Modeling transportation networks

Route choice models

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Introduction

Analysis of the supply and demand interactions

- supply: infrastructure, vehicles
- demand: travellers' decisions
  - origin and destination
  - transportation mode
  - departure time
  - route, itinerary
- Underlying mathematical structure: the network

In this lecture, we focus on one of the most complicated issue

Introduction

Analysis of the supply and demand interactions

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Route choice model

Given

- a mono- or multi-modal transportation network (nodes, links, origin, destination)
- an origin-destination pair
- link and path attributes

identify the route that a traveler would select.

Choice model

Assumptions about

1. the decision-maker: n
2. the alternatives
   - Choice set $\mathcal{C}_n$
   - $p \in \mathcal{C}_n$ is composed of a list of links (i, j)
3. the attributes
   - link-additive: length, travel time, etc.

$$x_{kp} = \sum_{(i,j) \in P} x_{k(i,j)}$$

   - non link-additive: scenic path, usual path, etc.
4. the decision-rules: $\Pr(p|\mathcal{C}_n)$

Shortest path

Decision-makers all identical

Alternatives

- all paths between O and D
- $\mathcal{C}_n = \mathcal{U} \quad \forall n$
- $\mathcal{U}$ can be unbounded when loops are present

Attributes one link additive generalized cost

$$c_p = \sum_{(i,j)\in P} c_{(i,j)}$$

- traveler independent
- link cost may be negative
- no loop with negative cost must be present so that $c_p > -\infty$ for all p

Shortest path

Decision-rules path with the minimum cost is selected

$$\Pr(p) = \begin{cases} K & \text{if } c_p \leq c_q \quad \forall c_q \in \mathcal{U} \\ 0 & \text{otherwise} \end{cases}$$

- K is a normalizing constant so that $\sum_{p \in \mathcal{U}} \Pr(p) = 1$.
- $K = 1/S$, where S is the number of shortest paths between O and D.
- Some methods select one shortest path p^*

$$\Pr(p) = \begin{cases} 1 & \text{if } p = p^* \\ 0 & \text{otherwise} \end{cases}$$

Shortest path

Advantages:

- well defined
- no need for behavioral data
- efficient algorithms (Dijkstra)

Disadvantages

- behaviorally irrealistic
- instability with respect to variations in cost
- calibration on real data is very difficult
  - inverse shortest path problem is NP complete
  - Burton, Pulleyblank and Toint (1997) The Inverse Shortest Paths Problem With Upper Bounds on Shortest Paths Costs *Network Optimization* , Series: Lecture Notes in Economics and Mathematical Systems , Vol. 450, P. M. Pardalos, D. W. Hearn and W. W. Hager (Eds.), pp. 156-171, Springer

Dial's approach

Dial R. B. (1971) A probabilistic multipath traffic assignment model which obviates path enumeration *Transportation Research* Vol. 5, pp. 83-111.

Decision-makers all identical

Alternatives efficient paths between O and D

Attributes link-additive generalized cost

Decision-rules multinomial logit model

Dial's approach

- Def 1: A path is efficient if every link in it has
  - its initial node closer to the origin than its final node, and
  - its final node closer to the destination than its initial node.
- Def 2: A path is efficient if every link in it has its initial node closer to the origin than its final node.

Efficient path: a path that does not backtrack.

Dial's approach

- Choice set $\mathcal{C}_n$ = set of efficient paths (finite, no loop)
- No explicit enumeration
- Every efficient path has a non zero probability to be selected
- Probability to select a path

$$\Pr(p) = \frac{e^{\theta(\sum_{(i,j)\in p^*} c(i,j) - \sum_{(i,j)\in p} c(i,j))}}{\sum_{q\in\mathcal{C}_n} e^{\theta(\sum_{(i,j)\in p^*} c(i,j) - \sum_{(i,j)\in p} q(i,j))}}$$

where p^* is the shortest path and θ is a parameter

Dial's approach

Note: the length of the shortest path is constant across $\mathcal{C}_n$

$$\Pr(p) = \frac{e^{-\theta \sum_{(i,j)\in p} c(i,j))}}{\sum_{q\in\mathcal{C}_n} e^{-\theta \sum_{(i,j)\in q} q(i,j))}} = \frac{e^{-\theta c_p}}{\sum_{q\in\mathcal{C}_n} e^{-\theta c_q}}$$

Multinomial logit model with

$$V_p = -\theta c_p$$

Dial's approach

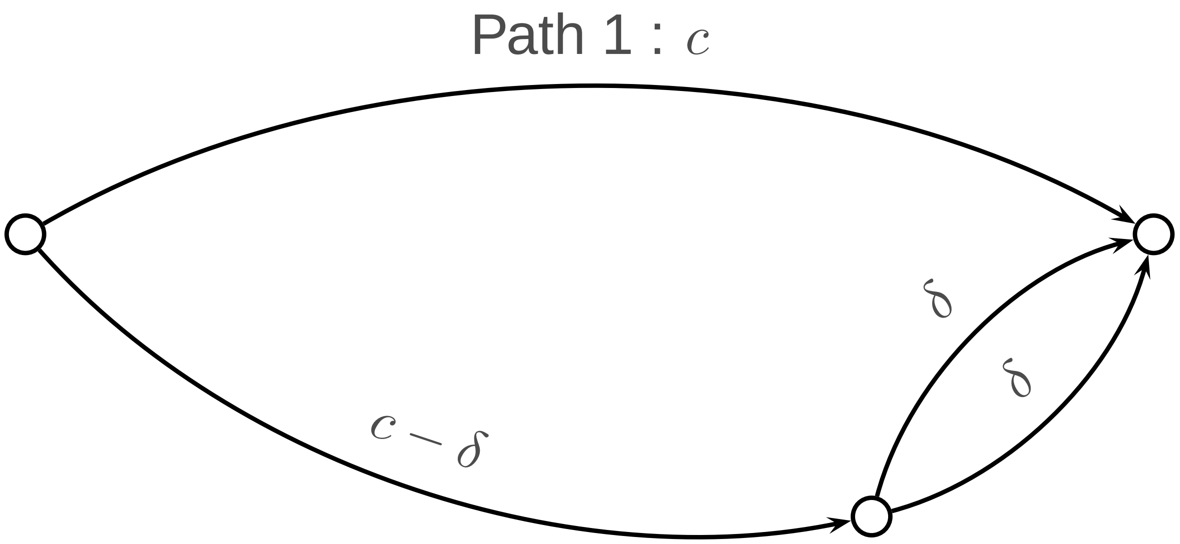
Advantages:

- probabilistic model, more stable
- calibration parameter θ
- avoid path enumeration
- designed for traffic assignment

Disadvantages:

- MNL assumes independence among alternatives
- efficient paths are mathematically convenient but not behaviorally motivated

Dial's approach



$$\Pr(1) = \frac{e^{-\theta c_1}}{\sum_{q \in \mathcal{C}} e^{-\theta c_q}} = \frac{e^{-\theta c}}{3e^{-\theta c}} = \frac{1}{3} \text{ for any } c, \delta, \theta$$

Path Size Logit

- With MNL, the utility of overlapping paths is overestimated
- When δ is large, there is some sort of "double counting"
- Idea: include a correction

$$V_p = -\theta c_p + \beta \ln \mathsf{PS}_p$$

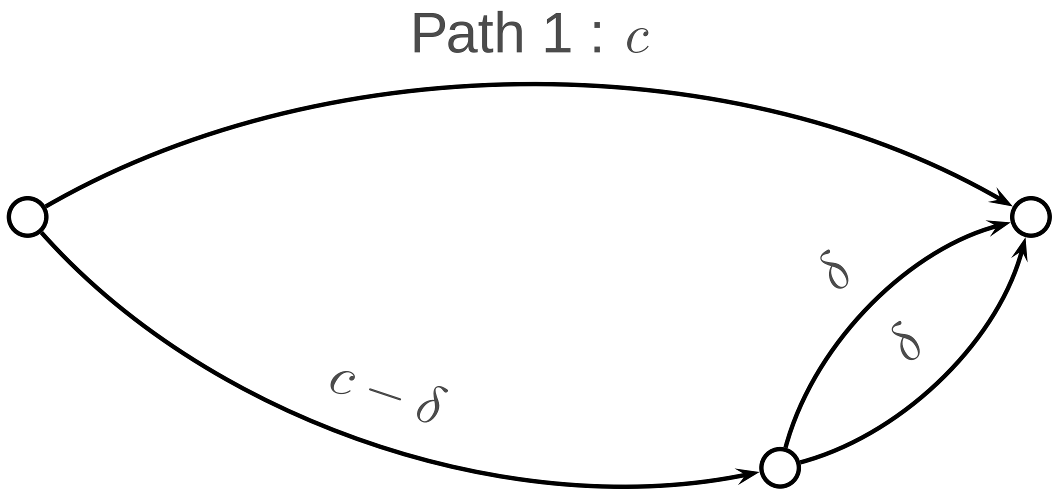
where

$$\mathsf{PS}_p = \sum_{(i,j)\in p} \frac{c_{(i,j)}}{c_p} \frac{1}{\sum_{q \in \mathcal{C}} \delta^q_{i,j}}$$

and

$$\delta^q_{i,j} = \begin{cases} 1 & \text{if link } (i,j) \text{ belongs to path } q \\ 0 & \text{otherwise} \end{cases}$$

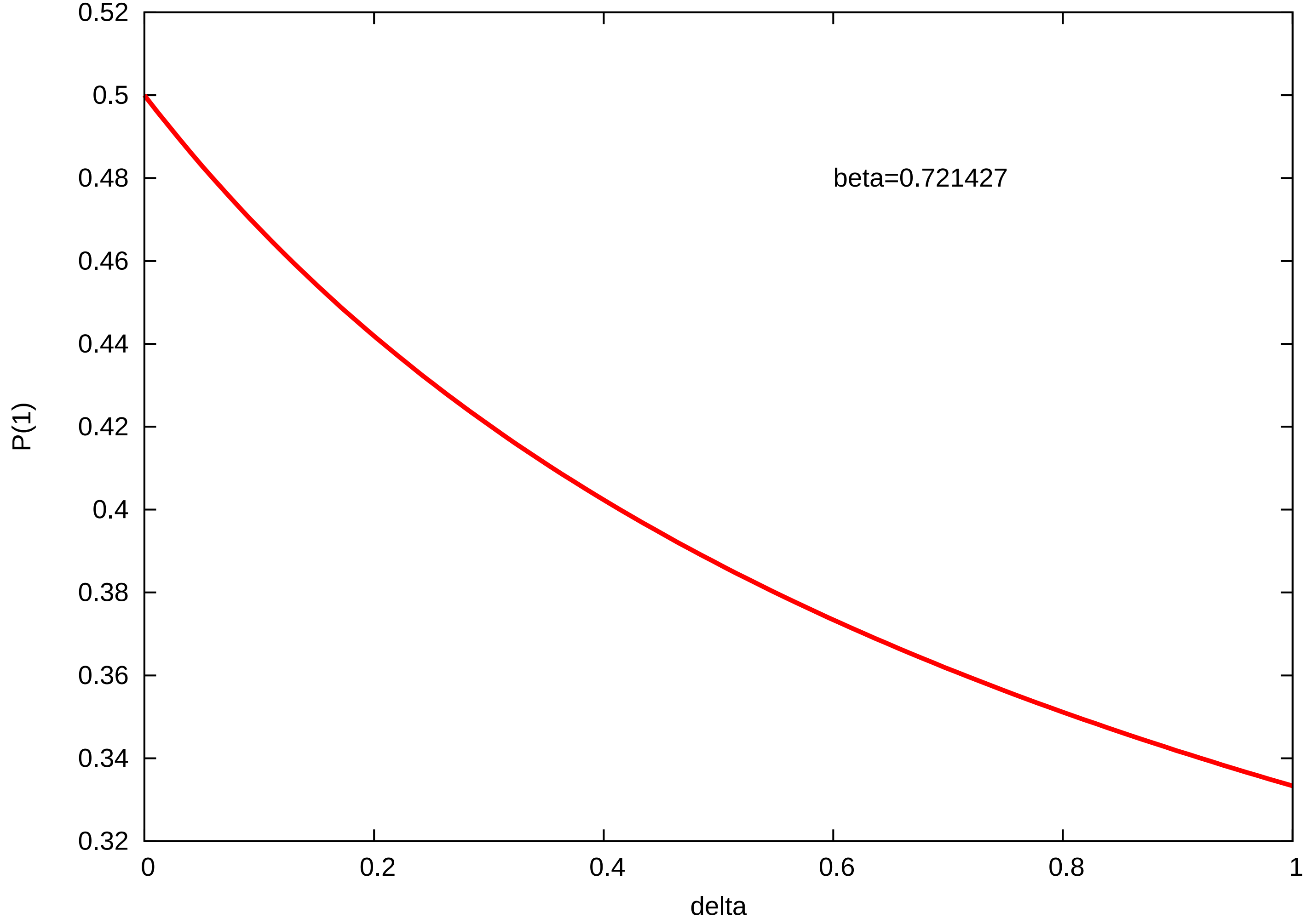
Path Size Logit



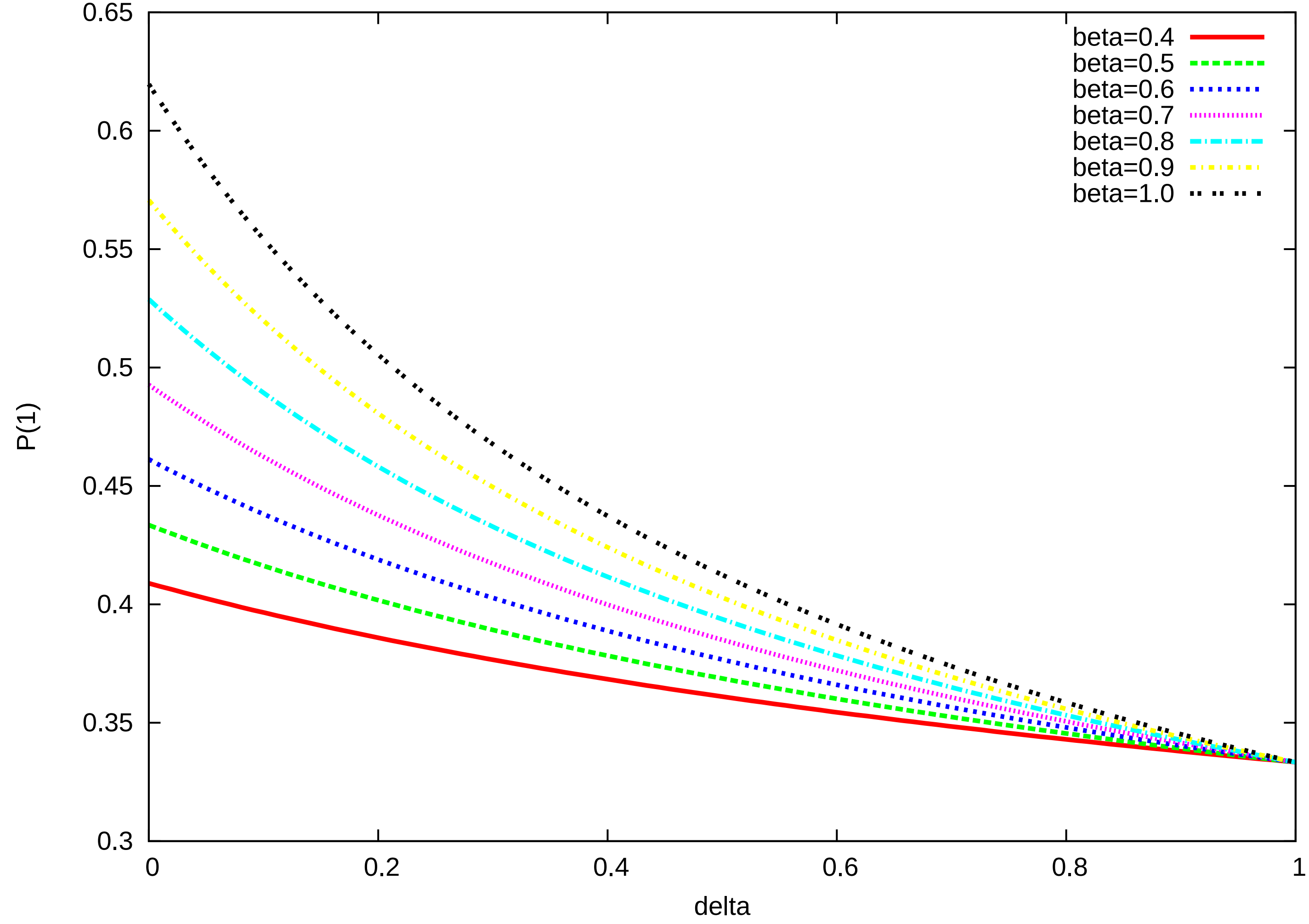
$$\text{PS}_1 = \frac{c}{c}\frac{1}{1} = 1$$

$$\text{PS}_2 = \text{PS}_3 = \frac{c-\delta}{c}\frac{1}{2} + \frac{\delta}{c}\frac{1}{1} = \frac{1}{2} + \frac{\delta}{2c}$$

Path Size Logit



Path Size Logit



Path Size Logit

Advantages:

- MNL formulation: simple
- Easy to compute
- Exploits the network topology
- Practical

Disadvantages:

- Derived from the theory on nested logit
- Several formulations have been proposed
- Correlated with observed and unobserved attributes
- May give biased estimates

Path Size Logit: readings

- Cascetta, E., Nuzzolo, A., Russo, F., Vitetta, A. 1996. A modified logit route choice model overcoming path overlapping problems. Specification and some calibration results for interurban networks. In Lesort, J.B. (Ed.), Proceedings of the 13th International Symposium on Transportation and Traffic Theory, Lyon, France.
- Ramming, M., 2001. Network Knowledge and Route Choice, PhD thesis, Massachusetts Institute of Technology.
- Ben-Akiva, M., and Bierlaire, M. (2003). Discrete choice models with applications to departure time and route choice. In Hall, R. (ed) *Handbook of Transportation Science, 2nd edition* pp.7-38. Kluwer.

Path Size Logit: readings

- Hoogendoorn-Lanser, S., van Nes, R. and Bovy, P. (2005) Path Size Modeling in Multimodal Route Choice Analysis. *Transportation Research Record* vol. 1921 pp. 27-34
- Frejinger, E., and Bierlaire, M. (2007). Capturing correlation with subnetworks in route choice models, *Transportation Research Part B: Methodological* 41(3):363-378. doi:10.1016/j.trb.2006.06.003

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Random utility models

Decision-makers with characteristics

- value of time
- access to information
- trip purpose

Alternatives explicit set of paths

Attributes both link-additive and path specific

Decision-rules RUM designed to capture correlations

Note: MNL is a random utility model, but the indendence assumption is inappropriate. We must relax it.

Random utility models

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Relax the independence assumption

$$\begin{pmatrix} U_{1n} \\ \vdots \\ U_{Jn} \end{pmatrix} = \begin{pmatrix} V_{1n} \\ \vdots \\ V_{Jn} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1n} \\ \vdots \\ \varepsilon_{Jn} \end{pmatrix}$$

that is

$$U_n = V_n + \varepsilon_n$$

and ε_n is a vector of random variables.

Assumption about the random term:
multivariate distribution

In the rest, we omit n

Relax the independence assumption

A multivariate random variable ε is represented by a density function

$$f(\varepsilon_1, \ldots, \varepsilon_J)$$

and

$$P(\varepsilon \leq x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_J} f(\varepsilon) d\varepsilon_J \ldots d\varepsilon_1$$

where $x \in \mathbb{R}^J$ is a $J \times 1$ vector of constants.

Multinomial probit model

$$U = \begin{pmatrix} U_1 \\ \vdots \\ U_J \end{pmatrix} = \begin{pmatrix} V_1 + \varepsilon_1 \\ \vdots \\ V_J + \varepsilon_J \end{pmatrix} = V + \varepsilon$$

Probability to choose path p

$$\Pr(p|\mathcal{C}) = \Pr(U_j - U_p \leq 0 \quad \forall j \in \mathcal{C})$$

Let Δ_i be a $(J-1 \times J)$ matrix obtained from the $(J-1 \times J-1)$ identity matrix, where a column of -1 has been added at index i. For $J=3$, we have

$$\Delta_1 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \Delta_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \Delta_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Multinomial probit model

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \Delta_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Therefore

$$\Delta_2 U = \begin{pmatrix} U_1 - U_2 \\ U_3 - U_2 \end{pmatrix}$$

In general, we obtain

$$\begin{aligned} \Pr(p|\mathcal{C}) &= \Pr(U_j - U_p \leq 0 \quad \forall j \in \mathcal{C}) \\ &= \Pr(\Delta_p U \leq 0) \end{aligned}$$

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Multinomial probit model

Assume that $U \sim N(V, \Sigma)$, where $\Sigma \in \mathbb{R}^{J \times J}$ is the variance-covariance matrix.

$$\Delta_p U \sim N(\Delta_p V, \Delta_p \Sigma \Delta_p^T)$$

$$\Pr(p|\mathcal{C}) =$$

$$\int_{\varepsilon_J=-\infty}^{0} \cdots \int_{\varepsilon_{p-1}=-\infty}^{0} \int_{\varepsilon_{p+1}=-\infty}^{0} \cdots \int_{\varepsilon_1=-\infty}^{0} f_p(\varepsilon) d\varepsilon_1 \ldots d\varepsilon_{p-1} d\varepsilon_{p+1} \ldots d\varepsilon_J$$

$$f_p(\varepsilon) = (2\pi)^{-\frac{J-1}{2}} \left|\Delta_p \Sigma \Delta_p^T\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\varepsilon - \Delta_p V)^T (\Delta_p \Sigma \Delta_p^T)^{-1} (\varepsilon - \Delta_p V)\right)$$

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Multinomial probit: issues

Variance-covariance matrix

- must be independent of the OD
- must capture the physical overlap of paths
- contains $J(J+1)/2$ unknown parameters
- Idea: use a structured variance-covariance matrix, with few parameters

`Yai, Iwakura and Morichi (1997) Multinomial probit with structured covariance for route choice behavior.` *`Transportation Research Part B`* `31(3), pp. 195-207.`

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Structured variance-covariance

$$\varepsilon = \varepsilon^\ell + \varepsilon^0$$

where ε^0 is a vector of independent r.v.
It is assumed that

$$\text{var}(\varepsilon^\ell_r) = c_r \sigma^2$$

where

- c_r is the length (or travel time) of path r
- σ is an unknown parameter to be estimated

The covariance is computed as follows:

$$\text{cov}(\varepsilon^\ell_r, \varepsilon^\ell_q) = \text{E}[\varepsilon^\ell_r \varepsilon^\ell_q] - \text{E}[\varepsilon^\ell_r]\,\text{E}[\varepsilon^\ell_q] = \text{E}[\varepsilon^\ell_r \varepsilon^\ell_q]$$

Structured variance-covariance

Let

$$\varepsilon_r^\ell = \varepsilon_{r\cap q} + \varepsilon_{r\setminus q}$$

where

- $\varepsilon_{r\cap q}$ corresponds to sections of r overlapping with q
- $\varepsilon_{r\setminus q}$ corresponds to sections of r not overlapping with q

$$\mathrm{E}[\varepsilon_r^\ell \varepsilon_q^\ell] = \mathrm{E}\left[(\varepsilon_{r\cap q} + \varepsilon_{r\setminus q})(\varepsilon_{r\cap q} + \varepsilon_{q\setminus r})\right] = \mathrm{E}[\varepsilon_{r\cap q}^2] = \mathrm{var}(\varepsilon_{r\cap q})$$

As before, we define

$$\mathrm{var}(\varepsilon_{r\cap q}) = c_{rq}\sigma^2$$

where c_{rq} is the length (or travel time) of the section common to r and q.

Structured variance-covariance

$$\varepsilon = \varepsilon^\ell + \varepsilon^0, \quad \Sigma = \Sigma^\ell + \Sigma^0$$

with

$$\Sigma^\ell = \sigma^2 \begin{pmatrix} c_1 & c_{12} & \cdots & c_{1J} \\ c_{12} & c_2 & \cdots & c_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1J} & c_{2J} & \cdots & c_J \end{pmatrix}$$

and

$$\Sigma^0 = \sigma_0^2 I$$

Note: only the ratio σ/σ_0 is identified

Multinomial probit: issues

Estimation

- Integral has no closed form
- Numerical integration is cumbersome and almost impossible, except when J is small
- Most efficient estimation method: `Bolduc (1999) A practical technique to estimate multinomial probit models in transportation` *`Transportation Research Part B`* `33, pp. 63-79.`
- Bolduc estimates a model with 9 alternatives
- Yai et. al. estimate a model with $J = 3$.
- In the route choice context, J can be much larger

Relax the independence assumption

- If the CDF $F(\varepsilon_1, \ldots, \varepsilon_J)$ of the distribution is known

$$f(\varepsilon_1, \ldots, \varepsilon_J) = \frac{\partial^J F}{\partial \varepsilon_1 \cdots \partial \varepsilon_J}(\varepsilon_1, \ldots, \varepsilon_J)$$

- The choice probability is

$$\begin{aligned}
\Pr(p) &= \Pr(V_1 + \varepsilon_1 \leq V_p + \varepsilon_p, \ldots, V_J + \varepsilon_J \leq V_p + \varepsilon_p) \\
&= P(\varepsilon_1 \leq V_p + \varepsilon_p - V_1, \ldots, \varepsilon_J \leq V_p + \varepsilon_p - V_J) \\
&= \int_{\varepsilon_p=-\infty}^{\infty} F_p(V_p + \varepsilon_p - V_1, \ldots, \varepsilon_p, \ldots, V_p + \varepsilon_p - V_J) d\varepsilon_p
\end{aligned}$$

MEV models

Family of models proposed by McFadden (1978) (called GEV)
Idea: a model is generated by a function

$$G : \mathbb{R}^J \to \mathbb{R}$$

From G, we can build

- The cumulative distribution function (CDF)
- The probability model
- The expected maximum utility

  McFadden, D. (1978). Modelling the Choice of Residential Location. In A. Karlquist et al. (eds.), *Spatial Interaction Theory and Residential Location*, North-Holland, Amsterdam, pp. 75-96.

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MEV models

1. G is homogeneous of degree $\mu > 0$, that is

$$G(\alpha y) = \alpha^{\mu} G(y)$$

2. $\lim_{y_i \to +\infty} G(y_1, \ldots, y_i, \ldots, y_J) = +\infty$, for each $i = 1, \ldots, J$,

3. the kth partial derivative with respect to k distinct y_i is non negative if k is odd and non positive if k is even, i.e., for all (distinct) indices $i_1, \ldots, i_k \in \{1, \ldots, J\}$, we have

$$(-1)^k \frac{\partial^k G}{\partial y_{i_1} \ldots \partial y_{i_k}}(y) \leq 0, \; \forall y \in \mathbb{R}^J_+.$$

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MEV models

- CDF: $F(\varepsilon_1, \ldots, \varepsilon_J) = e^{-G(e^{-\varepsilon_1}, \ldots, e^{-\varepsilon_J})}$
- Probability: $P(i|C) = \frac{e^{V_i + \ln G_i(e^{V_1}, \ldots, e^{V_J})}}{\sum_{j \in C} e^{V_j + \ln G_j(e^{V_1}, \ldots, e^{V_J})}}$ with $G_i = \frac{\partial G}{\partial x_i}$. This is a closed form
- Expected maximum utility: $V_C = \frac{\ln G(\ldots) + \gamma}{\mu}$ where γ is Euler's constant.
- Note: $P(i|C) = \frac{\partial V_C}{\partial V_i}$.

Euler's constant

$$\gamma = -\int_0^{+\infty} e^{-x} \ln x dx = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \approx 0,57722$$

Network representation of MEV models

Automatic way of defining G based on a network representation
Let (V, E) be a network with link parameters $\alpha_{(i,j)} \geq 0$
Assumptions:

1. No circuit.
2. One node without predecessor: *root*.
3. J nodes without successor: *alternatives*.
4. For each node v_i, there exists at least one path from the root to v_i such that $\prod_{k=1}^{P} \alpha_{(i_{k-1}, i_k)} > 0$.

Network MEV

For each node v_i, we define

- a set of indices $I_i \subseteq \{1, \ldots, J\}$ of J_i relevant alternatives,
- a homogeneous function $G^i : \mathbb{R}^{J_i} \longrightarrow \mathbb{R}$, and
- a parameter μ_i.

Recursive definition of I_i:

- $I_i = \{i\}$ for alternatives,
- $I_i = \bigcup_{j \in \text{succ}(i)} I_j$ for all other nodes.

Network MEV

Recursive definition of G^i:

For alternatives:

$$G^i \quad : \quad \mathbb{R} \longrightarrow \mathbb{R} \quad : \quad G^i(x_i) = x_i^{\mu_i} \quad i = 1, \ldots, J$$

For all others:

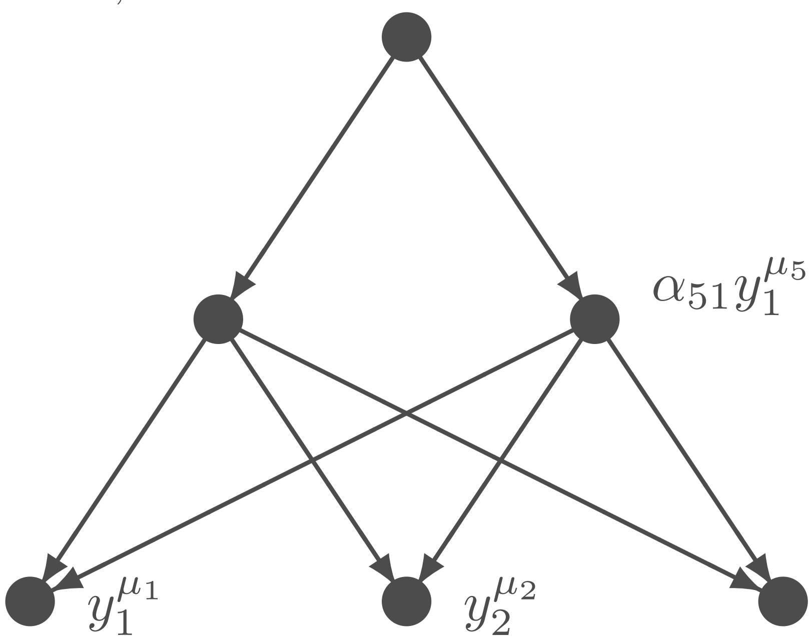
$$G^i \quad : \quad \mathbb{R}^{J_i} \longrightarrow \mathbb{R} \quad : \quad G^i(x) = \sum_{j \in \text{SUCC}(i)} \alpha_{(i,j)} G^j(x)^{\frac{\mu_i}{\mu_j}}$$

Network MEV

Example: Cross-Nested Logit

$$G = \sum_m \left(\sum_{j \in C} \alpha_{jm} y_j^{\mu_m} \right)^{\frac{\mu}{\mu_m}}$$

$\sum_{i=4,5} \alpha_{0i} \left(\alpha_{i1} y_1^{\mu_i} + \alpha_{i2} y_2^{\mu_i} + \alpha_{i3} y_3^{\mu_i}\right)^{\frac{\mu_0}{\mu_i}}$



$\alpha_{51} y_1^{\mu_5} + \alpha_{52} y_2^{\mu_5} + \alpha_{53} y_3^{\mu_5}$

$y_1^{\mu_1}$ $y_2^{\mu_2}$ $y_3^{\mu_3}$

Network MEV

- Daly, A., and Bierlaire, M. (2006). A General and Operational Representation of Generalised Extreme Value Models, *Transportation Research Part B: Methodological* 40(4):285-305. doi:10.1016/j.trb.2005.03.003
- Daly, A. (2001). Recursive nested EV model. ITS Working Paper 559, Institute for Transport Studies, University of Leeds.
- Bierlaire, M. (2002). The network GEV model. In: Proceedings of the 2nd Swiss Transportation Research Conference, Ascona, Switzerland.

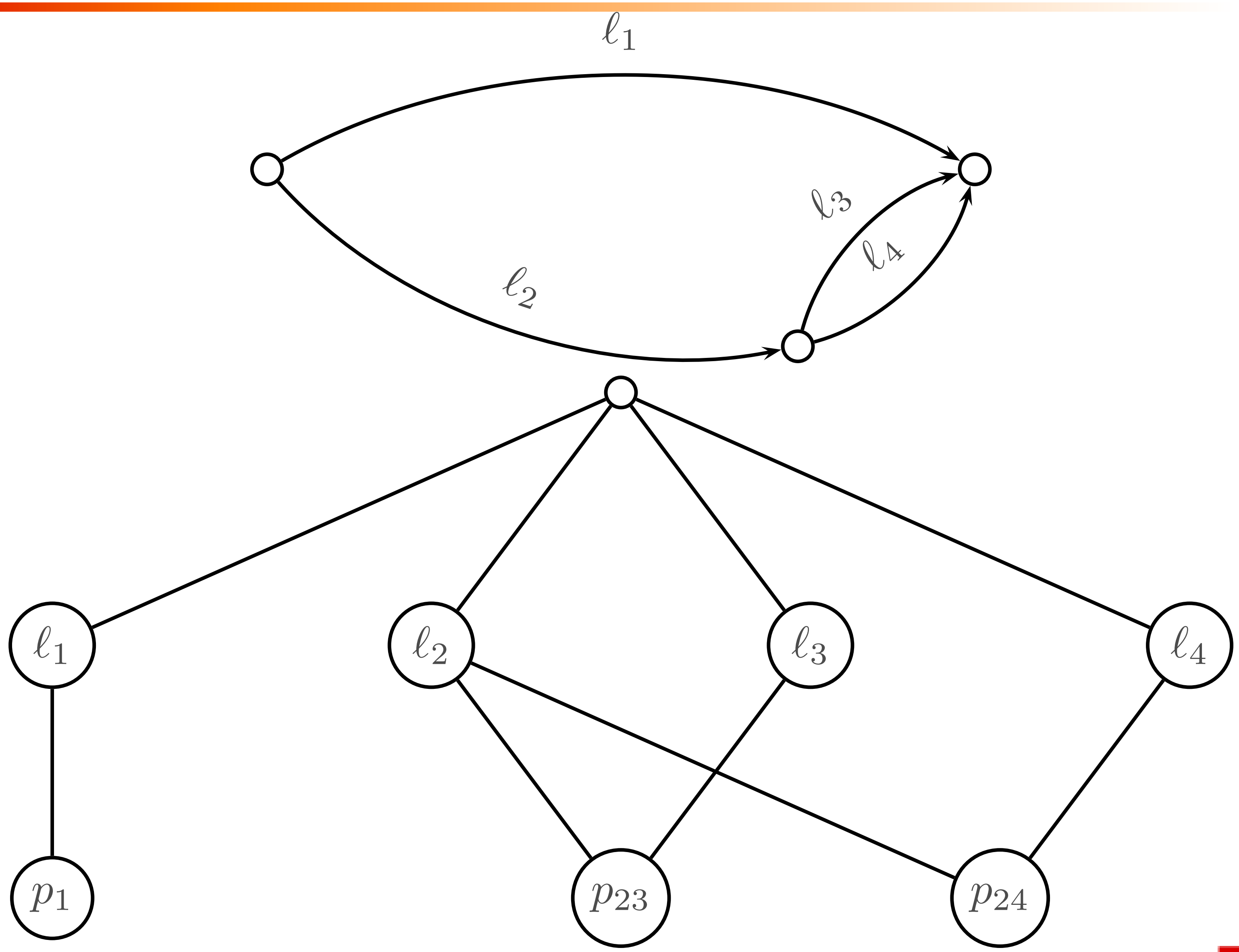
Link Nested Logit Model

Vovsha, P. and Bekhor, S., 1998 Link-nested logit model of route choice, Overcoming route overlapping problem, *Transportation Research Record* 1645, pp. 133-142.

Idea:

- Use the cross-nested model specification
- Alternatives = paths
- Nests = links

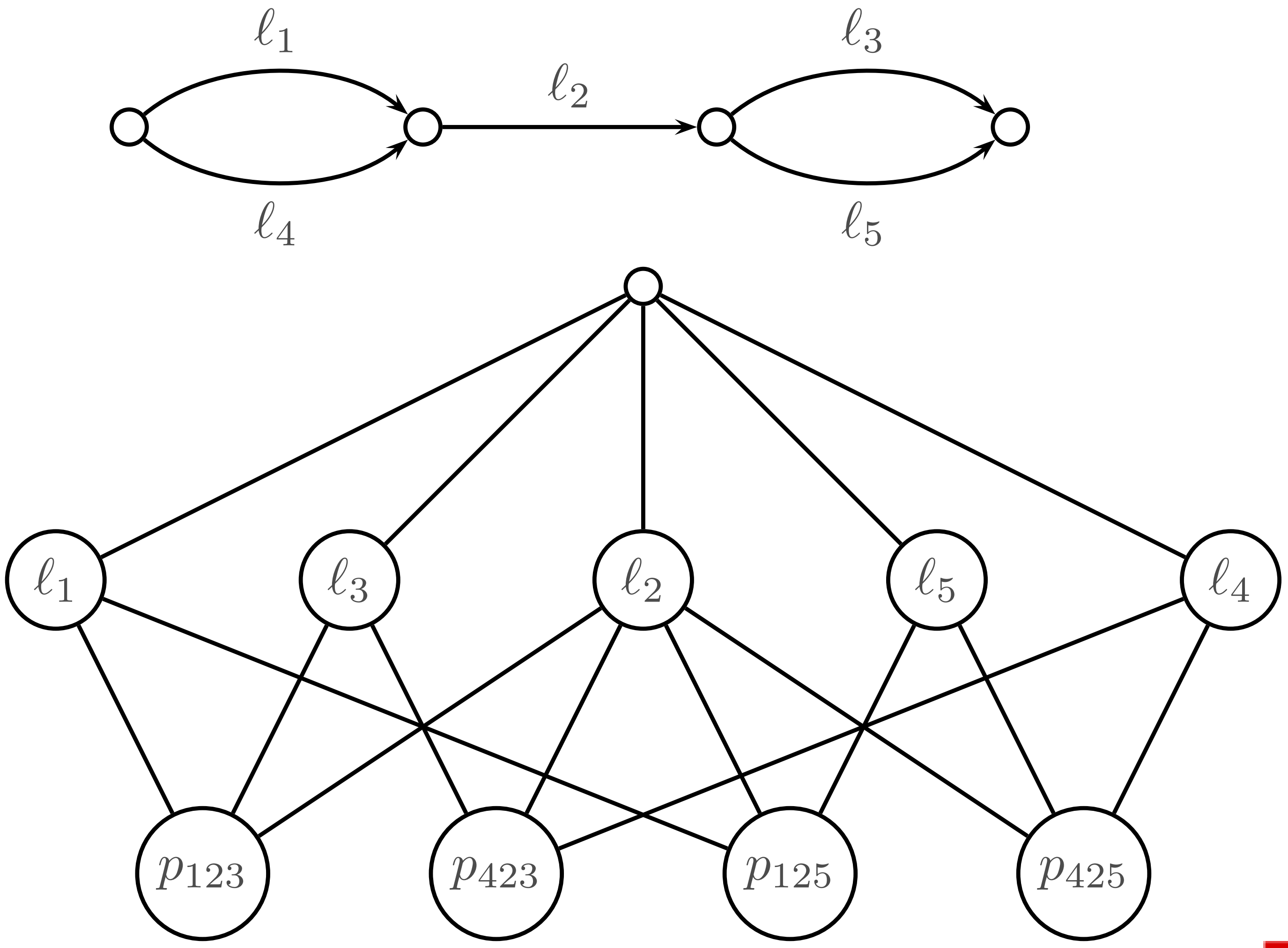
Link Nested Logit Model



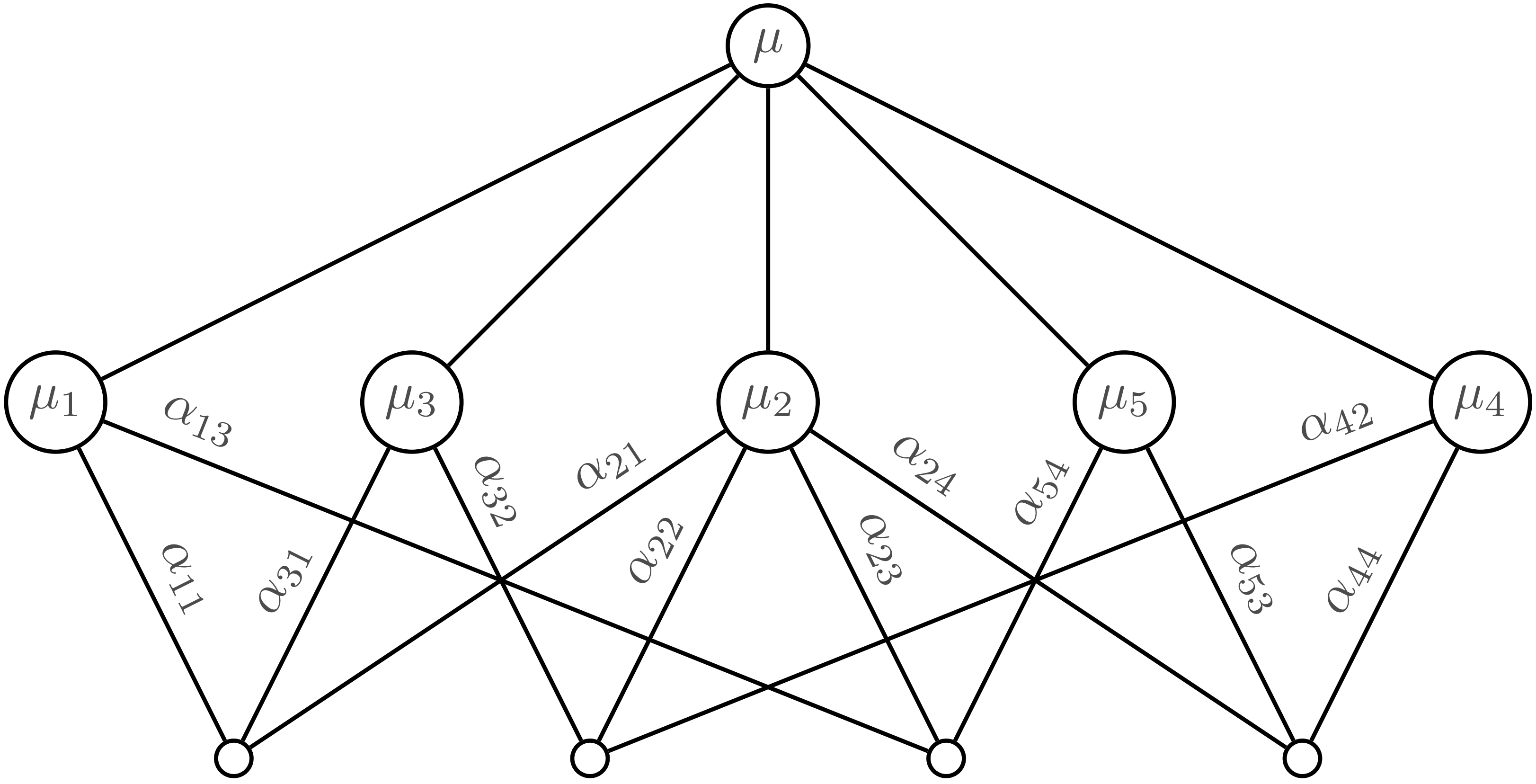
Link Nested Logit Model

- In this example, nests ℓ_1, ℓ_3 and ℓ_4 contain a single alternative
- The model collapses to a nested logit model

Link Nested Logit Model



Link Nested Logit Model



Link Nested Logit Model

- 5 + 12 = 17 parameters to capture the correlation
- Variance covariance matrix: $\Sigma \in \mathbb{R}^{4\times 4}$
- It is symmetric, so there are $J(J+1)/2 = 10$ entries
- For a given correlation matrix, the identification of the associated parameters is not straightforward
- Abbe, E., Bierlaire, M., and Toledo, T. (2007). Normalization and correlation of cross-nested logit models, *Transportation Research Part B: Methodological* 41(7):795-808. doi:10.1016/j.trb.2006.11.006

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Link Nested Logit Model

Advantages

- Closed form probability
- Rich correlation structure

Disadvantages

- High number of nests
- All correlations structures cannot be reproduced
- Identification of the associated parameters is not straightforward

Factor Analytic Specification

- Links: Bekhor, S., Ben-Akiva, M. and Ramming M.S. (2002). Adaptation of Logit Kernel to Route Choice Situation. *Transportation Research Record*, 1805, 78-85.
- Subnetworks: Frejinger, E., and Bierlaire, M. (2007). Capturing correlation with subnetworks in route choice models, *Transportation Research Part B: Methodological* 41(3):363-378. doi:10.1016/j.trb.2006.06.003

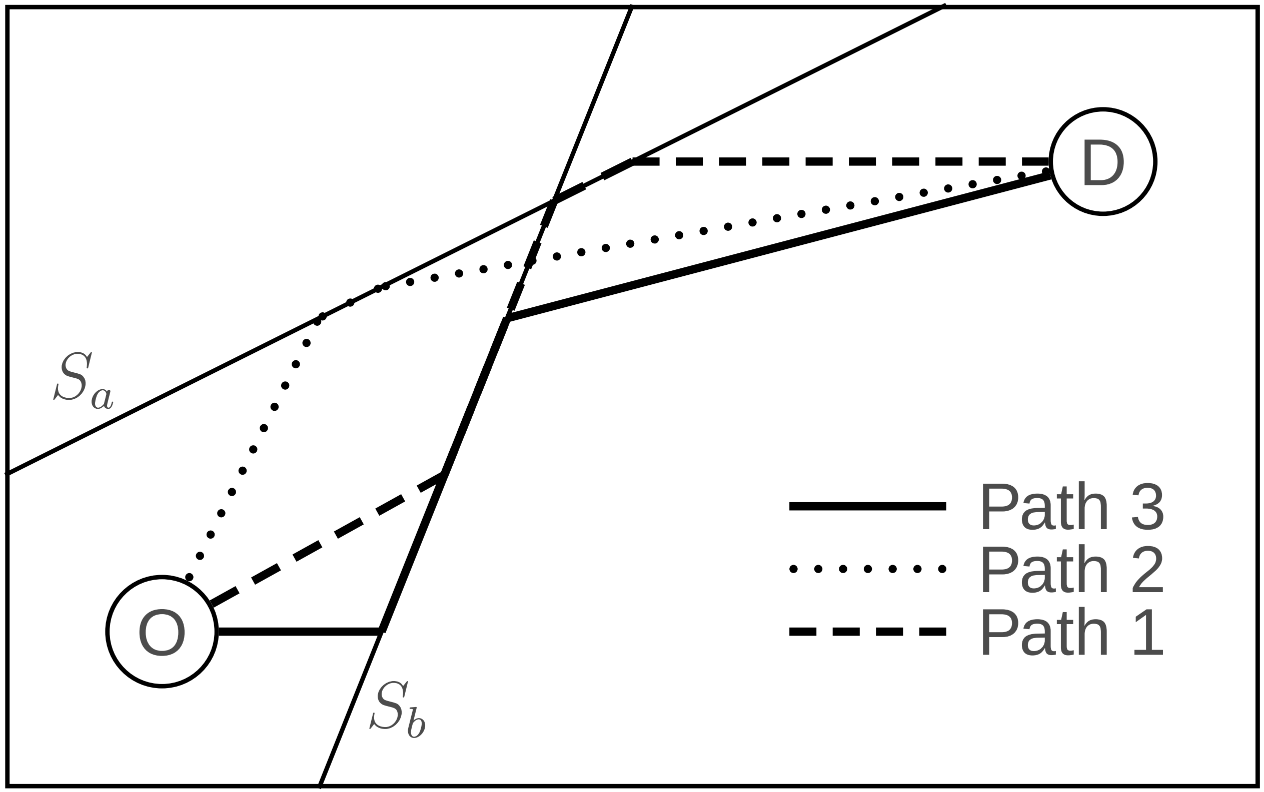
Subnetwork component

Sequence of links corresponding to a part of the network which can be easily labeled, and is behaviorally meaningful in actual route descriptions

- Champs-Elysées in Paris
- Fifth Avenue in New York
- Mass Pike in Boston
- City center in Lausanne

Paths sharing a subnetwork component are assumed to be correlated even if they are not physically overlapping

Subnetworks - Example



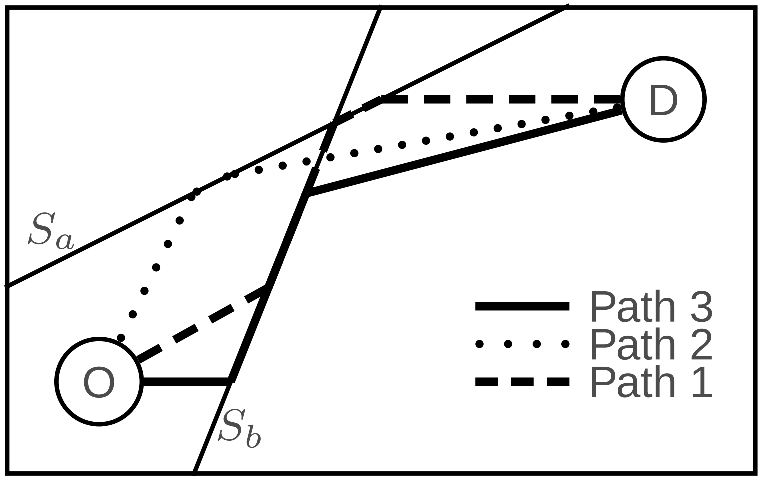
Subnetworks - Methodology

- Factor analytic specification of an error component model (based on model presented in Bekhor et al., 2002)

$$U_p = V_p + \sum_s \sqrt{c_{ps}}\sigma_s\zeta_s + \nu_p$$

  - c_{ps} is the length by which path p overlaps with subnetwork component s
  - σ_s is an unknown parameter
  - $\zeta_s \sim N(0,1)$
  - ν_p i.i.d. Extreme Value distributed

Subnetworks - Example



$$U_1 = \beta^T X_1 + \sqrt{l_{1a}}\sigma_a\zeta_a + \sqrt{l_{1b}}\sigma_b\zeta_b + \nu_1$$

$$U_2 = \beta^T X_2 + \sqrt{l_{2a}}\sigma_a\zeta_a + \nu_2$$

$$U_3 = \beta^T X_3 + \sqrt{l_{3b}}\sigma_b\zeta_b + \nu_3$$

$$\Sigma =$$

$$\begin{bmatrix} l_{1a}\sigma_a^2 + l_{1b}\sigma_b^2 & \sqrt{l_{1a}}\sqrt{l_{2a}}\sigma_a^2 & \sqrt{l_{1b}}\sqrt{l_{3b}}\sigma_b^2 \\ \sqrt{l_{1a}}\sqrt{l_{2a}}\sigma_a^2 & l_{2a}\sigma_a^2 & 0 \\ \sqrt{l_{3b}}\sqrt{l_{1b}}\sigma_b^2 & 0 & l_{3b}\sigma_b^2 \end{bmatrix}$$

Mixture of MNL

In statistics, a mixture density is a pdf which is a convex linear combination of other pdf's.

If $f(\varepsilon, \theta)$ is a pdf, and if $w(\theta)$ is a nonnegative function such that

$$\int_\theta w(\theta) d\theta = 1$$

then

$$g(\varepsilon) = \int_\theta w(\theta) f(\varepsilon, \theta) d\theta$$

is also a pdf. We say that g is a mixture of f.

If f is the pdf of a MNL model, it is a mixture of MNL

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Mixture of MNL

$$U_p = V_p + \sum_s \sqrt{c_{ps}}\sigma_s\zeta_s + \nu_p$$

If ζ is given,

$$\Pr(p|\zeta) = \frac{e^{V_p + \sum_s \sqrt{c_{ps}}\sigma_s\zeta_s}}{\sum_q e^{V_q + \sum_s \sqrt{c_{qs}}\sigma_s\zeta_s}}$$

ζ is distributed $N(0, I)$

$$\Pr(p) = \int_\zeta \frac{e^{V_p + \sum_s \sqrt{c_{ps}}\sigma_s\zeta_s}}{\sum_q e^{V_q + \sum_s \sqrt{c_{qs}}\sigma_s\zeta_s}}\phi(\zeta)d\zeta$$

Mixture of MNL

$$\Pr(p) = \int_{\zeta} \frac{e^{V_p + \sum_s \sqrt{c_{ps}}\sigma_s \zeta_s}}{\sum_q e^{V_q + \sum_s \sqrt{c_{qs}}\sigma_s \zeta_s}} \phi(\zeta) d\zeta$$

Not a closed form. Simulated Maximum Likelihood is to be used

- Train, K. (2003) Discrete Choice Methods with Simulation, Cambridge University Press

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Subnetworks

Advantages

- Rich correlation structure
- Flexibility between complexity and realism

Disadvantages

- Non closed form

Summary

- Shortest path
- Dial's efficient paths
- Path Size Logit
- Multinomial probit
- Link Nested Logit
- Subnetworks

Additionnal Reading

Prashker, J. N. and Bekhor, S. (2004) Route Choice Models Used in the Stochastic User Equilibrium Problem: A Review, *Transport Reviews*, 24:4, 437 - 463

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Additionnal Reading

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