The Maximum Capture Problem with Flexible Substitution Paterns

Sven Müller (Karlsruhe University of Applied Sciences)
Knut Haase (Hamburg Business School)

Outline

Problem Setting

- Firm modifies network of facilities (branches) in a geographical market

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- Problem also known as **maximum capture problem with random utilities** (MCPRU)

Generalization: Locations $\leftrightarrow$ alternatives

Decisions only about **alternatives** not about **attributes**
Assortment optimization, product line/portfolio planning problem
Extension: simultaneous decisions on discrete attributes: Krohn et al. (2016), *working paper*

Basic Notation of Data

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- q_i quantity of customers located in $i \in \mathcal{I}$
- u_{ij} utility of customers located in $i \in \mathcal{I}$ patronizing a facility located at $j \in \mathcal{M}$:

$$u_{ij} = v_{ij} + \epsilon_{ij} \tag{1}$$

with

- v_{ij} deterministic utility
- ϵ_{ij} stochastic utility

Customer Choice Behavior and Patronage

A customer located in $i \in \mathcal{I}$ chooses to patronize a facility located in $j \in \mathcal{M}$, iff

$$u_{ij} > u_{im} \; \forall \; m \in \mathcal{M}, m \neq j. \tag{2}$$

Note, u_{ij} is treated as a random quantity ($u_{ij} = v_{ij} + \epsilon_{ij}$)

General choice model

$$p_{ij} = \text{Prob}\left(u_{ij} > u_{im} \; \forall \; m \in \mathcal{M}, m \neq j\right) \tag{3}$$

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Assume stochastic component ϵ_{ij} is iid EV, the probability (3) is given by

Multinomial logit model (MNL)

$$p_{ij}^{\text{MNL}} = \frac{e^{v_{ij}}}{\sum_{m \in \mathcal{M}} e^{v_{im}}} \tag{4}$$

Maximum Capture Problem with Random Utilities

Problem Statement MCPRU

Select r facility locations from all potential locations $\mathcal{J}$ with $\mathcal{J} \subset \mathcal{M}$ such that the total expected patronage of the firm is maximized.

Y_j $= 1$ if a (new) facility is located at $j \in \mathcal{J}$ and 0 otherwise

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Problem **P0** (MCPRU)

$$\text{Maximize } F^{\mathsf{P0}} = \sum_{i \in \mathcal{I}} q_i \sum_{j \in \mathcal{J}} \frac{Y_j e^{v_{ij}}}{\sum_{m \in \mathcal{M}} Y_m e^{v_{im}}} \tag{5}$$

subject to

$$\sum_{j \in \mathcal{J}} Y_j = r \tag{6}$$

$$Y_j \in \{0, 1\} \qquad \forall\, j \in \mathcal{J} \tag{7}$$

Linear MIP reformulations of (5)-(7) in Haase & Müller (2014b), *EJOR*

Decomposition of Error Term

Recall

- $u_{ij} = v_{ij} + \epsilon_{ij}$
- $p_{ij} = \text{Prob}\,(u_{ij} > u_{im}\ \forall\ m \in \mathcal{M}, m \neq j)$

Assume

$$\epsilon_{ij} = \sum_{c \in \mathcal{C}} \eta_c h_{ijc} + \varepsilon_{ij} \tag{8}$$

with

$\mathcal{C}$	set of so-called error components
h_{ijc}	observable attributes related to demand point $i \in \mathcal{I}$ and facility location $j \in \mathcal{M}$ denoting the structure of substitution for error component $c \in \mathcal{C}$
η_c	a random term related to error component $c \in \mathcal{C}$
ε_{ij}	iid EV

Maximum Capture Problem with Flexible Substitution Patterns (MCPFS)

$f\left(\eta\,|\theta\,\right)$: $|\mathcal{C}|$-dimensional density function

Mixed multinomial logit model (MXL)

$$p_{ij}^{\mathsf{MXL}} = \int_{\eta} \left(\frac{e^{v_{ij} + \sum_{c \in \mathcal{C}} \eta_c h_{ijc}}}{\sum_{m \in \mathcal{M}} e^{v_{im} + \sum_{c \in \mathcal{C}} \eta_c h_{imc}}} \right) f\left(\eta\,|\theta\,\right) d\eta \tag{9}$$

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Problem **P1**: MCPFS

$$\text{Maximize } F^{\mathsf{P1}} = \sum_{i \in \mathcal{I}} q_i \sum_{j \in \mathcal{J}} \int_\eta \left(\frac{Y_j e^{v_{ij} + \sum_{c \in \mathcal{C}} \eta_c h_{ijc}}}{\sum_{m \in \mathcal{M}} Y_m e^{v_{im} + \sum_{c \in \mathcal{C}} \eta_c h_{imc}}} \right) f(\eta \,|\theta\,)\, d\eta \tag{10}$$

subject to (6) and (7)

Logit-Smoothed Simulation Procedure (A1)

Rewrite (9) as

$$p_{ij}^{\mathsf{MXL}} = \int_{\eta} \pi_{ij}\left(\eta\right) f\left(\eta\,|\theta\,\right) d\eta \tag{11}$$

where

$$\pi_{ij}\left(\eta\right) = \frac{e^{v_{ij}+\sum_{c\in\mathcal{C}}\eta_c h_{ijc}}}{\sum_{m\in\mathcal{M}} e^{v_{im}+\sum_{c\in\mathcal{C}}\eta_c h_{imc}}} \tag{12}$$

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Given θ, for each $i \in \mathcal{I}$:

- draw S different realizations of η
- compute $\pi_{ij}\left(\eta^s\right)$ for all draws

Simulated MXL choice probabilities (A1)

$$\check{p}_{ij}^{\mathsf{MXL}} = \frac{1}{S}\sum_{s=1}^{S} \pi_{ij}\left(\eta^s\right) \tag{13}$$

Concept

- For a given η (12) are MNL choice probabilities
- Apply linear reformulation of Haase (2009), *working paper*
- **A1** can be exploited based on idea of Müller & Haase (2014), *BuR*:
  - IIA applies to each draw, but not to the average over all draws

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Consider parameter "substitution with competitor"

$$\varphi_{ijs} = \frac{e^{v_{ij}+\sum_{c\in\mathcal{C}}\eta_c^s h_{ijc}}}{\sum_{k\in\mathcal{M}\setminus\mathcal{J}} e^{v_{ik}+\sum_{c\in\mathcal{C}}\eta_c^s h_{ikc}}} \tag{14}$$

and the non-negative variables

X_{ijs} probability of customers in $i \in \mathcal{I}$ patronizing a facility located at $j \in \mathcal{J}$ for draw s

$\tilde{X}_{is}$ denoting the cumulative choice probabilities of the facilities of the competitor(s) for a given demand point $i \in \mathcal{I}$ for draw s

Linear Deterministic Equivalent of MCPFS using A1

Problem **P2**: MCPFSA1

$$\text{Maximize } F^{\mathsf{P2}} = \sum_{i \in \mathcal{I}} q_i \sum_{j \in \mathcal{J}} \frac{1}{S} \sum_{s=1}^{S} X_{ijs} \tag{15}$$

subject to (6), (7), and

$$\tilde{X}_{is} + \sum_{j \in \mathcal{J}} X_{ijs} \leq 1 \quad \forall\, i \in \mathcal{I}, s = 1, \ldots, S \tag{16}$$

$$(1 + \varphi_{ijs})\, X_{ijs} - \varphi_{ijs} Y_j \leq 0 \quad \forall\, i \in \mathcal{I},\ j \in \mathcal{J}, s = 1, \ldots, S \tag{17}$$

$$X_{ijs} - \varphi_{ijs} \tilde{X}_{is} \leq 0 \quad \forall\, i \in \mathcal{I},\ j \in \mathcal{J}, s = 1, \ldots, S \tag{18}$$

$$X_{ijs} \geq 0 \quad \forall\, i \in \mathcal{I},\ j \in \mathcal{J}, s = 1, \ldots, S \tag{19}$$

$$\tilde{X}_{is} \geq 0 \quad \forall\, i \in \mathcal{I}, s = 1, \ldots, S \tag{20}$$

- $\sum_{s=1}^{S} X_{ijs}/S$ are the approximate MXL choice probabilities of (13)
- If $\mathcal{C} = \emptyset$ and $S = 1$, then **P2** becomes the linear reformulation of **P0**

Crude-Frequency Simulation Procedure (A2)

- The MXL (9) immediately follows from general choice model (3)
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- draw S different realizations of η and ε_{ij}
- compute

$$u_{ij}^s = v_{ij} + \sum_{c \in \mathcal{C}} \eta_c^s h_{ijc} + \varepsilon_{ij}^s \tag{21}$$

and

$$a_{ijs} = \begin{cases} 1, & \text{if } u_{ij}^s > u_{ik}^s \ \forall\, k \in \mathcal{M}, k \neq j \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

Simulated MXL choice probabilities (A2)

$$\check{p}_{ij}^{\text{MXL}} = \frac{1}{S} \sum_{s=1}^{S} a_{ijs} \tag{23}$$

Linear Deterministic Equivalent of MCPFS using A2

Based on Haase & Müller (2013), *Omega*, redefine parameter (22) as

$$a_{ijs} = \begin{cases} 1, & \text{if } u_{ij}^s > u_{ik}^s \ \forall\, k \in \mathcal{M} \setminus \mathcal{J}, k \neq j \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

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Problem P3: MCPFSA2

$$\text{Maximize } F^{\text{P3}} = \sum_{i \in \mathcal{I}} \frac{q_i}{S} \sum_{s=1}^{S} Z_{is} \tag{25}$$

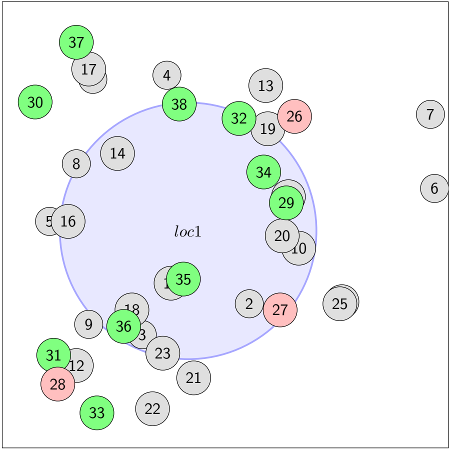
subject to (6), (7), and

$$Z_{is} - \sum_{j \in \mathcal{J}} a_{ijs} Y_j \leq 0 \qquad \forall\, i \in \mathcal{I}, s = 1, \ldots, S \tag{26}$$

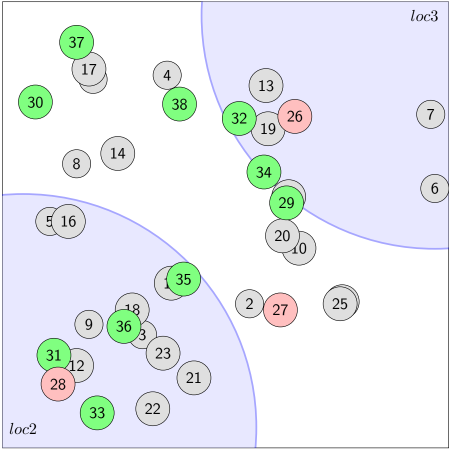
$$Z_{is} \in [0, 1] \qquad \forall\, i \in \mathcal{I}, s = 1, \ldots, S \tag{27}$$

Synthetic Data

$$\begin{aligned} v_{ij} &= -\beta \cdot d_{ij} \\ \eta_c^s &\sim \mathcal{N}\left(0, \vartheta^2\right) \end{aligned}$$



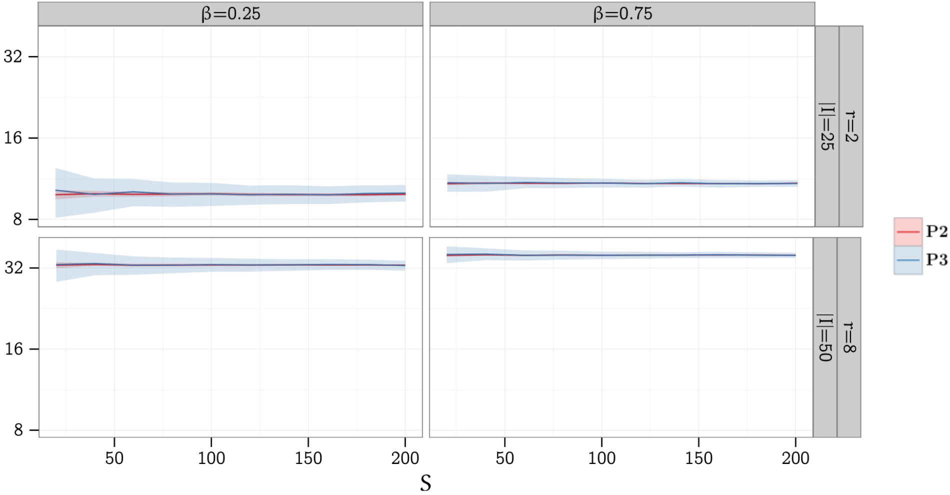
Spatial error component c=2 ($loc1$)



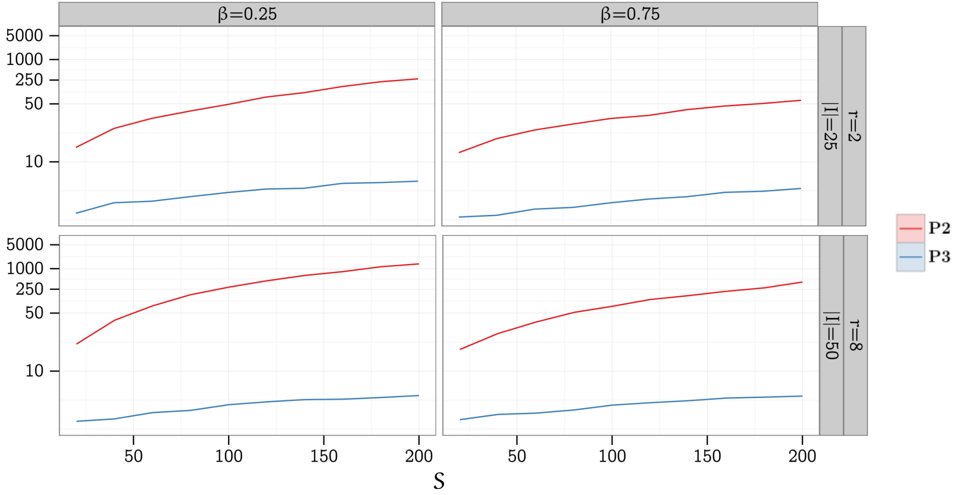
Spatial error components c=3 ($loc2$) and c=4 ($loc3$)

Sample Variance of Objective Function Value

- 10 instances
- $\vartheta = 1$, $|\mathcal{J}| = 10$
- $|\mathcal{C}| = \{1, 2\}$



Computational Effort in CPU-sec



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  - decision on continuous attributes

Q & A

Get in touch:
Sven Müller
Karlsruhe University of Applied Sciences
Transport Business Economics
www.spatial-management-science.org
@GeoOptimization

Lower Bound to P1

Optimal solution of a given problem **P#**:

$$\mathcal{J}^*\left(F^{\mathsf{P}\#*}\right) = \left\{j \in \mathcal{J} \left| y_j^* = 1 \right.\right\} \tag{28}$$

$\mathcal{M}^*$ is the corresponding set of located facilities Consider this set $\mathcal{M}^*$ in (12) such that

$$\pi_{ij}^*(\eta) = \frac{e^{v_{ij}+\sum_{c\in\mathcal{C}}\eta_c h_{ijc}}}{\sum_{m\in\mathcal{M}^*} e^{v_{im}+\sum_{c\in\mathcal{C}}\eta_c h_{imc}}}. \tag{29}$$

Now, increase in step 1 of procedure **A1** the number of draws from S to S' with $S' \gg S$. Replace $\pi_{ij}(\eta^s)$ by (29) in step 2 of **A1**. Then, compute the corresponding choice probabilities in (13). Finally,

$$F_{\mathsf{eval}}^{\mathsf{P}\#} = \sum_{i\in\mathcal{I}} q_i \sum_{j\in\mathcal{J}^*\left(F^{\mathsf{P}\#}\right)} \check{p}_{ij}^{\mathsf{MXL}} \tag{30}$$

is the evaluated objective function value for problem **P#**. If $F^{\mathsf{P}\#} \approx F_{\mathsf{eval}}^{\mathsf{P}\#}$ then S might be sufficiently large.

Solution Quality

Sample Variance

$$\sigma_S^2\left(\mathbf{P2}\right) = \frac{1}{S-1}\sum_{s=1}^{S}\left(F^{\mathsf{P2}*} - \sum_{i\in\mathcal{I}} q_i \sum_{j\in\mathcal{J}} x^*_{ijs}\right)^2 \tag{31}$$

$$\sigma_S^2\left(\mathbf{P3}\right) = \frac{1}{S-1}\sum_{s=1}^{S}\left(F^{\mathsf{P3}*} - \sum_{i\in\mathcal{I}} q_i z^*_{is}\right)^2 \tag{32}$$

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$\overline{F^{\mathsf{P}\#*}}$ average over N solutions

"**Solution Variance**" Romauch & Hartl (2005), *LNCS*

$$\sigma_{S,N}^2\,(\mathbf{P\#}) = \frac{1}{N\,(N-1)} \sum_{n=1}^{N} \left(F_n^{\mathsf{P}\#*} - \overline{F^{\mathsf{P}\#*}}\right)^2 \tag{33}$$

Data

- Cartesian coordinates randomly from uniform $[0, 30]$

d_{ij} rectangular distance between i and j

$v_{ij} = -\beta \cdot d_{ij}$

- $|\mathcal{C}| < 4$
- $\eta_c^s \sim \mathcal{N}\left(0, \vartheta^2\right)$

Error components

h_{im1} =1, if $m \in \mathcal{J}$ (firm nest)

h_{im2} =1, if $d_{loc1,m} < 0.55 \cdot R$ (city center nest)

h_{im3} =1, if $d_{loc2,m} < R$ (suburb A nest)

h_{im4} =1, if $d_{loc3,m} < R$ (suburb B nest)

With R as a radius to be specified

$|\mathcal{C}| = 1$, then $\mathcal{C} = \{1\}$

$|\mathcal{C}| = 2$, then $\mathcal{C} = \{1, 2\}$

$|\mathcal{C}| = 3$, then $\mathcal{C} = \{1, 3, 4\}$

Set Up Study 1

Hypothesis

P2 (MXL) and **P0** (MNL) yield identical, or at least similar results.

- 10 instances
- $|\mathcal{I}| = 50$
- $\beta = 0.25$
- $S = 50$
- **P0lin** linear reformulation of **P0**

Solution evaluation

$$F_{\text{eval}}^{\text{P0lin*}} = \sum_{i \in \mathcal{I}} q_i \sum_{j \in \mathcal{J}^*\left(F^{\text{P0lin}}\right)} \check{p}_{ij}^{\text{MXL}}\left(\text{MNL}\right).$$

Result Study 1

			P0lin		P2			
β	$\lvert\mathcal{C}\rvert$	ϑ	$F^{\text{P0lin}*}_{\text{eval}}$	CPU	$F^{\text{P2}*}$	CPU	DevOFV	DevSol
0.05	1	0.50	26.03	0.43	26.03	414.25	0.00	0.04
0.05	1	1.00	25.92	0.43	25.92	353.15	0.00	0.00
0.05	1	1.50	25.90	0.46	25.90	312.89	0.00	0.02
0.05	1	3.00	25.60	0.45	25.60	210.80	0.00	0.02
1.50	3	0.50	31.75	0.09	31.42	12.14	0.33	0.00
1.50	3	1.00	33.88	0.09	33.26	17.72	0.65	0.04
1.50	3	1.50	30.77	0.09	30.67	18.16	0.15	0.02
1.50	3	3.00	30.64	0.08	30.04	15.55	0.60	0.08

$$\text{DevOFV} = \left| F^{\text{P2}*} - F^{\text{P2}^{\text{MNL}}*} \right|, \ \text{DevSol} = \frac{\mathcal{J}^*\left(F^{\text{P2}*}\right) \triangle \mathcal{J}^*\left(F^{\text{P2}^{\text{MNL}}}{}_*\right)}{2 \cdot r}$$

Set Up Study 2

Hypothesis

P3$^{\text{MNL}}$ approximates **P2**$^{\text{MNL}}$ - and hence **P0** - arbitrarily close.

- 10 instances
- **P#**$^{\text{MNL}}$ MNL-variant of **P2** and **P3**
- **P3** evaluation:

$$F_{\text{eval}}^{\text{P3}^{\text{MNL}}*} = \sum_{i \in \mathcal{I}} q_i \sum_{j \in \mathcal{J}^*\left(F^{\text{P3}}\right)} \check{p}_{ij}^{\text{MNL}}$$

Result Study 2

$\lvert\mathcal{I}\rvert$	$\lvert\mathcal{J}\rvert$	r	β	S	**P2**$^{\text{MNL}}$ $\frac{F^{\text{P2}^{\text{MNL}}*}}{\lvert\mathcal{I}\rvert}$	**P2**$^{\text{MNL}}$ CPU	**P3**$^{\text{MNL}}$ $\frac{F^{\text{P3}^{\text{MNL}}*}}{\lvert\mathcal{I}\rvert}$	**P3**$^{\text{MNL}}$ $\frac{F_{\text{eval}}^{\text{P3}^{\text{MNL}}*}}{\lvert\mathcal{I}\rvert}$	**P3**$^{\text{MNL}}$ CPU	DevOFV	DevSol
25	10	3	0.25	10	0.57	9.28	0.60	0.57	5.37	0.00	0.17
25	10	3	0.25	50	0.57	9.28	0.57	0.57	6.51	0.00	0.27
25	10	3	0.25	100	0.57	9.28	0.57	0.57	5.15	0.00	0.10
25	10	3	0.75	10	0.66	4.67	0.65	0.65	0.03	0.01	0.27
25	10	3	0.75	50	0.66	4.67	0.66	0.66	0.03	0.00	0.00
25	10	3	0.75	100	0.66	4.67	0.66	0.66	1.00	0.00	0.00
25	10	8	0.25	10	0.50	6.98	0.51	0.50	0.03	0.00	0.06
25	10	8	0.25	50	0.50	6.98	0.51	0.50	0.84	0.00	0.02
25	10	8	0.25	100	0.50	6.98	0.50	0.50	0.03	0.00	0.01
100	25	7	0.75	10	0.66	7.85	0.66	0.65	3.23	0.00	0.26
100	25	7	0.75	50	0.66	7.85	0.65	0.65	0.48	0.00	0.17
100	25	7	0.75	100	0.66	7.85	0.66	0.65	1.92	0.00	0.10
100	25	19	0.25	10	0.53	49.48	0.54	0.52	0.04	0.01	0.15
100	25	19	0.25	50	0.53	49.48	0.53	0.53	2.96	0.00	0.08
100	25	19	0.25	100	0.53	49.48	0.53	0.53	0.22	0.00	0.06
100	25	19	0.75	10	0.55	8.84	0.56	0.55	0.04	0.00	0.08
100	25	19	0.75	50	0.55	8.84	0.55	0.55	0.88	0.00	0.05
100	25	19	0.75	100	0.55	8.84	0.55	0.55	0.15	0.00	0.03

$$\text{DevOFV} = \left| \frac{F_{\text{eval}}^{\text{P3}^{\text{MNL}}*}}{\lvert\mathcal{I}\rvert} - \frac{F^{\text{P2}^{\text{MNL}}*}}{\lvert\mathcal{I}\rvert} \right|, \quad \text{DevSol} = \frac{\mathcal{J}^*\left(F^{\text{P3}^{\text{MNL}}*}\right) \triangle \mathcal{J}^*\left(F^{\text{P2}^{\text{MNL}}*}\right)}{2 \cdot r}$$

Study 3

How fast converges $F^{\mathsf{P}\#}$ to $F^{\mathsf{P}\#}_{\mathsf{eval}}$ in S?

- 10 instances
- $\vartheta = 1$, $\beta = 1$, r=5, $S' = 5000$
- $|\mathcal{C}| = \{1, 2\}$

$\lvert\mathcal{I}\rvert$	$\lvert\mathcal{J}\rvert$	S	$F^{\mathsf{P2}}/\lvert\mathcal{I}\rvert$	$F^{\mathsf{P2}}_{\mathsf{eval}}/\lvert\mathcal{I}\rvert$	$\left\lvert F^{\mathsf{P2}}/\lvert\mathcal{I}\rvert - F^{\mathsf{P2}}_{\mathsf{eval}}/\lvert\mathcal{I}\rvert\right\rvert$
25	10	10	0.5451	0.5479	0.004360
25	10	50	0.5463	0.5478	0.003720
25	10	100	0.5470	0.5477	0.002280
25	10	150	0.5487	0.5488	0.002670
100	20	10	0.5651	0.5630	0.005470
100	20	50	0.5630	0.5633	0.001460
100	20	100	0.5630	0.5633	0.001050
100	20	150	0.5634	0.5632	0.001110

Study 5: Solution Variance

We set $|\mathcal{J}| = 10$, $r = 3$, $\beta = 1$, $|\mathcal{C}| = \{1, 2\}$, and $\vartheta = 1$