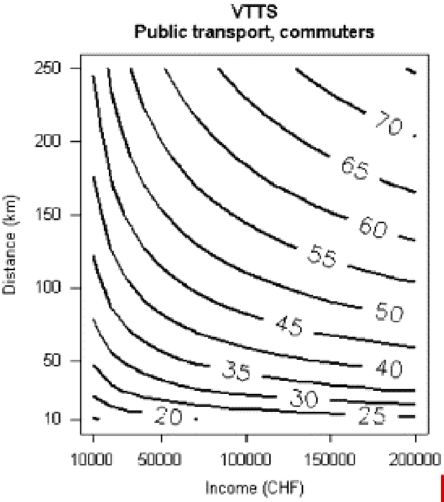
- travel time
- travel cost
- level of congestion (car)
- frequency (TC)
- number of transfers (TC)
- trip length
- income
- inertia
- car availability
- sex
- 1/2-fare CFF
- general subscription
- trip purpose

Results

	Business	Commute	Leisure	Shopping
Time TC (CHF/h)	49.57	27.81	21.84	17.73
Time car (CHF/h)	50.23	30.64	29.20	24.32
Headway (CHF/h)	14.88	11.18	13.38	8.48
CHF/transfer	7.85	4.89	7.32	3.52

Results


VTTS
Public transport, business travellers
250
200
150
100
50
10
Distance (km)
10000
50000
100000
150000
200000
Income (CHF)
10
20
30
40
50
60
70
80
90
100


VTTS
Public transport, commuters
250
200
150
100
50
10
Distance (km)
10000
50000
100000
150000
200000
Income (CHF)
20
25
30
35
40
45
50
55
60
65
70

Introduction

- Choice model captures demand
- Demand is elastic to price
- Predicted demand varies with price, if it is a variable of the model
- In principle, the probability to use/purchase an alternative decreases if the price increases.
- The revenue per user increases if the price increases.
- Question: what is the optimal price to optimize revenue?

In short

- Price$\uparrow \Rightarrow$ profit/passenger$\uparrow$ and number of passengers $\downarrow$
- Price$\downarrow \Rightarrow$ profit/passenger$\downarrow$ and number of passengers $\uparrow$
- What is the best trade-off?

Revenue calculation

Number of persons choosing alternative i in the population

$$\hat{N}(i) = \sum_{s=1}^{S} N_s P(i|x_s, p_{is})$$

where

- the population is segmented into S homogeneous strata
- p_{is} is the price of item i in segment s
- x_s gathers all other variables corresponding to segment s
- $P(i|x_s, p_{is})$ is the choice model
- N_s is the number of individuals in segment s

Revenue calculation

Total revenue from i

$$R_i = \sum_{s=1}^{S} N_s P(i|x_s, p_{is}) p_{is}$$

If the price is constant across segments...

$$R_i = p_i \sum_{s=1}^{S} N_s P(i|x_s, p_i)$$

Price optimization

Optimizing the price of product i is solving the problem

$$\max_{p_i} p_i \sum_{s=1}^{S} N_s P(i|x_s, p_i)$$

Notes:

- It assumes that everything else is equal
- In practice, it is likely that the competition will also adjust the prices

Illustrative example

A binary logit model with

$$\begin{aligned} V_1 &= \beta_p p_1 - 0.5 \\ V_2 &= \beta_p p_2 \end{aligned}$$

so that

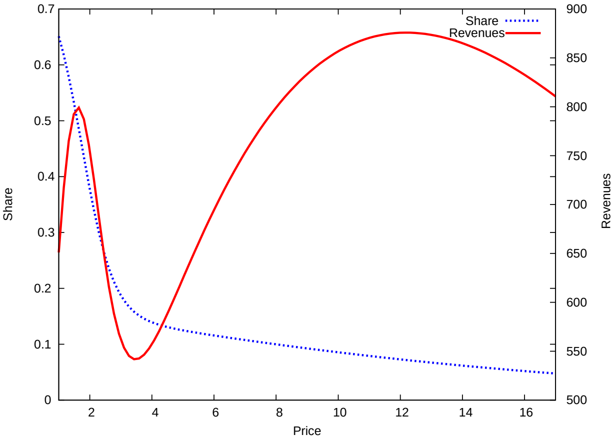
$$P(1|p) = \frac{e^{\beta_p p_1 - 0.5}}{e^{\beta_p p_1 - 0.5} + e^{\beta_p p_2}}$$

Two groups in the population:

- Group 1: $\beta_p = -2$, $N_s = 600$
- Group 2: $\beta_p = -0.1$, $N_s = 400$

Assume that $p_2 = 2$.

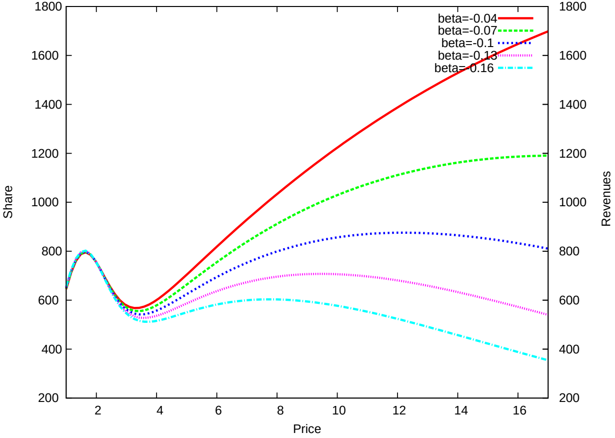
Illustrative example



Sensitivity analysis

- Parameters are estimated, we do not know the real value
- 95% confidence interval: $[\widehat{\beta}_p - 1.96\sigma, \widehat{\beta}_p + 1.96\sigma]$
- Perform a sensitivity analysis for β_p in group 2

Sensitivity analysis



Summary