Choice Probability Generating Functions
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Motivation

To fully characterize finite-mean RUM

It can be done analogously to how MEV models are introduced in McFadden (1978):

- Define a function $G$ satisfying some conditions
- Then $U = \log G(e^{V_1}, ..., e^{V_J}) + \gamma$ satisfies

$$U = E[\max_{j \in C} U_j]$$

where $\gamma = 0.57721566$ is Euler’s constant.

by relaxing some of the conditions of $G$. 
Conclusions

Absolutely continuous finite-mean ARUM

Theorem

Corollary 1

CPGF satisfying $[G0^*]-[G5^*]$

Corollary 2

CPS satisfying $[P0^*]-[P5^*]$
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Definitions

Concepts needed:

- Random utility model (RUM)
- Additive random utility model (ARUM)
- Transformed additive random utility model (TARUM)
- Choice probability generating function (CPGF)
- Choice probability system (CPS)
A random utility model (RUM) is a vector \( U = (U_1, ..., U_J) \in \mathbb{R}^J \) with CDF \( R(u_1, ..., u_J) \). It induces an observable choice probability:

\[
P_C(j) = \Pr(U_j > U_k \text{ for } j \neq k \in C) = \int_{-\infty}^{\infty} \nabla_j R(u, ... u) \, du
\]

- Given a continuously differentiable increasing transformation \( r : \mathbb{R} \rightarrow \mathbb{R} \), the image of a RUM is a RUM with the same associated choice probability.
- This defines a class.
- A representative can be chosen such that \( E(U_j) \) is finite \( \forall j \).
Additive random utility model

Given:
- A finite-mean RUM with CDF \( R(u_1, \ldots, u_J) \) and choice probability \( P_C^0(j) \)
- \( \mathbf{m} = (m_1, \ldots, m_J) \in \mathbb{R}^J \) a location vector

Then the family,
\[
U = (U_1, \ldots, U_J) = \mathbf{m} + U^0
\]
is an additive random utility model (ARUM). Each element is a finite-mean RUM with CDF \( R(u_1 - m_1, \ldots, u_J - m_J) \) and choice probability
\[
P_C(j|\mathbf{m}) = \Pr(U_j \geq U_k \text{ for } k \in C) = \Pr(U_j^0 + m_j \geq U_k^0 + m_k \text{ for } k \in C)
\]
Definitions

Transformed additive random utility model (TARUM)

Given

- $r_j$ a continuously differentiable increasing transformation with a continuously differentiable inverse,
- $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_J)$ with an absolutely continuous CDF $F(\varepsilon)$, and
- $m = (m_1, \ldots, m_J)$ a location vector.

A finite-mean family of RUM defined implicitly by $\varepsilon_j = r_j(U_j - m_j)$ is a finite-mean transformed additive random utility model (TARUM).
Every RUM has a representation that can be embedded in an ARUM with finite mean, or an observationally equivalent TARUM.
Choice probability system (CPS)

A complete choice probability system is a family of non-negative functions $P_c(j|m)$ for $j = 1, \ldots, J$ that sum up to one.

A partial choice probability system is a family of non-negative functions $P_c(j|m)$ defined for $j = 1, \ldots, k < J$ whose sum does not exceed 1.
A choice probability generating function (CPGF) is a function $G : [0, +\infty)^J \rightarrow [0, +\infty]$ with the following properties:

- **G1** Weak alternating signs property (ASP)
- **G2** Homogeneity
- **G3** Boundary
- **G4** Integrability

The gradient

$$P_c(j|m) = \frac{\partial \ln G(e^m)}{\partial m_j}$$

is the choice probability generated by the CPGF $G$.

Properties [G1]-[G4] are equivalent to another set of properties [G0*]-[G5*]
Theorem

Given

- A finite-mean TARUM $\varepsilon_j = r_j(U_j - m_j)$ with $\varepsilon = (\varepsilon_1, ..., \varepsilon_J)$ with CDF $F(\varepsilon)$
- An observationally equivalent ARUM $U_j = m_j + \zeta_j$ with $\zeta = (\zeta_1, ..., \zeta_J)$ with CDF $R(\zeta) \equiv F(r(U - m))$

Then, an associated CPGF exists and is defined by:

$$\ln G(e^m) \equiv E[\max_{j \in C} U_j] \equiv \int_{0}^{+\infty} [1 - R(u - m)] \, du - \int_{-\infty}^{0} R(u - m) \, du$$

The choice probability implied by the TARUM satisfies:

$$P_C(j|m) = \Pr(U_j > U_k \text{ for } k \neq j) = \frac{\partial \ln G(e^m)}{\partial m_j}$$

The reciprocal is also true.
Theorem

Absolutely continuous finite-mean ARUM

Theorem

Corollary 1

Corollary 2

CPGF satisfying $[G0^*]-[G5^*]$

CPS satisfying $[P0^*]-[P5^*]$
Corollary 1

A complete or partial CPS is ARUM-consistent if and only if it satisfies:

P1 Alternating signs property (ASP)
P2 Homogeneity
P3 Boundary
P4 Integrability

These can be expressed as another set of properties [P0*]-[P5*]
Corollary 1

Absolutely continuous finite-mean ARUM

Theorem

Corollary 1

CPGF satisfying \([G0^*]-[G5^*]\)

Corollary 2

CPS satisfying \([P0^*]-[P5^*]\)
Corollary 2

If $P_C(1|\mathbf{m})$ is the first probability in a CPS satisfying $[P0^*]-[P5^*]$ then

$$h(\mathbf{m}) = m_1 + \int_{m_1}^{+\infty} [1 - P(1|v_1, \mathbf{m}_{-1})]$$

is a CPGF satisfying $[G0^*]-[G5^*]$ and whose gradient gives the partial CPS.

Notation: Given $\mathbf{m} = (m_1, \ldots, m_J)$, $\mathbf{m}_{-j} = (m_1, \ldots, m_{j-1}, m_{j+1}, \ldots, m_J)$ denotes a vector excluding component $j$. 
Corollary 2

Absolutely continuous finite-mean ARUM

Theorem

Corollary 1

CPGF satisfying [G0*]-[G5*]

Corollary 2

CPS satisfying [P0*]-[P5*]
A choice probability $P_C(1|m)$ for a single alternative that satisfies $[P0^*]-[P5^*]$ (for consistency with an ARUM) can be used to determine the CPGF from which the choice probabilities for the rest of the alternatives can be determined.
Simple example: logit model

Notation:

\[ h(m) = \ln G(e^m) \]

Corollary 2:

\[ h(m) = m_1 + \int_{m_1}^{+\infty} [1 - P(1|v_1, m-1)] \]

Multinomial logit:

\[ P_C(1|m) = \frac{e^{m_1}}{\sum_{j=1}^{J} e^{m_j}} \]
Simple example: logit model

From one probability we can recover \( h(\mathbf{m}) \):

\[
h(\mathbf{m}) = m_1 + \int_{m_1}^{+\infty} 1 - P(1|\nu_1, m_{-1}) \, d\nu_1
\]

\[
= m_1 + \int_{m_1}^{+\infty} 1 - \frac{e^{\nu_1}}{e^{\nu_1} + \sum_{j=2}^{J} e^{m_j}} \, d\nu_1
\]

\[
= m_1 + \left[ \nu_1 - \ln \left( e^{\nu_1} + \sum_{j=2}^{J} e^{m_j} \right) \right]_{\nu_1 = m_1}^{\nu_1 = +\infty}
\]

\[
= m_1 - m_1 + \ln \left( e^{m_1} + \sum_{j=2}^{J} e^{m_j} \right)
\]

\[
= \ln \left( \sum_{j=1}^{J} e^{m_j} \right)
\]
Simple example: logit model

And from $h(m)$ the rest of probabilities:

$$P_C(i|m) = \frac{\partial}{\partial m_i} h(m) = \frac{\partial}{\partial m_i} \ln \left( \sum_{j=1}^{J} e^{m_j} \right) = \frac{e^{m_i}}{\sum_{j=1}^{J} e^{m_j}}$$
Conclusions revisited
Conclusions revisited

Conclusions

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Future work
Future work

To give an application to this theoretical work. But... how?

- To use the CPGF approach to propose new models
- To take advantage of the fact that full information on choice probabilities is contained in only one of them:
  1. Specify the choice probability for one alternative
  2. Computation of the CPGF and its gradient.
  3. This CPGF could then be used to calculate willingness to pay (WTP) for a policy change
Thanks for your attention!

Questions?
Properties [G1]-[G4]

[G1] Weak ASP
\( \ln G(y) \) satisfies the ASP, so that for any permutation \( \sigma \) of \( (1, \ldots, J) \) and \( k = 1, \ldots, J \), its mixed partial derivatives are independent of the order of differentiation and

\[
\chi_{\sigma:k}(\ln y) \equiv (-1)^k \nabla_{\sigma:k} \ln G(y) \geq 0
\]

[G2] Homogeneity
For each \( \lambda > 0 \) and \( y \in [0, +\infty]^J \), \( G(\lambda y) = \lambda G(y) \)
Properties [G1]-[G4]

[G3] Boundary
G(0)=0, and for $j = 1, \ldots, J$, if $1_{(j)}$ denotes a unit vector with the $j$th component equal to one, then $\lim_{\lambda \to \infty} G(\lambda 1_{(j)}) \to \infty$

[G4] Integrability
Let $\mathbf{m} = (m_1, \ldots, m_J)$, $\mathbf{e}^\mathbf{m} = (e^{m_1}, \ldots, e^{m_J})$, $L = \{\mathbf{m} \in \mathbb{R}^J | \sum_{j=1}^J m_j = 0\}$. Then the following holds:

$$\int_{L} \chi_{1, \ldots, J}(\mathbf{m}) \, d\mathbf{m} = J^{-1}$$

$$\int_{L} |m_j| \chi_{1, \ldots, J}(\mathbf{m}) \, d\mathbf{m} < +\infty$$

$$\max_{k \neq j} m_k - m_j \to \infty \implies \chi_j(\mathbf{m}) \to 0$$
Properties \([G0^*]-[G5^*]\)

Let \(h(m) = \ln G(e^m)\), then \([G1]-[G4]\) are equivalent to \([G0^*]-[G5^*]\), where:

\([G0^*]\) \(h(m)\) is a convex function whose mixed partial derivatives exist

\([G1^*] \) \((-1)^{k-1} h_{j_1, \ldots, j_k}(m) \geq 0\) for any \(k = 1, \ldots, J\) and distinct indices \(j_1, \ldots, j_k\)

\([G2^*]\) \(h(m - \gamma) = h(m) - \gamma\) for any scalar \(\gamma\)
Properties \([G0^*]-[G5^*]\)

\[\text{[G3^*]} \lim_{m_j \to -\infty} \frac{h(m)}{m_j} = 0, \]
\[\lim_{m_j \to -\infty} m_j h_j(m) = 0, \]
\[\lim_{m_j \to +\infty} \frac{h(m)}{m_j} = 1, \]
\[\lim_{m_j \to +\infty} m_j [1 - h_j(m)] = 0, \]
\[\int_{v=-\infty}^{0} h_j(m_j, m_{-j}) \, dm_j < +\infty, \text{ and} \]
\[\int_{m_j=0}^{+\infty} [1 - h_j(m_j, m_{-j})] \, dm_j < +\infty. \]
Properties $[G0^*]-[G5^*]$

$[G4^*]$  \(-1\)^{J-1} \int_{m_{\cdot j}=-\infty}^{+\infty} \frac{\partial^J h(m)}{\partial m_1 \cdots \partial m_{\cdot j}} \, dm_{\cdot j} = 1 \text{ for each } m_j$

$[G5^*]$  \(- \int_{m_k=-\infty}^{+\infty} |m_k| h_{jk}(-\infty, \ldots, -\infty, m_j, -\infty, \ldots, -\infty, m_k, -\infty, \ldots, -\infty) \, dm_k < +\infty \text{ for each } m_j \text{ and } k \neq j\)
Appendix

Properties [P0*]-[P5*]

[P0*]
- Mixed partial derivatives of $P_C(j|m)$ with respect to $m$ exist
- $P_C(j|m)$ is monotone, non-decreasing in $m_j$ and monotone non-increasing in $m_{-j}$
- $\partial P_C(j|m)/\partial m_i = \partial P_C(i|m)/\partial m_j$ for $1 \leq i, j \leq k$

[P1*] $(-1)^n \partial^n P_C(j_0|m)/\partial m_{j_1}...\partial m_{j_n} \geq 0$ for any $n$ and distinct indices $j_0,...,j_k$, with $j_0 \leq k$

[P2*] $P_C(j|m - \gamma) \equiv P_C(j|m)$ for any scalar $\gamma$
Properties [P0*]-[P5*]

[P3*] \( \lim_{m_j \to -\infty} m_j P_C(j|m) = 0, \)
\( \lim_{m_j \to +\infty} m_j [1 - P_C(j|m)] = 0, \)
\( \int_{-\infty}^{0} P_C(j|m_j, m_{-j}) \, dm_j < +\infty, \) and
\( \int_{0}^{+\infty} [1 - P_C(j|m_j, m_{-j})] \, dm_j < +\infty \)

[P4*] \((-1)^{J-1} \int_{m_{-j} = -\infty}^{+\infty} \frac{\partial^{J-1} P_C(j|m)}{\partial m_1 \partial m_{j-1} \partial m_{j+1} \ldots \partial m_J} \, dm_{-j} = 1 \) for each \( m_j \)

[P5*] \((-1)^{J-1} \int_{m_i = -\infty}^{+\infty} |m_i| \frac{\partial P_C(j|m)}{\partial m_k} \, dm_i < +\infty \) for each \( m_j \) and \( k \neq j \)
Definition: ARUM-consistent

A complete or partial CPS is **ARUM-consistent** if there exists an ARUM with CDF $F$ such that for $1 \leq j \leq k \leq J$,

$$P_C(j|m) = \text{Prob}(U_j + m_j \geq U_k + m_k \text{ for } k \neq j)$$

$$\equiv \int_{-\infty}^{+\infty} F_j(v - m_1, \ldots, v - m_j) \, dv$$