Human Behavior and Optimization

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Outline

1. Introduction
2. Microeconomics
3. The logit model
4. Profit optimization, facility location
5. Activity-based models
6. Conclusion
Motivation

Human dimension in
- engineering
- business
- marketing
- planning
- policy making

Need for
- behavioral *theories*
- quantitative *methods*
- operational mathematical *models*
Motivation

Concept of demand
- marketing
- transportation
- energy
- finance

Concept of choice
- brand, product
- mode, destination
- type, usage
- buy/sell, product
In this lecture...

Focus

- Individual behavior (vs. aggregate behavior)
- Theory of behavior which is
  - descriptive (how people behave) and not normative (how they should behave)
  - general: not too specific
  - operational: can be used in practice for forecasting
- Type of behavior: choice
In this lecture...

Microeconomics in a nutshell
In this lecture...

Microeconomics in a nutshell  →  Logit and MEV models
In this lecture...

- Microeconomics in a nutshell
- Logit and MEV models
- Profit maximization, facility location
In this lecture...

- Microeconomics in a nutshell
- Profit maximization, facility location
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- Activity-based models
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- Optimization
  - Microeconomics in a nutshell
  - Profit maximization, facility location
  - Logit and MEV models
  - Activity-based models
In this lecture...

- **Optimization**
  - Microeconomics in a nutshell
  - Profit maximization, facility location

- **Discrete choice**
  - Logit and MEV models
  - Activity-based models
In this lecture...

Tutorial

- Optimization
  - Microeconomics in a nutshell
  - Profit maximization, facility location

- Discrete choice
  - Logit and MEV models
  - Activity-based models

Introduction
In this lecture...

- **Tutorial**
  - Microeconomics in a nutshell

- **Research**
  - Profit maximization, facility location

- **Optimization**
- **Discrete choice**
  - Logit and MEV models
  - Activity-based models
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**Decision rule**

**Homo economicus**
Rational and narrowly self-interested economic actor who is optimizing her outcome

**Utility**

\[ U_n : C_n \longrightarrow \mathbb{R} : a \mapsto U_n(a) \]

- captures the attractiveness of an alternative
- measure that the decision maker wants to optimize

**Behavioral assumption**
- the decision maker associates a utility with each alternative
- the decision maker is a perfect optimizer
- the alternative with the highest utility is chosen
Microeconomic consumer theory

Continuous choice set

- Consumption bundle

\[ Q = \begin{pmatrix} q_1 \\ \vdots \\ q_L \end{pmatrix}; \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_L \end{pmatrix} \]

- Budget constraint

\[ p^T Q = \sum_{\ell=1}^{L} p_\ell q_\ell \leq I. \]

- No attributes, just quantities
Preferences

Operators $\succ$, $\sim$, and $\succeq$

- $Q_a \succ Q_b$: $Q_a$ is preferred to $Q_b$,
- $Q_a \sim Q_b$: indifference between $Q_a$ and $Q_b$,
- $Q_a \succeq Q_b$: $Q_a$ is at least as preferred as $Q_b$.

Rationality

- Completeness: for all bundles $a$ and $b$,
  \[ Q_a \succ Q_b \text{ or } Q_a \prec Q_b \text{ or } Q_a \sim Q_b. \]
- Transitivity: for all bundles $a$, $b$ and $c$,
  \[ \text{if } Q_a \succeq Q_b \text{ and } Q_b \succeq Q_c \text{ then } Q_a \succeq Q_c. \]
- “Continuity”: if $Q_a$ is preferred to $Q_b$ and $Q_c$ is arbitrarily “close” to $Q_a$, then $Q_c$ is preferred to $Q_b$. 

Utility function

- Parameterized function:

\[ \tilde{U} = \tilde{U}(q_1, \ldots, q_L; \theta) = \tilde{U}(Q; \theta) \]

- Consistent with the preference indicator:

\[ \tilde{U}(Q_a; \theta) \geq \tilde{U}(Q_b; \theta) \]

is equivalent to

\[ Q_a \succeq Q_b. \]

- Unique up to an order-preserving transformation
Optimization problem

\[
\max_Q \tilde{U}(Q; \theta)
\]
subject to

\[
p^T Q \leq I, \quad Q \geq 0.
\]

Demand function

- Solution of the optimization problem
- Quantity as a function of prices and budget

\[
Q^* = f(I, p; \theta)
\]
Example: Cobb-Douglas

\[ \tilde{U}(q_1, q_2) = \theta_0 q_1^{\theta_1} q_2^{\theta_2} \]
Example
Example

Optimization problem

\[
\max_{q_1, q_2} \tilde{U}(q_1, q_2; \theta_0, \theta_1, \theta_2) = \theta_0 q_1^{\theta_1} q_2^{\theta_2}
\]

subject to

\[
p_1 q_1 + p_2 q_2 = I.
\]

Lagrangian of the problem:

\[
L(q_1, q_2, \lambda) = \theta_0 q_1^{\theta_1} q_2^{\theta_2} + \lambda(I - p_1 q_1 - p_2 q_2).
\]

Necessary optimality condition

\[
\nabla L(q_1, q_2, \lambda) = 0
\]
Example

Necessary optimality conditions

\[ \theta_0 \theta_1 q_1^{\theta_1 - 1} q_2^{\theta_2} - \lambda p_1 = 0 \quad (\times q_1) \]
\[ \theta_0 \theta_2 q_1^{\theta_1} q_2^{\theta_2 - 1} - \lambda p_2 = 0 \quad (\times q_2) \]
\[ p_1 q_1 + p_2 q_2 - I = 0. \]

We have

\[ \theta_0 \theta_1 q_1^{\theta_1} q_2^{\theta_2} - \lambda p_1 q_1 = 0 \]
\[ \theta_0 \theta_2 q_1^{\theta_1} q_2^{\theta_2} - \lambda p_2 q_2 = 0. \]

Adding the two and using the third condition, we obtain

\[ \lambda I = \theta_0 q_1^{\theta_1} q_2^{\theta_2} (\theta_1 + \theta_2) \]

or, equivalently,

\[ \theta_0 q_1^{\theta_1} q_2^{\theta_2} = \frac{\lambda I}{(\theta_1 + \theta_2)} \]
Solution

From the previous derivation

\[ \theta_0 q_1^{\theta_1} q_2^{\theta_2} = \frac{\lambda I}{(\theta_1 + \theta_2)} \]

First condition

\[ \theta_0 \theta_1 q_1^{\theta_1} q_2^{\theta_2} = \lambda p_1 q_1. \]

Solve for \( q_1 \)

\[ q_1^* = \frac{I \theta_1}{p_1(\theta_1 + \theta_2)} \]

Similarly, we obtain

\[ q_2^* = \frac{I \theta_2}{p_2(\theta_1 + \theta_2)} \]
Optimization problem

\[ \begin{align*}
q_1^* & \quad q_2^* \\
\frac{l}{p_2} & \quad \frac{l}{p_1}
\end{align*} \]

Income constraint
Demand functions

Product 1

\[ q_1^* = \frac{l}{p_1} \frac{\theta_1}{\theta_1 + \theta_2} \]

Product 2

\[ q_2^* = \frac{l}{p_2} \frac{\theta_2}{\theta_1 + \theta_2} \]

Comments

- Demand decreases with price
- Demand increases with budget
- Demand independent of \( \theta_0 \), which does not affect the ranking
- Property of Cobb Douglas: the demand for a good is only dependent on its own price and independent of the price of any other good.
Demand curve (inverse of demand function)

- Good 1, Low income (1000)
- Good 1, High income (10000)
- Good 2, Low income (1000)
- Good 2, High income (10000)
Indirect utility

Substitute the demand function into the utility

\[ U(I, p; \theta) = \theta_0 \left( \frac{I}{p_1 \frac{\theta_1}{\theta_1 + \theta_2}} \right)^{\theta_1} \left( \frac{I}{p_2 \frac{\theta_2}{\theta_1 + \theta_2}} \right)^{\theta_2} \]

Indirect utility

Maximum utility that is achievable for a given set of prices and income

In discrete choice...
- only the indirect utility is used
- therefore, it is simply referred to as “utility”
Microeconomic theory of discrete goods

Expanding the microeconomic framework

- Continuous goods
- and discrete goods

The consumer

- selects the quantities of continuous goods: \( Q = (q_1, \ldots, q_L) \)
- chooses an alternative in a discrete choice set \( i = 1, \ldots, j, \ldots, J \)
- discrete decision vector: \( (y_1, \ldots, y_J) \), \( y_j \in \{0, 1\} \), \( \sum_j y_j = 1 \).

Note

- In theory, one alternative of the discrete choice combines all possible choices made by an individual.
- In practice, the choice set will be more restricted for tractability.
Utility maximization

Utility

\[ \tilde{U}(Q, y, \tilde{z}^T y; \theta) \]

- **Q**: quantities of the continuous good
- **y**: discrete choice
- \( \tilde{z}^T = (\tilde{z}_1, \ldots, \tilde{z}_i, \ldots, \tilde{z}_J) \in \mathbb{R}^{K \times J} \): \( K \) attributes of the \( J \) alternatives
- \( \tilde{z}^T y \in \mathbb{R}^K \): attributes of the chosen alternative
- **\( \theta \)**: vector of parameters
Utility maximization

Optimization problem

$$\max_{Q,y} \tilde{U}(Q, y, \tilde{z}^T y; \theta)$$

subject to

$$p^T Q + c^T y \leq I$$
$$\sum_j y_j = 1$$
$$y_j \in \{0, 1\}, \forall j.$$ 

where $c^T = (c_1, \ldots, c_i, \ldots, c_J)$ contains the cost of each alternative.

Solving the problem

- Mixed integer optimization problem
- No optimality condition
- Impossible to derive demand functions directly
Solving the problem

Step 1: condition on the choice of the discrete good

- Fix the discrete good, that is select a feasible $y$.
- The problem becomes a continuous problem in $Q$.
- Conditional demand functions can be derived:

$$q_{\ell|y} = f(I - c^Ty, p, \tilde{z}^Ty; \theta),$$

or, equivalently, for each alternative $i$,

$$q_{\ell|i} = f(I - c_i, p, \tilde{z}_i; \theta).$$

- $I - c_i$ is the income left for the continuous goods, if alternative $i$ is chosen.
- If $I - c_i < 0$, alternative $i$ is declared unavailable and removed from the choice set.
Solving the problem

Conditional indirect utility functions
Substitute the demand functions into the utility:

\[ U_i = U(I - c_i, p, \tilde{z}_i; \theta) \]
for all \( i \in C \).

Step 2: Choice of the discrete good

\[ \max_y U(I - c^T y, p, \tilde{z}^T y; \theta) \]

- Enumerate all alternatives.
- Compute the conditional indirect utility function \( U_i \).
- Select the alternative with the highest \( U_i \).
- Note: no income constraint anymore.
### Simple example: mode choice

#### Attributes

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Attributes</th>
<th>Attributes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car (1)</td>
<td>$t_1$</td>
<td>$c_1$</td>
</tr>
<tr>
<td>Bus (2)</td>
<td>$t_2$</td>
<td>$c_2$</td>
</tr>
</tbody>
</table>

#### Utility

\[ \tilde{U} = \tilde{U}(y_1, y_2), \]

where we impose the restrictions that, for $i = 1, 2$,

\[ y_i = \begin{cases} 1 & \text{if travel alternative } i \text{ is chosen,} \\ 0 & \text{otherwise;} \end{cases} \]

and that only one alternative is chosen: $y_1 + y_2 = 1$. 
Simple example: mode choice

Choice set

\[
\begin{array}{c}
\bullet (0, 1) \\
(1, 0)
\end{array}
\]
Simple example: mode choice

Utility functions

\[ U_1 = -\beta_t t_1 - \beta_c c_1, \]
\[ U_2 = -\beta_t t_2 - \beta_c c_2, \]

where \( \beta_t > 0 \) and \( \beta_c > 0 \) are parameters.

Equivalent specification

\[ U_1 = -\left(\frac{\beta_t}{\beta_c}\right) t_1 - c_1 = -\beta t_1 - c_1 \]
\[ U_2 = -\left(\frac{\beta_t}{\beta_c}\right) t_2 - c_2 = -\beta t_2 - c_2 \]

where \( \beta > 0 \) is a parameter.

Choice

- Alternative 1 is chosen if \( U_1 \geq U_2 \).
- Ties are ignored.
Simple example: mode choice

Choice

Alternative 1 is chosen if

\[-\beta t_1 - c_1 \geq -\beta t_2 - c_2\]

or

\[-\beta(t_1 - t_2) \geq c_1 - c_2\]

Alternative 2 is chosen if

\[-\beta t_1 - c_1 \leq -\beta t_2 - c_2\]

or

\[-\beta(t_1 - t_2) \leq c_1 - c_2\]

Dominated alternative

- If \(c_2 > c_1\) and \(t_2 > t_1\), \(U_1 > U_2\) for any \(\beta > 0\)
- If \(c_1 > c_2\) and \(t_1 > t_2\), \(U_2 > U_1\) for any \(\beta > 0\)
Simple example: mode choice

Trade-off
- Assume $c_2 > c_1$ and $t_1 > t_2$.
- Is the traveler willing to pay the extra cost $c_2 - c_1$ to save the extra time $t_1 - t_2$?
- Alternative 2 is chosen if

$$-\beta(t_1 - t_2) \leq c_1 - c_2$$

or

$$\beta \geq \frac{c_2 - c_1}{t_1 - t_2}$$

- $\beta$ is called the *willingness to pay* or *value of time*
Simple example: mode choice

\[ c_1 \beta t_1 + c_1 = c_2 + \beta t_2 \]

Alt. 1 is chosen
Alt. 2 is chosen

Alt. 1 is preferred
Alt. 2 is preferred

Alt. 1 is dominant
Alt. 2 is dominant
Behavioral validity of the utility maximization?

Assumptions

Decision-makers

- are able to process information
- have perfect discrimination power
- have transitive preferences
- are perfect maximizer
- are always consistent

Relax the assumptions

Use a probabilistic approach: what is the probability that alternative $i$ is chosen?
Random utility model

Probability model

\[ P(i|C_n) = \Pr(U_{in} \geq U_{jn}, \forall j \in C_n), \]

Random utility

\[ U_{in} = V_{in} + \varepsilon_{in} = \beta^T X_{in} + \varepsilon_{in}. \]

Similarity with linear regression

\[ Y = \beta^T X + \varepsilon \]

Here, \( U \) is not observed. Only the choice is observed.
Derivation

Joint distributions of $\varepsilon_n$

- Assume that $\varepsilon_n = (\varepsilon_{1n}, \ldots, \varepsilon_{Jn})$ is a multivariate random variable
- with CDF $F_{\varepsilon_n}(\varepsilon_1, \ldots, \varepsilon_{Jn})$
- and pdf $f_{\varepsilon_n}(\varepsilon_1, \ldots, \varepsilon_{Jn}) = \frac{\partial^{Jn} F}{\partial \varepsilon_1 \cdots \partial \varepsilon_{Jn}}(\varepsilon_1, \ldots, \varepsilon_{Jn})$.

The random utility model: $P_n(i|C_n) =$

$$\int_{\varepsilon=-\infty}^{+\infty} \frac{\partial F_{\varepsilon_{1n}, \varepsilon_{2n}, \ldots, \varepsilon_{Jn}}}{\partial \varepsilon_i}(\ldots, V_{in} - V_{(i-1)n} + \varepsilon, \varepsilon, V_{in} - V_{(i+1)n} + \varepsilon, \ldots) \, d\varepsilon$$
Random utility model

- The general formulation is complex.
- We can derive specific models based on simple assumptions.
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Road map

- Optimization
  - Microeconomics in a nutshell
- Discrete choice
  - Logit and MEV models
Error term

Random utility

\[ U_{in} = V_{in} + \varepsilon_{in}. \]

Assumptions about the distribution

- **Probit**: central limit theorem: the sum of many i.i.d. random variables approximately follows a normal distribution.
- **Logit**: Gumbel theorem: the maximum of many i.i.d. random variables approximately follows an Extreme Value distribution: \( \text{EV}(\eta, \mu) \).
The Extreme Value distribution $\text{EV}(\eta, \mu)$

**Probability density function (pdf)**

$$f(t) = \mu e^{-\mu(t-\eta)} e^{-e^{-\mu(t-\eta)}}$$

**Cumulative distribution function (CDF)**

$$P(c \geq \varepsilon) = F(c) = \int_{-\infty}^{c} f(t) dt = e^{-e^{-\mu(c-\eta)}}$$
The Extreme Value distribution

pdf EV(0,1)

CDF EV(0,1)
Properties

If

\[ \varepsilon \sim \text{EV}(\eta, \mu) \]

then

\[
\mathbb{E}[\varepsilon] = \eta + \frac{\gamma}{\mu} \quad \text{and} \quad \text{Var}[\varepsilon] = \frac{\pi^2}{6\mu^2}
\]

where \( \gamma \) is Euler’s constant.

Euler’s constant

\[
\gamma = \lim_{k \to \infty} \sum_{i=1}^{k} \frac{1}{i} - \ln k = -\int_{0}^{\infty} e^{-x} \ln x \, dx \approx 0.5772
\]
The logit model

The random utility model: \( P_n(i|C_n) = \)

\[
\int_{\varepsilon = -\infty}^{+\infty} \frac{\partial F_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_J}}{\partial \varepsilon_i} \left( \ldots, V_{in} - V_{i-1}n + \varepsilon, \varepsilon, V_{in} - V_{i+1}n + \varepsilon, \ldots \right) d\varepsilon
\]

CDF

Assumption: i.i.d. EV distributions:

\[
F_{\varepsilon_n}(\varepsilon_1, \ldots, \varepsilon_J) = \prod_{i=1}^{J_n} e^{-e^{-\mu \varepsilon_i}}.
\]

Logit model

\[
P_n(i|C_n) = \frac{y_{in}e^{V_{in}}}{\sum_{j=1}^{J} y_{jn}e^{V_{jn}}}.\]
Logit model

\[ U_{in} = V_{in} + \varepsilon_{in}. \]

**Why “logit”?**

If \( U_{in} \) and \( U_{jn} \) are EV distributed, \( U_{in} - U_{jn} \) follows a logistic distribution.

**Availability of alternatives**

\[ y_{in} = \begin{cases} 1 & \text{if } i \in C_n, \\ 0 & \text{otherwise}. \end{cases} \]

\( y_{in=1} \) if alternative \( i \) is available to individual \( n \).
Expected maximum utility

If $\varepsilon_{in}, i = 1, \ldots, J_n$ are i.i.d. $EV(0, \mu)$, then

$$E[\max_i U_{in}] = \frac{1}{\mu} \ln \sum_{i=1}^{J} y_{jn} e^{\mu V_{in}}.$$
The logit model

Example

Two alternatives

\[ V_{0n} = 0 \]
\[ V_{1n} = -10 \times \text{price} + 3 \]

Choice probability

\[
P_n(1|\text{price}) = \frac{e^{-10\times\text{price}+3}}{e^0 + e^{-10\times\text{price}+3}} = \frac{e^{-10\times\text{price}+3}}{1 + e^{-10\times\text{price}+3}}
\]
Example

The logit model

Choice probability

$P_n(1|\text{price})$

Price

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Beyond logit

- Other distributional assumptions can be used.
- Logit is not always consistent with observed behavior.
- Trade-off between model complexity and behavioral realism.
- Example: Multivariate Extreme Value models
The logit model

MEV models

Definition

\( \varepsilon_n = (\varepsilon_{1n}, \ldots, \varepsilon_{Jn}) \)

follows a multivariate extreme value distribution if it has the CDF

\[
F_{\varepsilon_n}(\varepsilon_{1n}, \ldots, \varepsilon_{Jn}) = e^{-G(e^{-\varepsilon_{1n}}, \ldots, e^{-\varepsilon_{Jn}})},
\]

where \( G : \mathbb{R}_+^{Jn} \to \mathbb{R}_+ \) is a positive function with positive arguments, that must verify some properties.
The logit model

MEV models

Choice model

\[ P_n(i) = \frac{e^{V_{in} + \ln G_i(e^V)}}{\sum_j e^{V_{jn} + \ln G_j(e^V)}}. \]

Expected maximum utility

\[ E[\max_{j \in C_n} U_{jn}] = \frac{1}{\mu} (\log G(e^{V_{1n}}, \ldots, e^{V_{Jnn}}) + \gamma), \]

where \( \gamma \) is Euler’s constant.
Road map

Optimization

Microeconomics in a nutshell

Discrete choice

Logit and MEV models

Profit maximization, facility location
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A simple example

Data
- \( C \): set of movies
- Population of \( N \) individuals
- Utility function:
  \[ U_{in} = \beta_{in} p_{in} + f(z_{in}) + \varepsilon_{in} \]

Decision variables
- What movies to propose? \( y_{in} \)
- What price? \( p_{in} \)
Profit maximization

Data

- Two alternatives: my theater \((m)\) and the competition \((c)\)
- We assume an homogeneous population of \(N\) individuals

\[
\begin{align*}
U_{cn} &= 0 + \varepsilon_{cn} \\
U_{mn} &= \beta_n p_m + c_{mn} + \varepsilon_{mn}
\end{align*}
\]

- \(\beta_n < 0\)
- Logit model: \(\varepsilon_{mn}\) i.i.d. EV
Heterogeneous population

Two groups in the population

\[ U_{mn} = \beta_n p_m + c_{mn} + \varepsilon_{mn} \]

\( n = 1 \): Young fans:
2/3
\( \beta_1 = -10, \ c_{1m} = 3 \)

\( n = 2 \): Others:
1/3
\( \beta_1 = -0.9, \ c_{1m} = 0 \)
Demand

- Total demand
- Young fans
- Others

Demand vs. Price graph

- X-axis: Price (0 to 2)
- Y-axis: Demand (0 to 1)

Legend:
- Purple line: Total demand
- Green line: Young fans
- Blue line: Others
Demand and revenues

![Graph showing demand and revenues over price]

- **Revenues**
- **Total demand**
- **Young fans**
- **Others**

<table>
<thead>
<tr>
<th>Price</th>
<th>Demand</th>
<th>Revenues</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.45</td>
<td>0.4</td>
</tr>
<tr>
<td>0.5</td>
<td>0.35</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.2</td>
</tr>
<tr>
<td>1.5</td>
<td>0.15</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>
Optimization

Profit maximization
- Non linear
- Non convex

Facility location
- Discrete
The main idea
The main idea

Linearization

- Hopeless to linearize the logit formula (we tried...)
- Anyway, we want to go beyond logit.
The main idea

Linearization

- Hopeless to linearize the logit formula (we tried...)
- Anyway, we want to go beyond logit.

First principles

Each customer solves an optimization problem
The main idea

Linearization
- Hopeless to linearize the logit formula (we tried...)
- Anyway, we want to go beyond logit.

First principles
Each customer solves an optimization problem

Solution
Use the utility and not the probability
A linear formulation

Utility function

\[ U_{in} = V_{in} + \varepsilon_{in} = \sum_{k} \beta_k x_{ink} + f(z_{in}) + \varepsilon_{in}. \]

Simulation

- Assume a distribution for \( \varepsilon_{in} \)
- E.g. logit: i.i.d. extreme value
- Draw \( R \) realizations \( \xi_{inr}, r = 1, \ldots, R \)
- The choice problem becomes deterministic
Draws

- Draw $R$ realizations $\xi_{inr}$, $r = 1, \ldots, R$
- We obtain $R$ scenarios

$$U_{inr} = \sum_k \beta_k x_{ink} + f(z_{in}) + \xi_{inr}.$$ 

- For each scenario $r$, we can identify the largest utility.
- It corresponds to the chosen alternative.
Capacities

- Demand may exceed supply
- Each alternative $i$ can be chosen by maximum $c_i$ individuals.
- An exogenous priority list is available.
- Can be randomly generated, or according to some rules.
- The numbering of individuals is consistent with their priority.
Choice set

Variables

\[ y_i \in \{0, 1\} \quad \text{operator decision} \]
\[ y_{in}^d \in \{0, 1\} \quad \text{customer decision (data)} \]
\[ y_{in} \in \{0, 1\} \quad \text{product of decisions} \]
\[ y_{inr} \in \{0, 1\} \quad \text{capacity restrictions} \]

Constraints

\[ y_{in} = y_{in}^d y_i \quad \forall i, n \]
\[ y_{inr} \leq y_{in} \quad \forall i, n, r \]
Utility

Variables

\[ U_{inr} \]

\[ z_{inr} = \begin{cases} 
U_{inr} & \text{if } y_{inr} = 1 \\
\ell_{nr} & \text{if } y_{inr} = 0 
\end{cases} \]

(\( \ell_{nr} \) smallest lower bound)

Constraint: utility

\[ U_{inr} = \beta_{in} p_{in} + q_{d}(x_{d}) + \xi_{inr} \quad \forall i, n, r \]
Utility (ctd)

Constraints: discounted utility

\[ \ell_{inr} \leq z_{inr} \quad \forall i, n, r \]
\[ z_{inr} \leq \ell_{nr} + M_{inr} y_{inr} \quad \forall i, n, r \]
\[ U_{inr} - M_{inr} (1 - y_{inr}) \leq z_{inr} \quad \forall i, n, r \]
\[ z_{inr} \leq U_{inr} \quad \forall i, n, r \]
Profit optimization, facility location

Choice

Variables

\[ U_{nr} = \max_{i \in C} z_{inr} \]

\[ w_{inr} = \begin{cases} 1 & \text{if } z_{inr} = U_{nr} \\ 0 & \text{otherwise} \end{cases} \]

Constraints

\[ z_{inr} \leq U_{nr} \quad \forall i, n, r \]

\[ U_{nr} \leq z_{inr} + M_{nr}(1 - w_{inr}) \quad \forall i, n, r \]

\[ \sum_{i} w_{inr} = 1 \quad \forall n, r \]

\[ w_{inr} \leq y_{inr} \quad \forall i, n, r \]
Profit optimization, facility location

Capacity

Capacity cannot be exceeded ⇒ $y_{inr} = 1$

\[
\sum_{m=1}^{n-1} w_{imr} \leq (c_i - 1)y_{inr} + (n - 1)(1 - y_{inr}) \quad \forall i > 0, n > c_i, r
\]

Capacity has been reached ⇒ $y_{inr} = 0$

\[
c_i(y_{in} - y_{inr}) \leq \sum_{m=1}^{n-1} w_{imr}, \quad \forall i > 0, n, r
\]
Family of models

Constraints
- Set of linear constraints characterizing choice behavior
- Can be included in any relevant optimization problem.

Examples
- Profit maximization
- Facility location

Difficulties
- big $M$ constraints
- large dimensions
Profit maximization

Profit

If $p_{in}$ is the price paid by individual to purchase option $i$, the revenue generated by this option is

$$\frac{1}{R} \sum_{r=1}^{R} \sum_{n=1}^{N} p_{in} w_{inr}.$$  

Linearization

If $a_{in} \leq p_{in} \leq b_{in}$, we define $\eta_{inr} = p_{in} w_{inr}$, and the following constraints:

$$a_{in} w_{inr} \leq \eta_{inr}$$

$$\eta_{inr} \leq b_{in} w_{inr}$$

$$p_{in} - (1 - w_{inr}) b_{in} \leq \eta_{inr}$$

$$\eta_{inr} \leq p_{in} - (1 - w_{inr}) a_{in}$$
A case study

Challenge

- Take a choice model from the literature.
- It cannot be logit.
- It must involve heterogeneity.
- Show that it can be integrated in a relevant MILP.
A case study

Challenge

- Take a choice model from the literature.
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Parking choice

- [Ibeas et al., 2014]
Parking choices [Ibeas et al., 2014]

Alternatives
- Paid on-street parking
- Paid underground parking
- Free street parking

Model
- \( N = 50 \) customers
- \( C = \{ \text{PSP, PUP, FSP} \} \)
- \( C_n = C \quad \forall n \)
- \( p_{in} = p_i \quad \forall n \)
- Capacity of 20 spots
- Mixture of logit models
General experiments

Uncapacitated vs Capacitated case
- Maximization of revenue
- Unlimited capacity
- Capacity of 20 spots for PSP and PUP

Price differentiation by population segmentation
- Reduced price for residents
- Two scenarios
  1. Subsidy offered by the municipality
  2. Operator is forced to offer a reduced price
Uncapacitated vs Capacitated case

Uncapacitated

Capacitated
## Computational time

<table>
<thead>
<tr>
<th>$R$</th>
<th>Uncapacitated case</th>
<th>Capacitated case</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sol time</td>
<td>PSP</td>
</tr>
<tr>
<td>5</td>
<td>2.58 s</td>
<td>0.54</td>
</tr>
<tr>
<td>10</td>
<td>3.98 s</td>
<td>0.53</td>
</tr>
<tr>
<td>25</td>
<td>29.2 s</td>
<td>0.54</td>
</tr>
<tr>
<td>50</td>
<td>4.08 min</td>
<td>0.54</td>
</tr>
<tr>
<td>100</td>
<td>20.7 min</td>
<td>0.54</td>
</tr>
<tr>
<td>250</td>
<td>2.51 h</td>
<td>0.54</td>
</tr>
</tbody>
</table>
Facility location

Data
- \( U_{in} \): exogenous,
- \( C_i \): fixed cost to open a facility,
- \( c_i \): operational cost per customer to run the facility.

Objective function

\[
\min \sum_{i \in C_k} C_i y_i + \frac{1}{R} \sum_r \sum_i \sum_n c_i w_{inr}
\]
Benders decomposition

\[
\min \sum_{i \in C_k} C_i y_i + \frac{1}{R} \sum_r \sum_i \sum_n c_i w_{inr}
\]

subject to

\[
\max_w U_{nr} = \sum_i U_{inr} w_{inr}
\]

\[
\sum_i w_{inr} \leq 1
\]

\[
w_{inr} \leq y_i
\]

\[
w_{inr} \geq 0
\]

\[
w_{inr}, y_i \in \{0, 1\}.
\]
Benders decomposition

Customer subproblem: fix $y_i^*$

$$\max_w U_{nr} = \sum_i U_{inr} w_{inr}$$

subject to

$$\sum_i w_{inr} = 1$$

$$w_{inr} \leq y_i^*$$

$$w_{inr} \geq 0.$$
Benders decomposition

Primal

\[ \min_w U = - \sum_i U_i w_i \]
subject to
\[ \sum_i w_i = 1 \]
\[ w_i \leq y_i^* \quad \forall i \]
\[ w_i \geq 0. \]

Dual

\[ \max_{\lambda, \mu} \lambda + \sum_i \mu_i y_i^* \]
subject to
\[ \lambda + \mu_i \leq -U_i \quad \forall i \]
\[ \mu_i \leq 0 \quad \forall i \]
Bender decomposition

Ongoing work

- Exploit the duality results to generate cuts for the master problem.
- Investigate the use of Benders for other problems.
  - profit maximization,
  - maximum likelihood estimation of the parameters.
Road map

**Optimization**
- Microeconomics in a nutshell
- Profit maximization, facility location

**Discrete choice**
- Logit and MEV models
- Activity-based models
Outline

1. Introduction
2. Microeconomics
3. The logit model
4. Profit optimization, facility location
5. Activity-based models
6. Conclusion
Travel demand is derived from activity demand.

Activity demand is influenced by socio-economic characteristics, social interactions, cultural norms, basic needs, etc. [Chapin, 1974]

Activity demand is constrained in space and time [Hägerstraand, 1970].
Econometric models

Rule-based models
State of the art: econometric approach

[Bhat, 2005]
- Multiple Discrete Continuous Extreme Value
- Based on first principles.
- Decision-maker solves an optimization problem, with a time budget.
- Several alternatives may be chosen.
- Model derived from KKT conditions.
State of practice

**Sequence of decisions**  
*Source: [Scherr et al., 2020]*

![Sequence of decisions diagram](image)

- **Permanent choices**
  - Owner-ship: car, PT subscription
  - Location choice (primary: W, E)
  - Tour frequency choice
  - Stop frequency choice
  - Activity choice (secondary)
  - Destination choice (secondary)

- **Daily choices**
  - Mode choice
  - Activity duration and start time
  - Plan scheduling
  - Discretisation of destinations

- **Discrete choice models**
- **Rule-based iterative plan refinement**
Research question

Relax the *series of discrete choice models* approach

- The interactions of all decisions is complex.
- Sequence of models is most of the time arbitrary.

Integrated approach

Develop a model involving many decisions:

- activity participation,
- activity location,
- activity duration,
- activity scheduling,
- travel mode,
- travel path.
Research objectives

- Integrated approach based on first principles.
- Theoretical framework: utility maximization.
- Individuals solve a scheduling problem.
- Important aspects: trade-offs on activity sequence, duration and starting time.
- Again, we replace the error terms by draws.
Decision variables for individual $n$ and draw $r$

For each (potential) activity $a$:

- Activity participation: $w_{anr} \in \{0, 1\}$.
- Starting time: $x_{anr} \in \{0, \ldots, T\}$.
- Duration: $\tau_{anr} \in \{0, \ldots, T\}$.
- Scheduling: $z_{abnr} \in \{0, 1\}$: 1 if activity $b$ immediately follows $a$. 
Objective function

Additive utility

\[
\max \sum_{a \in A} w_{anr} U_{anr} + \theta_t \sum_{a \in A} \sum_{b \in A} z_{abnr} \rho(s_a, s_b, m_a, p_a).
\]
Constraints

**Time budget**

\[
\sum_{a \in A} w_{anr} \tau_{anr} + \sum_{a \in A} \sum_{b \in A} z_{abnr} \rho(s_a, s_b, m_a, p_a) = T, \ \forall n, r.
\]

**Time windows**

\[
0 \leq \gamma_a^- \leq x_{anr} \leq x_{anr} + \tau_{anr} \leq \gamma_a^+ \leq T, \ \forall a, n, r.
\]
Constraints

Precedence constraints

\[ z_{abnr} + z_{banr} \leq 1, \ \forall a, b, n, r. \]

Single successor/predecessor

\[
\sum_{b \in A \setminus \{a\}} z_{abnr} = w_{anr}, \ \forall a, n, r,
\]

\[
\sum_{b \in A \setminus \{a\}} z_{banr} = w_{anr}, \ \forall a, n, r.
\]
 Constraints

Consistent timing

\[(z_{abnr} - 1) T \leq x_{anr} + \tau_{anr} + t_{anr} - x_{bnr} \leq (1 - z_{abnr}) T, \ \forall a, b, n, r.\]

where

\[t_{anr} = \sum_{b \in A} z_{abnr} \rho(s_a, s_b, m_a, p_a).\]

Other constraints...

- mode of transportation
- route
- car availability
- etc.

see [Pougala et al., 2021] for details
Optimization problem

Simulation-based optimization

- For each realization of the error terms, we have an optimal schedule.
- It includes all the choice dimensions (activity participation, location, duration, scheduling, and mode and route).
- We can generate an empirical distribution of chosen schedules.
Real data

Dataset
- 2015 Swiss Mobility and Transport Microcensus.
- Daily trip diaries for 57’000 individuals.
- Records of activities, visited location, mode/path choice.
Real data

Assumptions

- Desired start times and durations are the recorded ones.
- Feasible time windows: percentiles start and end times from out of sample distribution.
- Only the recorded locations are considered.
- Uniform flexibility profile across population.
Individual 1 (weekday)

Optimal schedules generated for random draws of $\varepsilon_{an}$
Individual 2 (weekday)

Optimal schedules generated for random draws of $\varepsilon_{an}$
Individual 3 (weekday)

Optimal schedules generated for random draws of $\varepsilon_{an}$
Validation

Activity profiles for full-time workers, Lausanne area

**Simulation**

**Microcensus**

Source: SBB. Acknowledgment to Patrick Manser.
Validation

Activity profiles for individuals older than 65, Lausanne area

**Simulation**

**Microcensus**

Source: SBB. Acknowledgment to Patrick Manser.
Validation

Activity profiles for students, Lausanne area

Source: SBB. Acknowledgment to Patrick Manser.
Validation

Activity profiles for primary school pupils, Lausanne area

Validation

Microcensus

Source: SBB. Acknowledgment to Patrick Manser.
Activity-based models

Ongoing work

- Synthetic population
- Estimation of the parameters
- Social interactions
## Outline

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Conclusion

Optimization

- Microeconomics in a nutshell

Discrete choice

- Logit and MEV models

Tutorial

- Profit maximization, facility location

Research

- Activity-based models
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Readings
- [Pacheco Paneque, 2020]
- [Pacheco et al., 2021]
- [Bortolomiol et al., forta]
- [Bortolomiol et al., fortb]
- [Pougala et al., 2021]
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What about people in regional science?
*Papers in Regional Science.*

