A Spatial Branch and Benders Decomposition Algorithm for Continuous Pricing with Advanced Discrete Choice Demand Modeling

Tom Haering[a] Ivana Ljubic[b] Michel Bierlaire[a]

[a]Transport and Mobility Laboratory
School of Architecture, Civil and Environmental Engineering
Ecole Polytechnique Fédérale de Lausanne
[b]Dpt of Information Systems, Decision Sciences and Statistics
ESSEC Business School

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Outline

Motivation

Random utility models

- Workhorse of disaggregate demand models.
- Detailed characterization of complex behavioral patterns.
- Large literature with estimated and validated models.
- Almost never used in the operations research literature.

Example: mixture of logit

$$\Pr(i) = \int_x \frac{e^{V_i(x)}}{\sum_j e^{V_j(x)}} f_X(x) dx.$$

Motivation

Choice-based optimization

- Optimization problem involving a choice model.
- In this talk: a pricing problem.
- Main principle: weak assumptions on the choice model, so that most published models can potentially be considered.

Context

The problem

- A competitive market with $J + K$ products.
- An operator who controls the price of J products.
- Customers freely choose their preferred product.
- Pricing problem: maximize the revenues of the operator.

The model

- The decisions of the competitors is known and considered fixed.
- Customers' choices are characterized by a random utility model.
- Customers may have different tastes and preferences.
- The utility of the controlled alternatives is a linear function of price.
- The utility is a random variable with a known distribution.

Utility

Controlled alternatives

$$U_{in}(p) = U_{in}(p_i) = \beta_{in} p_i + c'_{in} + \varepsilon_{in}, \; i = 1, \dots, J$$

where

- p_i is the price of alternative i,
- $\beta_{in} < 0$ is the price coefficient for individual n (potentially a r.v.),
- c'_{in} is the fixed part of the utility observed by the analyst,
- ε_{in} is the fixed part of the utility unobserved by the analyst.

Uncontrolled alternatives

$$U_{jn}(p) = U_{jn} = c'_{jn} + \varepsilon_{jn}, \; j = 1 - K, \dots, 0$$

Simulation

Draws from the distributions

For $r = 1, \ldots, R$,

- β_{inr} are draws from β_{in},
- ε_{inr} are draws from ε_{in}.

Utility functions

$$
\begin{aligned}
u_{inr}(p) &= \beta_{inr} p_i + c'_{in} + \varepsilon_{inr}, & i &= 1, \ldots, J, \\
u_{jnr}(p) &= c'_{jn} + \varepsilon_{jnr}, & j &= 1 - K, \ldots, 0,
\end{aligned}
$$

or

$$
\begin{aligned}
u_{inr}(p) &= \beta_{inr} p_i + c_{inr}, & i &= 1, \ldots, J, \\
u_{jnr}(p) &= c_{jnr}, & j &= 1 - K, \ldots, 0.
\end{aligned}
$$

Choice model

Random utility model

$$P(i|p) = \Pr(U_{in}(p) \geq U_{jn}(p),\ \forall j).$$

Simulated random utility model

$$P(i|p) \approx \frac{1}{R} \sum_{r=1}^{R} \mathbb{1}\left[u_{inr}(p) \geq u_{jnr}(p),\ \forall j\right].$$

Outline

Choice model as a knapsack problem

For a given n and r:

Primal

$$\max_{w_{nr}} \sum_{i=1-K}^{J} w_{inr} u_{inr}(p)$$

subject to

$$\sum_{i=1-K}^{J} w_{inr} = 1,$$

$$w_{inr} \geq 0, \; i = 1-K, \ldots, J.$$

Dual

$$\min_{h_{nr}} h_{nr}$$

subject to

$$h_{nr} \geq u_{inr}, \; i = 1-K, \ldots, J.$$

h_{nr} is the largest utility.

Choice model as a knapsack problem

Integrality property

If $u_{inr} \neq u_{jnr}$, for each i, j, the solution of the knapsack problem is binary.

Optimality conditions: strong duality

$$
\begin{aligned}
h_{nr} &= \sum_{i=1-K}^{J} w_{inr} u_{inr}(p), \\
h_{nr} &\geq u_{inr},\ i = 1-K, \ldots, J, \\
\sum_{i=1-K}^{J} w_{inr} &= 1, \\
w_{inr} &\geq 0,\ i = 1-K, \ldots, J.
\end{aligned}
$$

The pricing problem

$$\min_{p,w,u,h} -\frac{1}{R}\sum_{r=1}^{R}\sum_{n=1}^{N}\sum_{i=1}^{J} p_i w_{inr}$$

subject to

$$h_{nr} = \sum_{i=1-K}^{J} w_{inr} u_{inr}, \qquad \forall n, r,$$

$$h_{nr} \geq u_{inr}, \qquad \forall i, n, r,$$

$$\sum_{i=1-K}^{J} w_{inr} = 1, \qquad \forall n, r,$$

$$w_{inr} \geq 0, \qquad \forall i, n, r,$$

$$u_{inr} = \beta_{inr} p_i + c_{inr}, \qquad i = 1, \ldots, J, \forall n, r,$$

$$u_{jnr} = c_{jnr}, \qquad j = 1-K, \ldots, 0, \forall n, r.$$

The pricing problem

Two difficulties

- Nonlinearity: $p_i w_{inr}$.
- Number of constraints: order of $J \times N \times R$.

First simplification

$$
\begin{aligned}
h_{nr} &\geq u_{inr}, && \forall i, n, r, \\
u_{jnr} &= c_{jnr}, && j = 1 - K, \ldots, 0, \forall n, r.
\end{aligned}
$$

- For each n and r, among the non controlled alternatives, only the best one matters.
- It can be safely assumed that there is only one “opt-out” alternative.

The pricing problem

Preprocessing

- For each n, r define

$$c_{0nr} = \max_{j=1-K,\ldots,0} c_{jnr}$$

- We redefine the problem with $K = 1$.

Specification

- Substitute

$$\begin{aligned} u_{inr} &= \beta_{inr} p_i + c_{inr}, & i = 1, \ldots, J, \forall n, r, \\ u_{0nr} &= c_{0nr}, & \forall n, r. \end{aligned}$$

- Define

$$\eta_{inr} = p_i w_{inr}.$$

The pricing problem

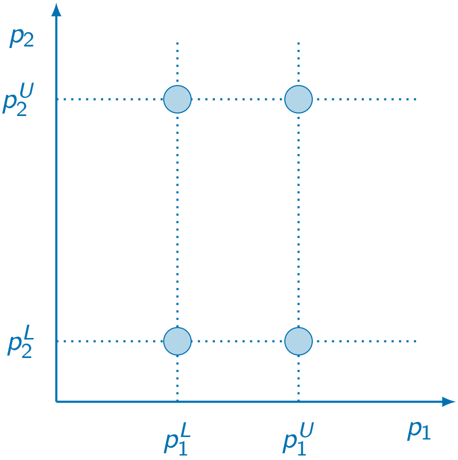
$$\min_{p,w,h,\eta} -\frac{1}{R}\sum_{r=1}^{R}\sum_{n=1}^{N}\sum_{i=1}^{J}\eta_{inr}$$

subject to

$$\begin{aligned}
h_{nr} &= c_{0nr}w_{0nr} + \sum_{i=1}^{J}\beta_{inr}\eta_{inr} + c_{inr}w_{inr}, && \forall n, r,\\
h_{nr} &\geq \beta_{inr}p_i + c_{inr}, && i = 1,\dots,J, \forall n, r,\\
h_{nr} &\geq c_{0nr}, && \forall n, r,\\
\eta_{inr} &= p_i w_{inr}, && i = 1,\dots,J, \forall n, r,\\
\sum_{i=0}^{J} w_{inr} &= 1, && \forall n, r,\\
w_{inr} &\geq 0, && \forall i, n, r,\\
p_i &\in [p_i^L, p_i^U] && i = 1,\dots,J.
\end{aligned}$$

Outline

Simplification



Simplification

Observations

- The number of controlled prices is generally low (usually, one or two).
- There are 2^J combinations of lower and upper bounds.

Procedure for each n and r

- For each combination, identify the best alternative.
- If alternative i is never the best, set $w_{inr} = 0$.
- If alternative i is always the best, set $w_{inr} = 1$.

Note

This happens often when bounds are tight.

Outline

McCormick envelopes

$p_i \in [p_i^L, p_i^U]$, $w_{inr} \in [0, 1]$

$$
\begin{aligned}
a_L &= p_i - p_i^L \geq 0, \\
a_U &= p_i^U - p_i \geq 0, \\
b_L &= w_{inr} \geq 0, \\
b_U &= 1 - w_{inr} \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
a_L b_L &= w_{inr} p_i - w_{inr} p_i^L = \eta_{inr} - w_{inr} p_i^L \geq 0, \\
a_L b_U &= p_i - p_i^L - w_{inr} p_i + w_{inr} p_i^L = p_i - p_i^L - \eta_{inr} + w_{inr} p_i^L \geq 0, \\
a_U b_L &= w_{inr} p_i^U - w_{inr} p_i = w_{inr} p_i^U - \eta_{inr} \geq 0, \\
a_U b_U &= p_i^U - p_i - w_{int} p_i^U + w_{inr} p_i = p_i^U - p_i - w_{int} p_i^U + \eta_{inr} \geq 0.
\end{aligned}
$$

McCormick envelopes

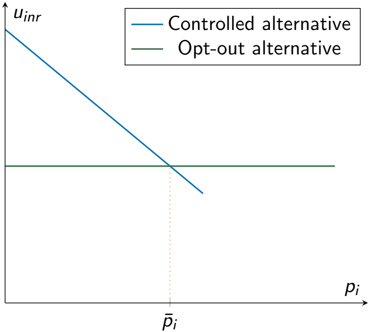
$$
\begin{aligned}
\eta_{inr} &\geq w_{inr} p_i^L, \\
\eta_{inr} &\leq p_i - p_i^L - w_{inr} p_i^L, \\
\eta_{inr} &\leq w_{inr} p_i^U, \\
\eta_{inr} &\geq -p_i^U + p_i + w_{int} p_i^U.
\end{aligned}
$$

Note

As we maximize on η_{inr}, the constraints setting lower bounds are not necessary.

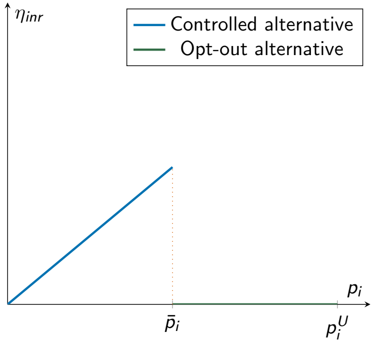
Break points

Competing with opt-out: utility



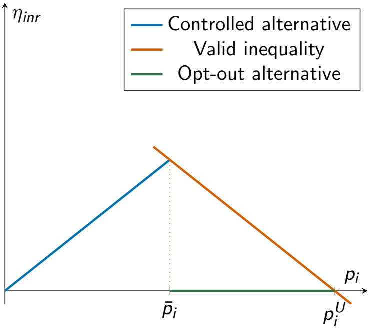
Break points

Competing with opt-out: revenue



Break points

Competing with opt-out: valid inequality



Valid inequalities based on break points

Competing with opt-out

$$\eta_{inr} \leq \frac{\bar{p}_i(p_i^U - p_i)}{p_i^U - \bar{p}_i}.$$

Competing with another controlled alternative

$$\eta_{inr} \leq \frac{\beta_j p_i^U p_j - c_i p_i^U + c_j p_i^U - p_i \left(\beta_j p_j^L - c_i + c_j\right)}{\beta_i p_i^U - \beta_j p_j^L + c_i - c_j}.$$

and

$$\eta_{inr} \leq \frac{\beta_j p_i^U p_j - c_i p_i^U + c_j p_i^U - p_i \left(\beta_j p_j^U - c_i + c_j\right)}{\beta_i p_i^U - \beta_j p_j^U + c_i - c_j}.$$

Outline

Dealing with nonlinearity

Spatial Branch & Bound

- Start with reasonable bounds on p_i: $[p_i^L, p_i^U]$.
- Relaxation: ignore the constraint $\eta_{inr} = p_i w_{inr}$.
- At each node, solve the relaxation: upper bound.
- Fix the price, identify the choices to obtain a feasible solution: lower bound.
- Split the price interval and branch.

Outline

Dealing with the large size

Observations

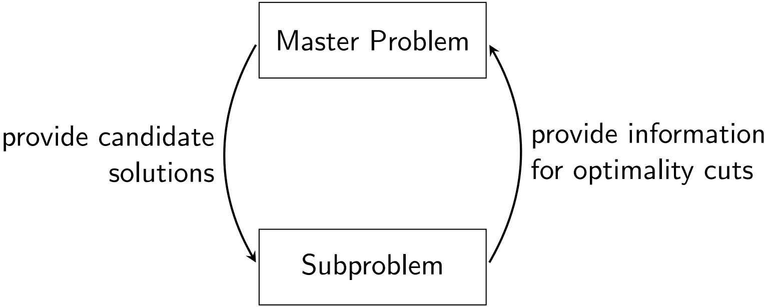
- The relaxation is a large LP.
- If the prices p are fixed, the problem is fully decomposed across n and r.
- Therefore, we consider Benders decomposition.

Benders decomposition

- Complicating variables: p.
- Benders subproblem: for each n and r.

Benders Decomposition

Decomposition scheme



- Iterative procedure.
- Candidate solutions provide upper bounds on the objective.
- Achieved objective values in the Master problem provide lower bounds.

Benders Decomposition

Subproblem(n, r)

$$\min_{p,w,h,\eta} -\frac{1}{R}\sum_{i=1}^{J}\eta_{inr}$$

$$\begin{aligned}
\text{s.t.} \quad & h_{nr} = c_{0nr}w_{0nr} + \sum_{i=1}^{J}\beta_{inr}\eta_{inr} + c_{inr}w_{inr} && \\
& h_{nr} \geq \beta_{inr}p_i + c_{inr}, && i = 1,\ldots,J, \\
& h_{nr} \geq c_{0nr}, && \\
& \eta_{inr} \in \text{McCormick}[p_i, w_{inr}, p_i^L, p_i^U], && i = 1,\ldots,J, \\
& \sum_{i=0}^{J} w_{inr} = 1, && \\
& w_{inr} \geq 0, && \forall i, \\
& p_i = p_i^c \quad (\varphi_{inr}^c) && i = 1,\ldots,J.
\end{aligned}$$

- Computes dual values (φ_i^c) for optimality cuts.

Benders Decomposition

Master problem

$$
\begin{aligned}
\min_{\mathcal{P},p} \quad & -\sum_{nr} \mathcal{P}_{nr} \\
\text{s.t.} \quad & \mathcal{P}_{nr} \leq \mathcal{P}^c_{nr} - \sum_{i=1}^{J} \varphi^c_{inr}(p_i - p^c_i), && \forall c \in \mathcal{C} \quad \forall n, r, \\
& \mathcal{P}_{nr} \leq \sum_{i=1}^{J} \mathrm{V}(\eta_{inr}), && \forall n, r, \\
& \sum_{nr} \mathcal{P}_{nr} \leq \mathcal{P}^{\text{best}}
\end{aligned}
$$

- Computes candidate solutions for the price.
- Fully disaggregated optimality cuts $\mathcal{C}$.
- Includes valid inequalities (V).

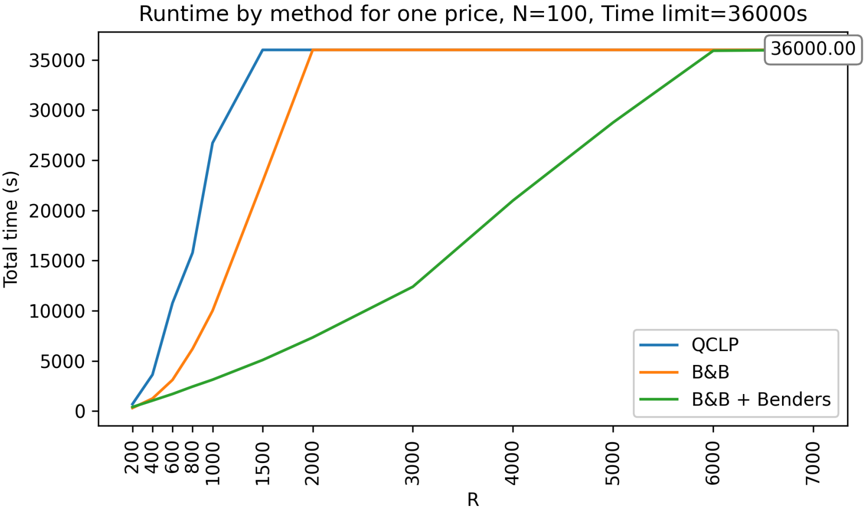
Outline

Case Study

Parking space operator [Ibeas et al., 2014]

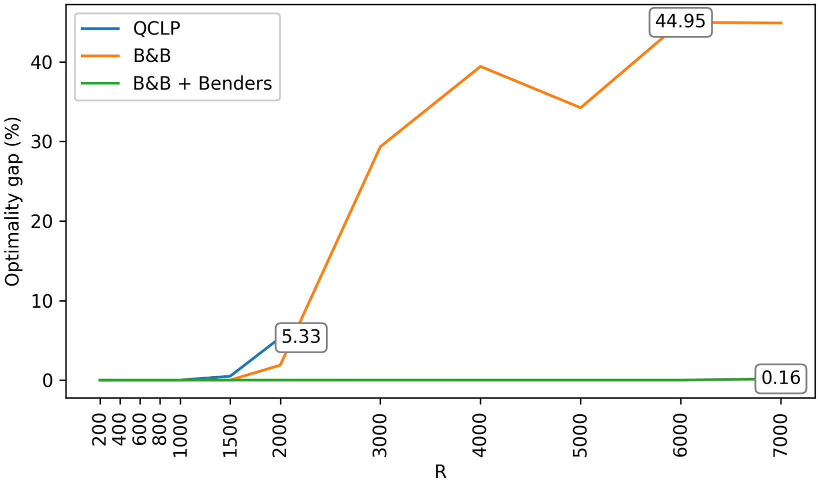
- **Alternatives**: Paid-Street-Parking (PSP), Paid-Underground-Parking (PUP) and Free-Street-Parking (FSP).
- Optimize prices for PSP and PUP, FSP is the **opt-out** alternative.
- **Socio-economic characteristics**: trip origin, vehicle age, driver income, residence area.
- **Product attributes**: access time to parking, access time to destination, and parking fee (price).
- Choice model is a **Mixed Logit**, $\beta_{\text{fee}}, \beta_{\text{time\_parking}} \sim \mathcal{N}(\mu, \sigma)$.

Computational results

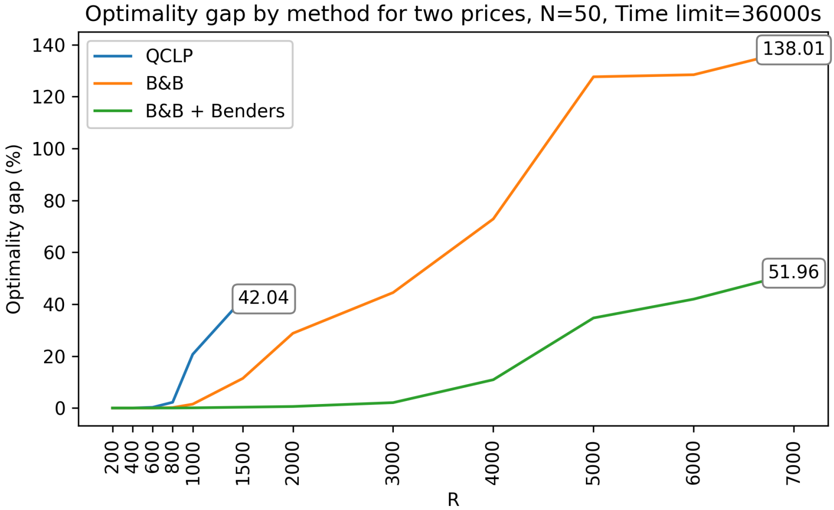


Computational results

Optimality gap by method for one price, N=100, Time limit=36000s



Computational results



Computational results

Table: Solve time (seconds) for single-price optimization (large-scale)

		QCLP		B&B		B&B + Benders	
N	R	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)
100	200	698	0.01	310	0.00	409	0.01
100	400	3629	0.01	1255	0.01	1050	0.01
100	600	10775	0.01	3110	0.01	1707	0.01
100	800	15784	0.01	6206	0.01	2444	0.01
100	1000	26727	0.01	10007	0.01	3131	0.01
100	1500	36000	0.49	22892	0.01	5093	0.01
100	2000	36000	5.33	36000	1.88	7341	0.01
100	3000	36000	-	36000	29.33	12396	0.01
100	4000	36000	-	36000	39.42	20990	0.01
100	5000	36000	-	36000	34.22	28768	0.01
100	6000	36000	-	36000	44.95	35917	0.01
100	7000	36000	-	36000	44.88	36000	0.16

Computational results

Table: Solve time (seconds) for two-price optimization (large-scale)

		QCLP		B&B		B&B + Benders	
N	R	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)
50	200	3338	0.01	2426	0.01	5498	0.01
50	400	23325	0.01	11746	0.01	21838	0.01
50	600	36000	0.26	26662	0.01	35367	0.01
50	800	36000	2.21	36000	0.16	35938	0.01
50	1000	36000	20.68	36000	1.48	36000	0.07
50	1500	36000	42.04	36000	11.41	36000	0.32
50	2000	36000	-	36000	28.79	36000	0.58
50	3000	36000	-	36000	44.48	36000	2.08
50	4000	36000	-	36000	72.86	36000	10.90
50	5000	36000	-	36000	127.64	36000	34.70
50	6000	36000	-	36000	128.44	36000	41.96
50	7000	36000	-	36000	138.01	36000	51.96

Simplifications + Valid inequalities

Table: One-price and two-price optimization runtime (seconds) when using simplifications (S) + valid inequalities (V1 and V2). Time limit = 36000s

N	R	QCLP	B&B	B&BD	B&BD+S	B&BD+S+V1	B&BD+S+V2
100	100	107	29	98	30	33	41
100	500	4739	625	851	252	673	519
100	1000	27586	10007	3387	1865	3329	2388
100	3000	-	25950	5606	3337	5019	3905

N	R	QCLP	B&B	B&BD	B&BD+S	B&BD+S+V1	B&BD+S+V2
50	100	840	660	1925	416	11253	18447
50	500	30600	16826	19904	4686	0.40%	1.01%
50	1000	20.68%	1.59%	0.07%	15066	1.87%	4.68%
50	3000	-	42.88%	2.07%	0.06%	3.54%	8.71%

Conclusions

- A pricing problem involving any choice model with utility linear in price.
- Exploiting the special structure of the problem helps a lot.
- Simplifications and valid inqualities.
- Branch & bound and Benders.
- We solve instances to optimality before GUROBI finds a first feasible solution.
- It is possible to solve problems with a large number of draws.

Appendix

Table: Utility parameters reported in [Ibeas et al., 2014]

Parameter	Value
ASC_{FSP}	0.0
ASC_{PSP}	32.0
ASC_{PUP}	34.0
Fee (€)	$\sim \mathcal{N}(-32.328, 14.168)$
Fee PSP - low income (€)	-10.995
Fee PUP - low income (€)	-13.729
Fee PSP - resident (€)	-11.440
Fee PUP - resident (€)	-10.668
Access time to parking (min)	$\sim \mathcal{N}(-0.788, 1.06)$
Access time to destination (min)	-0.612
Age of vehicle (1/0)	4.037
Origin (1/0)	-5.762

Appendix

Table: Solve time (seconds) for single-price optimization (small-scale)

N	R	MILP	QCQP	QCLP	B&B	B&BD
100	100	2849	392	242	174	216
100	150	7534	1087	708	378	574
100	200	8549	1746	1018	701	603
100	250	25333	2698	1713	1032	1012
100	300	37396	4346	3416	1511	1066
100	350	45362	6715	3927	1795	1169
100	400	65065	8986	5896	2104	1485

Table: Optimal profit and price for single-price optimization (small-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
100	100	54.134	[0.661]	54.134	[0.661]	54.134	[0.661]	54.133	[0.661]	54.133	[0.661]
100	150	54.233	[0.67]	54.233	[0.67]	54.233	[0.67]	54.233	[0.67]	54.232	[0.67]
100	200	54.599	[0.662]	54.599	[0.662]	54.599	[0.662]	54.598	[0.663]	54.596	[0.662]
100	250	54.622	[0.673]	54.622	[0.673]	54.622	[0.673]	54.619	[0.673]	54.618	[0.673]
100	300	54.48	[0.67]	54.48	[0.67]	54.479	[0.67]	54.479	[0.67]	54.478	[0.67]
100	350	54.449	[0.657]	54.448	[0.657]	54.449	[0.657]	54.448	[0.657]	54.447	[0.657]
100	400	54.389	[0.664]	54.389	[0.664]	54.389	[0.664]	54.389	[0.669]	54.388	[0.664]

Appendix

Table: Solve time (seconds) for two-price optimization (small-scale)

		MILP		QCQP	QCLP	B&B	B&BD
N	R	Time	Gap (%)	Time	Time	Time	Time
50	20	1238	0.01	60	32	32	184
50	50	3275	0.01	487	199	201	933
50	80	34907	0.01	1516	564	488	2051
50	100	251466	0.01	2475	843	614	2099
50	150	192213	0.01	2105	2404	1651	5614
50	200	252000	23.92	3023	3384	2438	5402

Appendix

Table: Optimal profit and price for two-price optimization (small-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
50	20	27.417	[0.609, 0.653]	27.417	[0.609, 0.653]	27.417	[0.609, 0.653]	27.416	[0.609, 0.653]	27.414	[0.609, 0.653]
50	50	26.71	[0.556, 0.654]	26.71	[0.556, 0.654]	26.71	[0.556, 0.654]	26.71	[0.556, 0.654]	26.707	[0.556, 0.654]
50	80	27.413	[0.57, 0.648]	27.413	[0.57, 0.648]	27.413	[0.57, 0.648]	27.412	[0.57, 0.648]	27.41	[0.57, 0.648]
50	100	27.546	[0.608, 0.704]	27.546	[0.608, 0.704]	27.546	[0.608, 0.704]	27.544	[0.608, 0.704]	27.544	[0.608, 0.704]
50	150	27.29	[0.562, 0.668]	27.289	[0.562, 0.668]	27.29	[0.562, 0.668]	27.29	[0.562, 0.668]	27.288	[0.562, 0.667]
50	200	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.995	[0.546, 0.679]	26.996	[0.546, 0.679]

Appendix

Table: Solve time (seconds) for single-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)
100	200	8348	0.01	1059	0.01	698	0.01	310	0.00	409	0.01
100	400	36000	20.39	5013	0.01	3629	0.01	1255	0.01	1050	0.01
100	600	36000	27.0	14796	0.01	10775	0.01	3110	0.01	1707	0.01
100	800	36000	113.12	21626	0.01	15784	0.01	6206	0.01	2444	0.01
100	1000	36000	122.21	36000	0.04	26727	0.01	10007	0.01	3131	0.01
100	1500	36000	121.82	36000	16.69	36000	0.49	22892	0.01	5093	0.01
100	2000	36000	124.91	36000	300.05	36000	5.33	36000	1.88	7341	0.01
100	3000	36000	125.44	36000	-	36000	-	36000	29.33	12396	0.01
100	4000	36000	149.07	36000	-	36000	-	36000	39.42	20990	0.01
100	5000	36000	-	36000	-	36000	-	36000	34.22	28768	0.01
100	6000	36000	-	36000	-	36000	-	36000	44.95	35917	0.01
100	7000	36000	-	36000	-	36000	-	36000	44.88	36000	0.16

Appendix

Table: Optimal profit and price for single-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
100	200	54.599	[0.662]	54.599	[0.662]	54.599	[0.662]	54.598	[0.663]	54.596	[0.662]
100	400	54.385	[0.664]	54.389	[0.664]	54.389	[0.664]	54.389	[0.669]	54.388	[0.664]
100	600	54.019	[0.625]	54.295	[0.667]	54.295	[0.667]	54.295	[0.667]	54.294	[0.667]
100	800	54.319	[0.662]	54.327	[0.653]	54.326	[0.653]	54.325	[0.653]	54.326	[0.653]
100	1000	54.421	[0.663]	54.429	[0.661]	54.429	[0.661]	54.429	[0.661]	54.429	[0.661]
100	1500	54.488	[0.67]	49.33	[0.971]	54.514	[0.654]	54.53	[0.659]	54.529	[0.659]
100	2000	54.511	[0.656]	22.469	[1.379]	53.966	[0.613]	54.54	[0.667]	54.541	[0.666]
100	3000	54.439	[0.664]	-	-	-	-	52.387	[0.801]	54.448	[0.661]
100	4000	54.422	[0.668]	-	-	-	-	51.175	[0.856]	54.428	[0.669]
100	5000	-	-	-	-	-	-	53.144	[0.764]	54.394	[0.661]
100	6000	-	-	-	-	-	-	49.207	[0.971]	54.399	[0.663]
100	7000	-	-	-	-	-	-	49.229	[0.97]	54.41	[0.669]

Appendix

Table: Solve time (seconds) for two-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)	Time	Gap (%)
50	200	36000	39.22	3098	0.01	3338	0.01	2426	0.01	5498	0.01
50	400	36000	100.58	17774	0.01	23325	0.01	11746	0.01	21838	0.01
50	600	36000	217.26	36000	0.18	36000	0.26	26662	0.01	35367	0.01
50	800	36000	138.07	36000	1.75	36000	2.21	36000	0.16	35938	0.01
50	1000	36000	185.45	36000	9.52	36000	20.68	36000	1.48	36000	0.07
50	1500	36000	345.36	36000	42.8	36000	42.04	36000	11.41	36000	0.32
50	2000	36000	393.41	36000	258.89	36000	-	36000	28.79	36000	0.58
50	3000	36000	-	36000	263.73	36000	-	36000	44.48	36000	2.08
50	4000	36000	-	36000	280.59	36000	-	36000	72.86	36000	10.90
50	5000	36000	-	36000	-	36000	-	36000	127.64	36000	34.70
50	6000	36000	-	36000	-	36000	-	36000	128.44	36000	41.96
50	7000	36000	-	36000	-	36000	-	36000	138.01	36000	51.96

Table: Optimal profit and price for two-price optimization (large-scale)

		MILP		QCQP		QCLP		B&B		B&BD	
N	R	Profit	Price	Profit	Price	Profit	Price	Profit	Price	Profit	Price
50	200	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.997	[0.546, 0.679]	26.995	[0.546, 0.679]	26.996	[0.546, 0.679]
50	400	21.689	[0.789, 0.956]	27.174	[0.556, 0.665]	27.174	[0.556, 0.665]	27.172	[0.556, 0.665]	27.172	[0.556, 0.665]
50	600	13.801	[1.087, 1.281]	27.243	[0.561, 0.682]	27.245	[0.563, 0.671]	27.246	[0.562, 0.683]	27.246	[0.562, 0.683]
50	800	16.993	[0.877, 1.203]	27.072	[0.578, 0.668]	27.06	[0.559, 0.659]	27.082	[0.573, 0.667]	27.089	[0.574, 0.667]
50	1000	13.987	[1.2, 1.212]	26.968	[0.59, 0.684]	26.208	[0.584, 0.797]	27.012	[0.571, 0.667]	27.031	[0.573, 0.67]
50	1500	10.144	[1.415, 1.485]	26.319	[0.584, 0.799]	26.322	[0.584, 0.799]	26.982	[0.582, 0.698]	27.052	[0.569, 0.667]
50	2000	9.255	[1.239, 1.866]	11.82	[1.199, 1.395]	-	-	26.718	[0.632, 0.712]	27.094	[0.565, 0.661]
50	3000	-	-	11.849	[1.198, 1.397]	-	-	25.983	[0.5, 0.756]	27.144	[0.571, 0.677]
50	4000	-	-	11.844	[1.199, 1.396]	-	-	24.707	[1.242, 0.766]	27.078	[0.582, 0.699]
50	5000	-	-	-	-	-	-	18.988	[1.0, 1.0]	26.012	[0.5, 0.755]
50	6000	-	-	-	-	-	-	18.915	[1.0, 1.0]	25.973	[0.5, 0.757]
50	7000	-	-	-	-	-	-	18.926	[1.0, 1.0]	24.681	[1.231, 0.766]

Bibliography I

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