Benders decomposition for choice-based optimization problems with discrete upper-level variables

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Motivation

- In many transportation problems suppliers can benefit from a disaggregate model of demand to capture observed and unobserved heterogeneity.
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- Many classes of discrete choice models cannot be easily integrated in mixed integer optimization models → **choice-based optimization**.
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- The majority of the works in the literature sacrifice complexity either at **demand level** or at supply level for the sake of tractability.
Motivation

- In many transportation problems suppliers can benefit from a disaggregate model of demand to capture **observed and unobserved heterogeneity**.

- Many classes of discrete choice models cannot be easily integrated in mixed integer optimization models → **choice-based optimization**

- The majority of the works in the literature sacrifice complexity either at **demand level** or at supply level for the sake of tractability.

- Alternative approach: trying to circumvent issues related to non-linearity and non-convexity of the demand function using **simulation**.
Simulation-based linearization of choice probabilities

- Let $I$ be the universal choice set and $N$ be the set of heterogeneous customers.

- Random utility models:
  \[ U_{in} = V_{in} + \epsilon_{in} \quad \forall i \in I, \forall n \in N. \]

- Choice probabilities:
  \[ P_{in} = \Pr[V_{in} + \epsilon_{in} = \max_{j \in I}(V_{jn} + \epsilon_{jn})]. \]

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- Linearization:
  \[ U_{inr} = V_{in} + \xi_{inr} \quad \forall i \in I, \forall n \in N, \forall r \in R, \]
  \[ x_{inr} = \begin{cases} 
  1 & \text{if } U_{inr} = \max_{j \in I} U_{jnr}, \\
  0 & \text{otherwise} \end{cases} \quad \forall i \in I, \forall n \in N, \forall r \in R, \]
  \[ P_{in} = \frac{1}{|R|} \sum_{r \in R} x_{inr} \quad \forall i \in I, \forall n \in N. \]

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Previous research

Applications:

- Optimizing prices for uncapacitated and capacitated services.\(^2\)
- Computing approximate equilibrium solutions for competitive markets.\(^3\)
- Determining optimal price-based regulation of transport markets.\(^4\)

Open questions:

- Scalability.
- Extension to variables other than prices.

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\(^3\)Bortolomiol, Lurkin, and Bierlaire, “A simulation-based heuristic to find approximate equilibria with disaggregate demand models” (2021).

\(^4\)Bortolomiol, Lurkin, and Bierlaire, “Price-based regulation of oligopolistic markets under discrete choice models of demand” (2021).
Continuous Pricing Problem (CPP)

- A supplier wants to maximize profits obtained by controlling alternatives \( I_k \subset I \).
- The utilities of the customers are price-dependent variables:

\[
U_{inr} = \beta_{p,inr} p_i + \hat{q}_{inr} + \xi_{inr} \quad \forall i \in I, \forall n \in N, \forall r \in R.
\]
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$$\max_p \quad \pi = \sum_{i \in I_k} \sum_{n \in N} \sum_{r \in R} \frac{1}{|R|} \theta_n p_i x_{inr},$$  

s.t.  

$$\sum_{i \in I} x_{inr} = 1 \quad \forall n \in N, \forall r \in R, \quad (2)$$

$$\sum_{j \in I} U_{jnr} x_{jnr} \geq U_{inr} \quad \forall i \in I, \forall n \in N, \forall r \in R, \quad (3)$$

$$0 \leq p_i \leq M_i^p \quad \forall i \in I, \quad (4)$$

$$x_{inr} \in \{0, 1\} \quad \forall i \in I, \forall n \in N, \forall r \in R. \quad (5)$$

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- The linearization of the product \( p_i \cdot x_{inr} \) (continuous and binary) can be done using big-M constraints.
Discrete Pricing Problem (DPP)

- For each alternative \( i \in I_k \) we constrain prices \( p_i \) to the set \( Q_i = \{ p_i^1, p_i^2, \ldots, p_i^{\mid Q \mid} \} \).
- Utilities become parameters of the optimization model: \( \hat{U}_{inr} = \beta_{p,inr} \hat{p}_i + \hat{q}_{inr} + \xi_{inr} \).
Discrete Pricing Problem (DPP)

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\[
\begin{align*}
\max_{\pi} \quad & \sum_{i \in I_k} \sum_{n \in N} \sum_{r \in R} \frac{1}{|R|} \theta_n \hat{p}_i x_{inr}, \\
\text{s.t.} \quad & \sum_{j \in I_k^{exp}} y_j = 1 \quad \forall i \in I, \\
& \sum_{i \in I^{exp}} x_{inr} = 1 \quad \forall n \in N, \forall r \in R, \\
& x_{inr} \leq y_i \quad \forall i \in I^{exp}, \forall n \in N, \forall r \in R, \\
& \sum_{j \in I^{exp}} \hat{U}_{jnr} x_{jnr} \geq \hat{U}_{inr} y_i \quad \forall i \in I^{exp}, \forall n \in N, \forall r \in R, \\
& x_{inr} \in \{0, 1\} \quad \forall i \in I^{exp}, \forall n \in N, \forall r \in R, \\
& y_i \in \{0, 1\} \quad \forall i \in I^{exp}.
\end{align*}
\]
Numerical experiments

| $|R|$ | CPP |  | DPP |  | Gap |
|---|---|---|---|---|---|
|   | |   |   |   |   |
|   | Time | Opt | $|I^\text{exp}|$ | Time | Opt |   |
| 20 | 0.45 | 71774.95 | 21 | 1.42 | 70390.50 | 1.93% |
|   | 51 | 7.18 | 71316.20 | 0.64% |
|   | 101 | 8.89 | 71379.90 | 0.55% |
| 50 | 10.46 | 72423.71 | 21 | 14.59 | 71889.00 | 0.74% |
|   | 51 | 31.51 | 72106.36 | 0.44% |
|   | 101 | 89.91 | 72185.30 | 0.33% |
| 100 | 101.64 | 66452.18 | 21 | 34.48 | 66118.40 | 0.50% |
|   | 51 | 161.03 | 66255.90 | 0.30% |
|   | 101 | 395.86 | 66341.32 | 0.17% |
| 200 | 288.89 | 70788.17 | 21 | 139.17 | 69859.60 | 1.31% |
|   | 51 | 415.90 | 70489.95 | 0.42% |
|   | 101 | 1829.24 | 70571.67 | 0.31% |

Table: High-speed rail pricing: solving CPP and DPP to optimality with CPLEX.
**Assortment and Continuous Pricing Problem (ACPP)**

- We include the decision about whether or not to offer any given product $i \in I_k$ to the customers.
- The actual utility for the customer is $U_{inr}^a = U_{inr} \cdot y_i$. 

 Customers must choose the alternative with the highest utility among those that are made available by the supplier:

$$U_{inr} = \beta_{inr} \cdot p_i + q_{inr} + \xi_{inr} \ \forall \ i \in I, \ \forall \ n \in N, \ \forall \ r \in R, \ (13)$$

$$U_{a} \leq U_{inr} \ \forall \ i \in I, \ \forall \ n \in N, \ \forall \ r \in R, \ (14)$$

$$U_{inr} \leq U_{a} + M U_{inr}(1 - y_i) \ \forall \ i \in I, \ \forall \ n \in N, \ \forall \ r \in R, \ (15)$$

$$U_{a} \leq M U_{inr} y_i \ \forall \ i \in I, \ \forall \ n \in N, \ \forall \ r \in R, \ (16)$$
Assortment and Continuous Pricing Problem (ACPP)

- We include the decision about whether or not to offer any given product $i \in I_k$ to the customers.
- The actual utility for the customer is $U^a_{inr} = U_{inr} \cdot y_i$.
- Customers must choose the alternative with the highest utility among those that are made available by the supplier:

$$U_{inr} = \beta_{p, inr} p_i + q_{inr} + \xi_{inr} \quad \forall i \in I, \forall n \in N, \forall r \in R, \quad (13)$$

$$U^a_{inr} \leq U_{inr} \quad \forall i \in I, \forall n \in N, \forall r \in R, \quad (14)$$

$$U_{inr} \leq U^a_{inr} + M^U_{inr} (1 - y_i) \quad \forall i \in I, \forall n \in N, \forall r \in R, \quad (15)$$

$$U^a_{inr} \leq M^U_{inr} y_i \quad \forall i \in I, \forall n \in N, \forall r \in R, \quad (16)$$
The formulation of the DPP still applies, with a small change:

\[
\max_y \pi = \sum_{i \in I^\text{exp}} \sum_{n \in N} \sum_{r \in R} \frac{1}{|R|} \theta_n \hat{p}_i x_{inr},
\]

\[
\sum_{j \in I^\text{exp}} y_j = 1 \\
\sum_{i \in I^\text{exp}} x_{inr} = 1 \\
x_{inr} \leq y_i \\
\sum_{j \in I^\text{exp}} \hat{U}_{jnr} x_{jnr} \geq \hat{U}_{inr} y_i \\
x_{inr} \in \{0, 1\} \\
y_i \in \{0, 1\}
\]
**Numerical experiments**

| $|R|$ | ACPP | ADPP | Gap |
|-----|------|------|-----|
|     | Time | Opt  | $|l_{i}^{\text{opt}}|$ | Time | Opt  |     |
| 10  | 11706 | 907.8 | 16  | 132  | 864.0 | 4.82% |
|     | 31    |       | 31  | 800   | 876.0 | 3.50% |
| 20  | 129600* | 877.0* | 16  | 429  | 842.0 | 3.99% |
|     | 31    |       | 31  | 778   | 862.5 | 1.65% |
| 50  | 129600* | 842.8* | 16  | 837  | 816.4 | 3.13% |
|     | 31    |       | 31  | 12191 | 830.4 | 1.47% |
| 100 | 129600* | 844.0* | 16  | 3419 | 828.2 | 1.87% |
|     | 31    |       | 31  | 39425 | 831.8 | 1.45% |

**Table:** Parking assortment and pricing: solving ACPP and ADPP to optimality with CPLEX.
What about Benders and discrete supply variables?

- Let’s fix the discrete supply variables of the supplier to $y^*$.
- The lower-level utility maximization problem for a single customer $n$ and scenario $r$ is as follows:

\[
\begin{align*}
\max_x & \quad U = \sum_{i \in I} \hat{U}_i x_i, \\
\text{s.t.} & \quad \sum_{i \in I} x_i = 1, \\
& \quad x_i \leq y_i^* \quad \forall i \in I, \\
& \quad x_i \geq 0 \quad \forall i \in I.
\end{align*}
\]

This is a continuous knapsack problem, where the knapsack’s capacity is equal to 1 and each item (alternative) $i$ has a weight of 1 and a value of $\hat{U}_{inr}$.
Benders decomposition

1. Initialize $UB = \infty$ and $LB = -\infty$ of the master problem (MP).
Benders decomposition

1. Initialize $UB = \infty$ and $LB = -\infty$ of the master problem (MP).
2. Initialize the restricted master problem (RMP):

$$\min_{y,z} z$$

s.t.

- Domain constraints on the $y$ variables
- $z \geq LB_z$. (28) (29) (30)

3. Solve current RMP. Save the solution $y^*, z^*$. Let $f(y^*, z^*)$ be the optimal objective value. Update $LB = f(y^*, z^*)$.

4. Given $y^*$, compute $f(y^*)_{ADPP}$ for the original problem by deriving the choices for all customers and scenarios. Update $UB = \min\{UB, f(y^*)_{ADPP}\}$.

5. If $UB - LB \leq \epsilon$, then stop. Else, solve the dual worker problem for $y = y^*$. Using the optimal dual variables, add to the master problem an optimality cut of the following form:

$$z \geq \sum_{n \in N} \sum_{r \in R} (\sum_{i \in I} m_i y_i + q)$$

and go to step 3.

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\[
\min_{y,z} \quad z \\
\text{s.t.} \quad \text{Domain constraints on the y variables} \\
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1. Initialize $UB = \infty$ and $LB = -\infty$ of the master problem (MP).

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   \hspace{1cm} (28)

   \text{s.t.} \quad \text{Domain constraints on the y variables}$\quad (29)$

   $z \geq LB_z.$ \hspace{1cm} (30)

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   and go to step 3.
Solving the master problem at each iteration is inefficient.

Benders cuts can be inserted while processing the branch-and-bound tree of the master problem.\(^5\)

Preliminary results

| $|R|$ | $|i_i^{\text{exp}}|$ | Opt | CPLEX | BBC |
|---|---|---|---|---|
| 5  | 3  | 2399.20 | 13.20 | 42.81 |
| 5  | 6  | 2526.20 | 165.68 | 230.17 |
| 5  | 12 | 2641.60 | 3685.49 | 1793.00 |
| 10 | 3  | 2330.20 | 87.79 | 106.40 |
| 10 | 6  | 2727.20 | 703.47 | 587.74 |
| 10 | 12 | 2795.10 | 10931.09 | 7627.22 |
| 20 | 3  | 2333.90 | 363.22 | 256.94 |
| 20 | 6  | 2585.15 | 1066.06 | 1669.50 |
| 20 | 12 | 2638.08 | 54336.94 | 27043.61 |

**Table:** Parking assortment and pricing, $|N| = 80$, $|l_k| = 12$: solving ADPP to optimality with CPLEX and BBC algorithm (single-thread).
Enhancements and future work

- Classical Benders cuts provide slow convergence → efficient cuts are key to the success of this approach:
  - Pareto-optimal cuts;\(^6\)
  - minimal infeasible subset cuts;
  - partial Benders decomposition.\(^7\)

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\(^7\) Crainic et al., “Partial benders decomposition: general methodology and application to stochastic network design” (2021).
Supply problems with advanced discrete choice models of demand can be written as stochastic optimization problems by relying on simulation.

Choice-based optimization problems with discrete upper-level variables exhibit a block-diagonal structure which make them particularly suitable to the use of decomposition techniques such as Benders.

We are working on efficient enhancements for our Benders approach to generate tighter cuts and reduce computational times.

The trade-off between the increased realism of the demand model and the computational complexity of the resulting optimization/equilibrium problem must be evaluated on a case-by-case basis.