A new mathematical formulation to integrate supply and demand within a choice-based optimization framework

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1. Introduction

2. Customer behavioral models

3. Linear formulation

4. Demand based revenues maximization

5. Case study

6. Conclusions
Introduction

Customer behavioral models

Operations research

Demand

Supply

Transportation
Demand and supply

Customer behavioral models
- Given the configuration of the system ⇒ predict the demand
- Maximize satisfaction
- **Here:** discrete choice models

Operations Research
- Given the demand ⇒ configure the system
- Minimize costs
- **Here:** MILP

Discrete choice models in optimization problems
- Integrated choice model ⇒ source of nonconvexity
- Many techniques to convexify and linearize. **Here:** different approach
  - Nonconvex representation of choice probabilities
  - Include a wide class of discrete choice models
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Customer behavioral models

Utilities

Demand and supply
- Population of $N$ individuals
- Set of products $C$ in the market
  - artificial "opt-out" product
- $C_n \subseteq C$ subset of available products to individual $n$

Utility

$U_{in}$ associated score to alternative $i$ by individual $n$: $U_{in} = V_{in} + \varepsilon_{in}$
- $V_{in}$: deterministic part
- $\varepsilon_{in}$: error term

Behavioral assumption: $n$ chooses $i$ if $U_{in}$ is the highest in $C_n$
**Probabilistic model**

**Choice**

\[ w_{in} = \begin{cases} 
1 & \text{if } n \text{ chooses } i \\
0 & \text{otherwise} 
\end{cases} \]

\( \forall n, \forall i \in C \)

**Availability**

\[ y_{in} = \begin{cases} 
1 & \text{if } i \in C_n \\
0 & \text{otherwise} 
\end{cases} \]

\( \forall n, \forall i \in C \)

\[ w_{in} = 1 \iff y_{in} = 1 \text{ and } U_{in} \geq U_{jn}, \forall j \in C_n \]

**Probabilistic model**

- \( \Pr(w_{in} = 1) = \Pr(U_{in} \geq U_{jn}, \forall j \in C_n) \)
- \( D_i = \sum_{n=1}^{N} \Pr(w_{in} = 1) \)
Non linearity

- $D_i$ is in general non linear
- **Example:** $\Pr(w_{in} = 1) = \frac{y_{in}e^{V_{in}}}{\sum_{j\in C} y_{jn}e^{V_{jn}}}$ (logit model)

Simulation

- Assume a distribution for $\varepsilon_{in}$
- Generate $R$ draws $\xi_{in1} \ldots \xi_{inR}$
- $r$ behavioral scenario
- The choice problem becomes **deterministic**
Demand model

\[ U_{inr} = V_{in} + \xi_{inr} = \sum_k \beta_k x_{ink} + f(z_{in}) + \xi_{inr} \]  

\[ \Rightarrow U_{inr} \text{ is not a random variable} \]

Endogenous part of \( V_{in} \)
- Linear in the variables \( x_{ink} \)
- Decision variables (involved in the optimization problem)
- Assumption for the integration in a MILP

Exogenous part of \( V_{in} \)
- Depends on other variables \( z_{in} \)
- \( f \) not necessarily linear
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Availability of alternatives (I)

Variables

- $y_{in}$ decision of the operator
  \[ y_{in} = 0 \quad \forall i \notin C_n, n \] (2)

- $y_{inr}$ availability at scenario level (e.g. demand exceeding capacity)
  \[ y_{inr} \leq y_{in} \quad \forall i, n, r \] (3)

Idea: linearization of $U_{inr}y_{inr}$

\[ \nu_{inr} = \begin{cases} U_{inr} & \text{if } y_{inr} = 1 \\ l_{inr} & \text{if } y_{inr} = 0 \end{cases} \]

Where $l_{inr} = \min\{U_{inr}\}$

$m_{inr} = \max\{U_{inr}\}$
Availability of alternatives (II)

Constraints

\[
\begin{align*}
\nu_{inr} & \leq \nu_{inr}, & \forall i, n, r \quad (4) \\
\nu_{inr} & \leq l_{inr} + (m_{inr} - l_{inr})y_{inr}, & \forall i, n, r \quad (5) \\
U_{inr} + (l_{inr} - m_{inr})(1 - y_{inr}) & \leq \nu_{inr}, & \forall i, n, r \quad (6) \\
\nu_{inr} & \leq U_{inr} & \forall i, n, r \quad (7)
\end{align*}
\]

- \( y_{inr} = 1 \) \( \Rightarrow \) Binding constraints: (6) and (7) \( \Rightarrow \) \( \nu_{inr} = U_{inr} \)
- \( y_{inr} = 0 \) \( \Rightarrow \) Binding constraints: (4) and (5) \( \Rightarrow \) \( \nu_{inr} = l_{inr} \)
Highest utility among the available alternatives

Linearization of the maximum of variables

\[ U_{nr} = \max_{j \in C_n} \{ U_{jnr} \} \]

Highest utility for individual \( n \) in scenario \( r \):

\[ \mu_{inr} = \begin{cases} 1 & \text{if } U_{nr} = U_{inr} \\ 0 & \text{otherwise} \end{cases} \]

\[ \nu_{inr} \leq U_{nr} \quad \forall i, n, r \] (8)

\[ U_{nr} \leq \nu_{inr} + M_{inr}(1 - \mu_{inr}) \quad \forall i, n, r \] (9)

\[ \sum_{i \in C} \mu_{inr} = 1 \quad \forall n, r \] (10)

where \( M_{inr} = \max_{j \in C} m_{jnr} - l_{inr} \)

- \( \mu_{inr} = 1 \Rightarrow U_{nr} = \nu_{inr} = U_{inr} \)
- \( \mu_{inr} = 0 \Rightarrow \nu_{nr} = l_{inr} \)
Choice and availability

Constraints

\[
\begin{align*}
\mu_{inr} & \leq y_{inr} & \forall i, n, r \\
\omega_{inr} & \leq \mu_{inr} & \forall i, n, r \\
\omega_{inr} & \leq y_{inr} & \forall i, n, r \\
\sum_{i \in C} \omega_{inr} & = 1 & \forall n, r
\end{align*}
\]

- (11) An unavailable alternative cannot be the one with highest utility
- (12) An alternative without the highest utility cannot be chosen
- (13) An unavailable alternative cannot be chosen
- (14) Only one alternative is chosen
Modeling framework

Model (1)-(14)

- Linear in the variables
  - Any variable appearing linearly in $U_{inr}$
  - The availability variables $y_{in}$, $y_{inr}$ and $v_{inr}$
  - The preference variables $\mu_{inr}$
  - The choice variables $w_{inr}$

- Demand within the market

$$D_i = \frac{1}{R} \sum_{n=1}^{N} \sum_{r=1}^{R} w_{inr}$$

- Further specifications
  - Capacity?
  - Price?
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Maximization of revenues

Application

- Operator selling services to a market, each offered service:
  - Price
  - Capacity (number of customers)
- Demand is price elastic and heterogenous
- **Goal**: best strategy in terms of capacity allocation and pricing

Revenues

- $p_{in}$ price that individual $n$ has to pay to access service $i$

\[
R_i = \frac{1}{R} \sum_{n=1}^{N} p_{in} \sum_{r=1}^{R} w_{inr}
\]

- $p_{in}$ endogenous variable $\Rightarrow R_i$ non linear
Pricing (I)

Linearization of $R_i$

- Discretization of the price $\Rightarrow p_{in}^1, \ldots, p_{in}^{L_{in}}$
- Binary variables $\lambda_{inl}$ such that $p_{in} = \sum_{l=1}^{L_{in}} \lambda_{inl} p_{in}^l$ and

$$\sum_{l=1}^{L_{in}} \lambda_{inl} = 1 \quad \forall i, n \quad (15)$$

- Revenues for alternative $i$

$$R_i = \frac{1}{R} \sum_{n=1}^{N} \sum_{l=1}^{L_{in}} \lambda_{inl} p_{in}^l \sum_{r=1}^{R} w_{inr}$$

- Still non linear $\Rightarrow \alpha_{inrl} = \lambda_{inl} w_{inr}$ to linearize it
Pricing (II)

Constraints

\[ \lambda_{inl} + w_{inr} \leq 1 + \alpha_{inrl} \quad \forall i, n, r, l \]  
\[ \alpha_{inrl} \leq \lambda_{inl} \quad \forall i, n, r, l \]  
\[ \alpha_{inrl} \leq w_{inr} \quad \forall i, n, r, l \]

Objective function

\[ \max R_i = \max \frac{1}{R} \sum_{n=1}^{N} \sum_{l=1}^{L_{in}} \alpha_{inrl} p_{in}^l \]
Capacity (I)

Priority list

- Who has access?
- We assume a priority list

\[ y_{inr} \geq y_{i(n+1)r} \quad \forall i, n, r \]  

(19)

Constraints (I)

\[ c_i(1 - y_{inr}) \leq \sum_{m=1}^{n-1} w_{imr} + (1 - y_{in})c_{max} \quad \forall i, n, r \]  

(20)

- \( y_{inr} = 0 \) and \( y_{in} = 1 \) \( \Rightarrow \) \( c_i \leq \sum_{m=1}^{n-1} \) (capacity is reached)
- \( y_{inr} = y_{in} = 1 \) and \( y_{inr} = y_{in} = 0 \) \( \Rightarrow \) always verified

Capacity

- \( c_i \) capacity of service \( i \)
- \( c_{max} = \max_i c_i, c_{min} = \min_i c_i \)
- \( K_n = \max(n, c_{max}) \)
Capacity (II)

Constraints (II)

\[
\sum_{m=1}^{n-1} w_{imr} + (1 - y_{in}) c_{\text{max}} = (c_i - 1) y_{inr} + K_n (1 - y_{inr}) \quad \forall i, n \leq c_{\text{min}}, r
\]  

- \( y_{inr} = y_{in} = 1 \Rightarrow 1 + \sum_{m=1}^{n-1} w_{imr} \leq c_i \)  
  (capacity must not be exceeded by the individuals choosing \( i + n \))

- \( y_{inr} = y_{in} = 0 \) and \( y_{inr} = 0, y_{in} = 1 \Rightarrow \) always verified
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Parking choices

Original experiment

- [Ibeas et al., 2014] *Modelling parking choices considering user heterogeneity*
- Stated preferences survey (197 respondents)
- Analyze viability of an underground car park
- 8 scenarios suggested

Free on-Street Parking (FSP)  
Free

Paid on-Street Parking (PSP)  
Price levels: 0.6 and 0.8

Paid Underground Parking (PUP)  
Price levels: 0.8 and 1.5
Choice model and preliminary experiments

Mixed Logit model

- **Attributes:** time to reach the destination
- **Socioeconomic characteristics:** residence, age of the vehicle
- **Interactions:** price and low income, price and residence
- **Random parameters:** access time and price

Preliminary experiment

- Subset of individuals
- Fixed capacity for the 3 alternatives
### Results

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</table>
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Conclusions and future work

Conclusions
- High dimensionality of the problem
- Any assumption can be made for the $\varepsilon_{in}$

Future work
- Design of scenarios $\Rightarrow$ more experiments!
- Speed up the computational results
  - Preprocessing in particular cases (e.g. dominant alternatives)
  - Decomposition techniques (e.g. by scenario)
- Introduce new features (e.g. N as a group of individuals)
Questions?