CIVIL-557

# Decision Aid Methodologies In Transportation

## Lecture 2: Linear Programming, Duality

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# Container Storage Problem

## Third model: the good one

- Input data:

$a_i$: Initial number of stored containers in block $i$

$N$: Number of new containers expected to arrive for storage in this period

$B$: Total number of blocks in the storage yard

$A$: Number of storage positions in each block

- Decision variables:

$x_i$: Number of arriving containers in this period to be stored to block $i$

# Container Storage Problem

## Third model: the good one

- Objective function:
  - The fill-ratio in the whole yard at the end of this period will be:

$$F = \frac{N + \sum_i a_i}{A \times B}$$

  - If the fill-ratios in all the blocks at the end of this period are all equal, they will all be equal to $F$
  - This policy determines $x_i$ to guarantee that the fill-ratio in each block $i$ will be as close to $F$ as possible by the end of this period

$$f_i = \frac{a_i + \boldsymbol{x_i}}{A}$$

# Container Storage Problem

## Third model: the good one

- Objective function:
  - Fill-ratio in each block $i$ ($\boldsymbol{f_i}$) should be as close as possible to $\boldsymbol{F}$:

Minimize $|\boldsymbol{f_i} - \boldsymbol{F}|$ for all $i$

$$\boldsymbol{f_i} = \frac{a_i + \boldsymbol{x_i}}{A} \qquad \boldsymbol{F} = \frac{N + \sum_i a_i}{A \times B}$$

$\longleftrightarrow$ Minimize $|\frac{\boldsymbol{a_i}+\boldsymbol{x_i}}{\boldsymbol{A}} - \boldsymbol{F}|$ for all $i$

$\longleftrightarrow$ Minimize $|\frac{\boldsymbol{a_i}+\boldsymbol{x_i}-\boldsymbol{A}\times\boldsymbol{F}}{\boldsymbol{A}}|$ for all $i$

$\longleftrightarrow$ Minimize $|\boldsymbol{a_i} + \boldsymbol{x_i} - \boldsymbol{A}\times\boldsymbol{F}|$ for all $i$

$\longleftrightarrow$ Minimize $\sum_i |\boldsymbol{a_i} + \boldsymbol{x_i} - \boldsymbol{A}\times\boldsymbol{F}|$

# Container Storage Problem

## Third model: the good one

- Objective function:

Minimize $\sum_i |a_i + x_i - A\times F|$

Minimize $\sum_i (\boldsymbol{\mu}_i^+ + \boldsymbol{\mu}_i^-)$

- Constraints:

$$\sum_i x_i = N$$

$$x_i \geq 0 \quad \forall i$$

$$\sum_i x_i = N$$

$$(a_i + x_i - A\times F) = \boldsymbol{\mu}_i^+ - \boldsymbol{\mu}_i^- \quad \forall i$$

$$x_i, \boldsymbol{\mu}_i^+, \boldsymbol{\mu}_i^- \geq 0 \quad \forall i$$

**Nonlinear function**

**How to make it linear ?**

# Linear programming

## Canonical form of Linear Program (LP):

$$\min_{x \in} c^T x$$ Linear objective function

subject to $Ax \leq b$ Linear inequalities

$x \geq 0,$ Non-negativity constraints

where $A \in \mathbb{R}^{m \times n}$, $\mathrm{b} \in \mathbb{R}^m$, $\mathrm{c} \in \mathbb{R}^n$ — $n$ variables, $m$ constraints

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# Linear programming

## Standard form of Linear Program (LP):

$$\min_{x \in \mathbb{R}^n} c^T x$$ Linear objective function

subject to $Ax = b$ Linear equalities

$x \geq 0,$ Non-negativity constraints

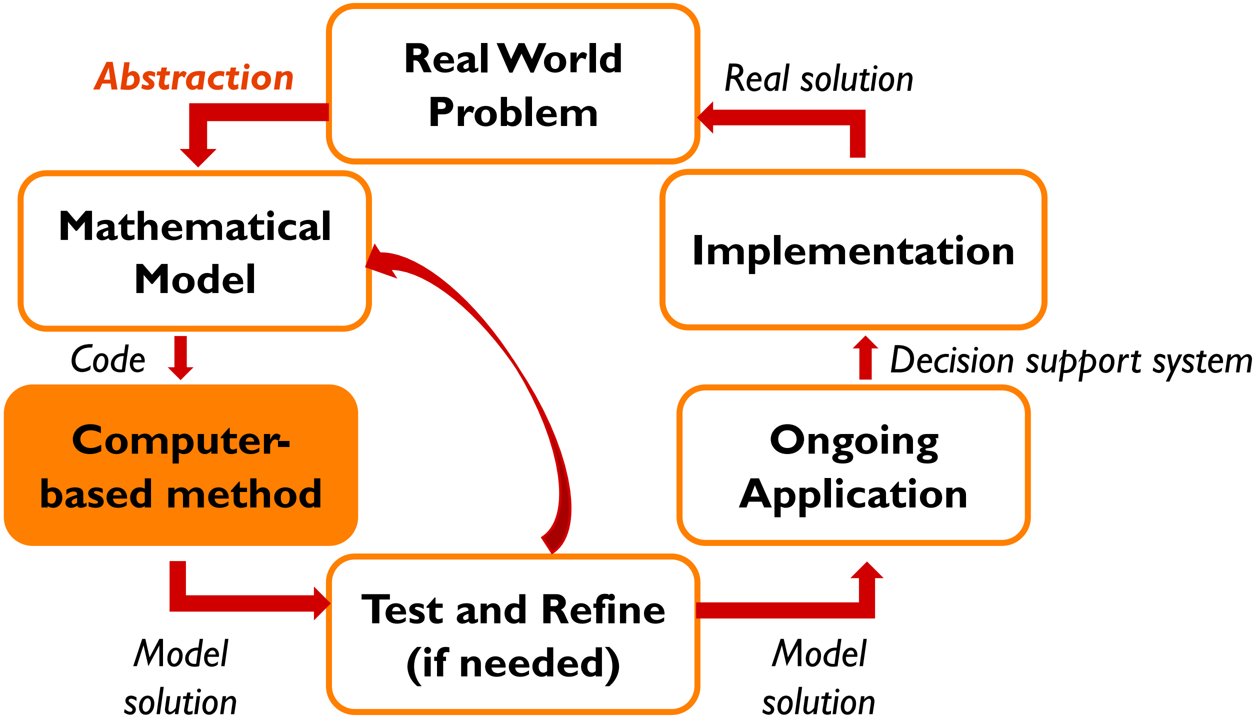
where $A \in \mathbb{R}^{m \times n}$, b $\in \mathbb{R}^m$, c $\in \mathbb{R}^n$ — $n$ variables, $m$ constraints

- Slack variables:

$$Ax \leq b \longleftrightarrow \begin{cases} Ax + y = b \\ y \geq 0 \end{cases} \qquad Ax \geq b \longleftrightarrow \begin{cases} Ax - y = b \\ y \geq 0 \end{cases}$$

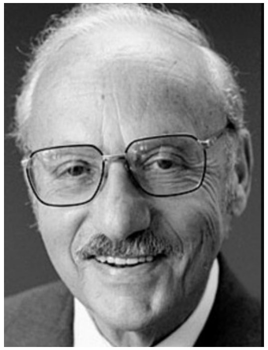
**How to solve a LP?**

# 6-steps O.R. Modeling Approach



# Linear programming

## Simplex method:



*"True optimization is the revolutionary contribution of modern research to decision processes"*

George B. Dantzig

- Dantzig (1947), *"father of linear programming"*
- The Simplex is an optimization algorithm designed to solve linear optimization problems

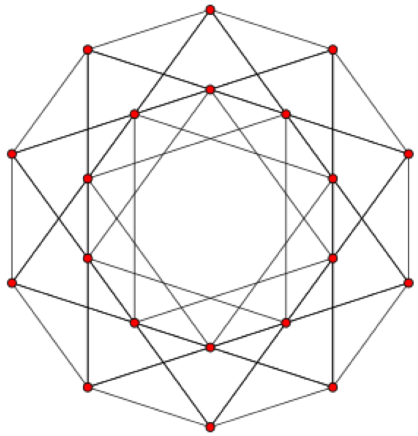
# Simplex method

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$Ax = b$$
$$x \geq 0,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$



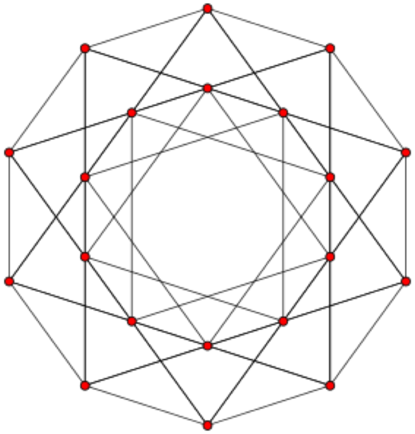
- A feasible solution is a solution that satisfies all constraints
- The feasible region is the set of all feasible solutions
- The feasible region forms a polyhedron
- An optimal solution is a feasible solution with the best objective function value

# Simplex method

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to

$$Ax = b$$
$$x \geq 0,$$



where $A \in \mathbb{R}^{m \times n}$, b $\in \mathbb{R}^m$, c $\in \mathbb{R}^n$

**Theorem:**

If the linear optimization problem has an optimal solution, there exists an **optimal vertex** of the constraint polyhedron

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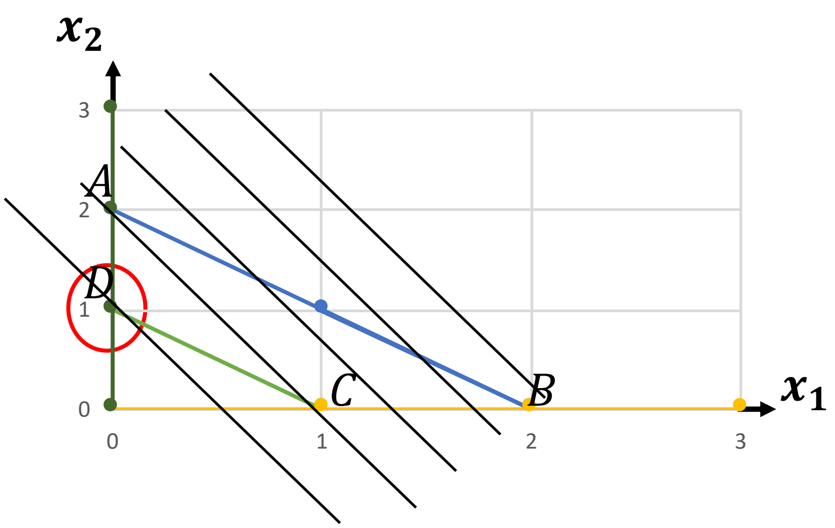
# How to find the optimal vertex?

## Graphical solution (2D example)

$$\min 2x_1 + x_2 \qquad\qquad \min 2x_1 + x_2$$
$$\mathbf{x_1 + x_2 \leq 2} \quad \rightarrow \quad \mathbf{x_1 + x_2 + y_1 = 2}$$
$$\mathbf{x_1 + x_2 \geq 1} \qquad\qquad \mathbf{x_1 + x_2 - y_2 = 1}$$
$$\mathbf{x_1 \geq 0}, x_2 \geq 0 \qquad\qquad \mathbf{x_1 \geq 0}, x_2 \geq 0$$
$$\mathbf{y_1, y_2 \geq 0}$$




## Complete enumeration

- At each vertex $n - m$ variables, called *non basic* variables, are set to 0
- The $m$ other variables, are said to be *basic*

$\boldsymbol{A}$: $x_1 = 0, x_2 = 2, y_1 = 0, y_2 = 1$ $\qquad$ $\boldsymbol{C}$: $x_1 = 1, x_2 = 0, y_1 = 1, y_2 = 0$

$\boldsymbol{B}$: $x_1 = 2, x_2 = 0, y_1 = 0, y_2 = 1$ $\qquad$ $\boldsymbol{D}$: $x_1 = 0, x_2 = 1, y_1 = 1, y_2 = 0$

The choice of a vertex of the constraint polyhedron amounts to the choice of the $n-m$ variables that are set to 0

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# Simplex method

## Back to our case study

Suppose 2 new containers are expected to arrive for storage in the next planning period of a terminal. Suppose there are only 2 blocks in the terminal, each with 20 storage spaces. For the moment, there are 6 containers in block 1 and 12 containers in block 2.

$\boldsymbol{N = 2}$

$\mathbf{B = 2}$

$\mathbf{A = 20}$

$\boldsymbol{a_1 = 6}$

$\boldsymbol{a_2 = 12}$

$\boldsymbol{F = \frac{6+12+2}{2\times 20} = 0.5}$

$$
\begin{aligned}
& \textit{Minimize} && u_1^+ + u_1^- + u_2^+ + u_2^- \\
& \textit{subject to} && \\
& && 6 + x_1 - (u_1^+ - u_1^-) = 20\times 0.5 \\
& && 12 + x_2 - (u_2^+ - u_2^-) = 20\times 0.5 \\
& && x_1 + x_2 = 2 \\
& && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{aligned}
$$

$$
\begin{aligned}
& \textit{Minimize} && u_1^+ + u_1^- + u_2^+ + u_2^- \\
& \textit{subject to} && \\
& && x_1 - u_1^+ + u_1^- = 4 \\
& && -x_2 + u_2^+ - u_2^- = 2 \\
& && x_1 + x_2 = 2 \\
& && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{aligned}
$$

$\boldsymbol{m = 3\ basic\ variables}$

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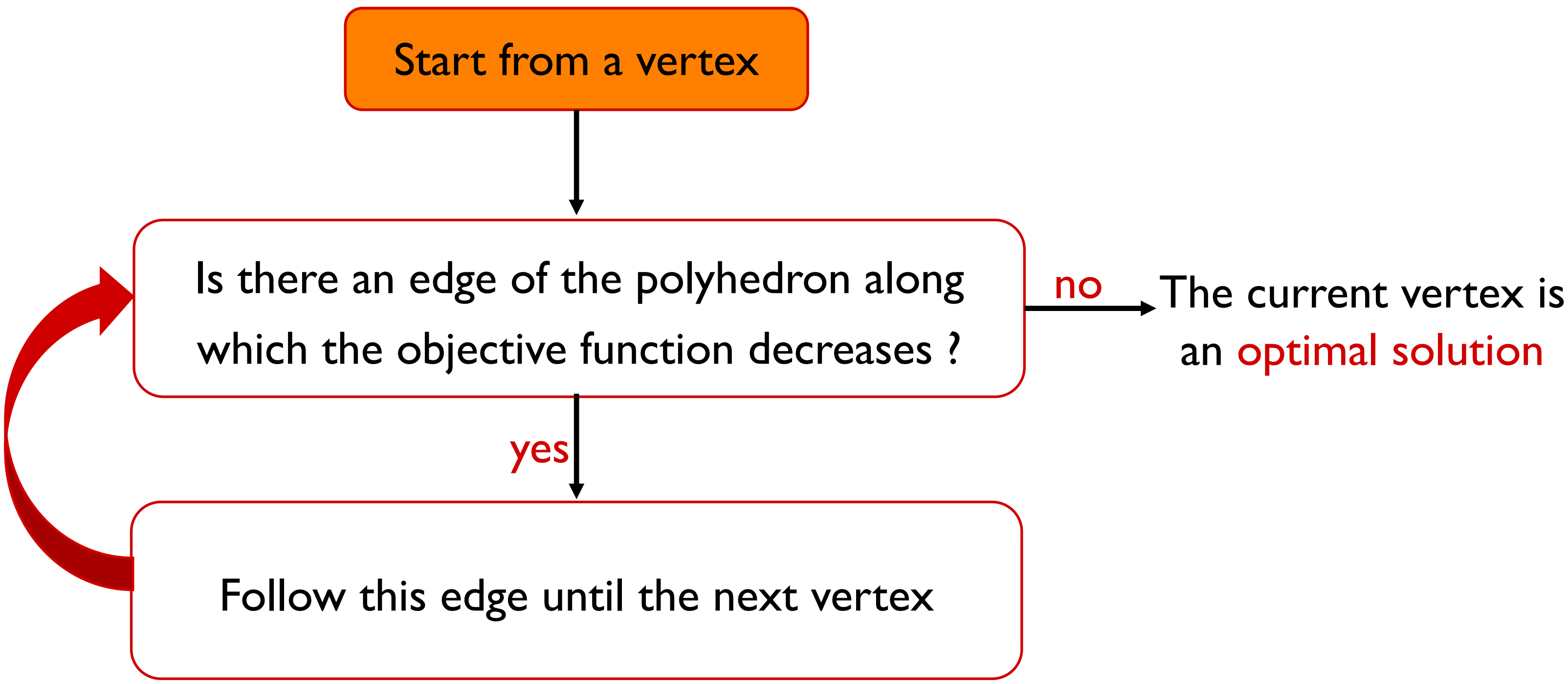
# Simplex method

## Simplex algorithm (minimization problem)

- The simplex method goes **intelligently** through the vertices



# Simplex method

## Simplex algorithm

$$\begin{array}{ll} \textit{Minimize} & u_1^+ + u_1^- + u_2^+ + u_2^- \\ \textit{subject to} & \end{array}$$

$$x_1 - u_1^+ + u_1^- = 4$$
$$-x_2 + u_2^+ - u_2^- = 2$$
$$x_1 + x_2 = 2$$
$$x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0$$

$n = 6$ variables

$m = 3$ constraints

Each vertex of the corresponding polyhedron represents **a basic <u>feasible</u> solution**

- $n - m$ variables are set to 0 *(non basic* variables*)*
- $m$ variables are > 0 *(basic* variables*)*

$$x_1 = 0$$
$$x_2 = 2$$
$$u_1^+ = 0$$
$$u_1^- = 4$$
$$u_2^+ = 4$$
$$u_2^- = 0$$

| | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | |
|---|---|---|---|---|---|---|---|
| | $B^{-1}A$ | | | | | | $B^{-1}b$ |
| | $\bar{c} = c^T - c_B^T B^{-1} A$ | | | | | | $-c_B^T B^{-1} b$ |

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# Simplex method

## Simplex algorithm

$$\begin{aligned} &\textit{Minimize} && u_1^+ + u_1^- + u_2^+ + u_2^- \\ &\textit{subject to} && \\ & && x_1 - u_1^+ + u_1^- = 4 \\ & && -x_2 + u_2^+ - u_2^- = 2 \\ & && x_1 + x_2 = 2 \\ & && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0 \end{aligned}$$

$n = 6$ variables

$m = 3$ constraints

Each vertex of the corresponding polyhedron represents **a basic feasible solution**

- $n - m$ variables are set to 0 *(non basic* variables*)*
- $m$ variables are > 0 *(basic* variables*)*

$$x_1 = 0,\quad x_2 = 2,\quad u_1^+ = 0,\quad u_1^- = 4,\quad u_2^+ = 4,\quad u_2^- = 0$$

| | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | |
|---|---|---|---|---|---|---|---|
| $u_1^-$ | 1 | 0 | -1 | 1 | 0 | 0 | 4 |
| $u_2^+$ | 1 | 0 | 0 | 0 | 1 | -1 | 4 |
| $x_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 2 |
| | | | | | | | |

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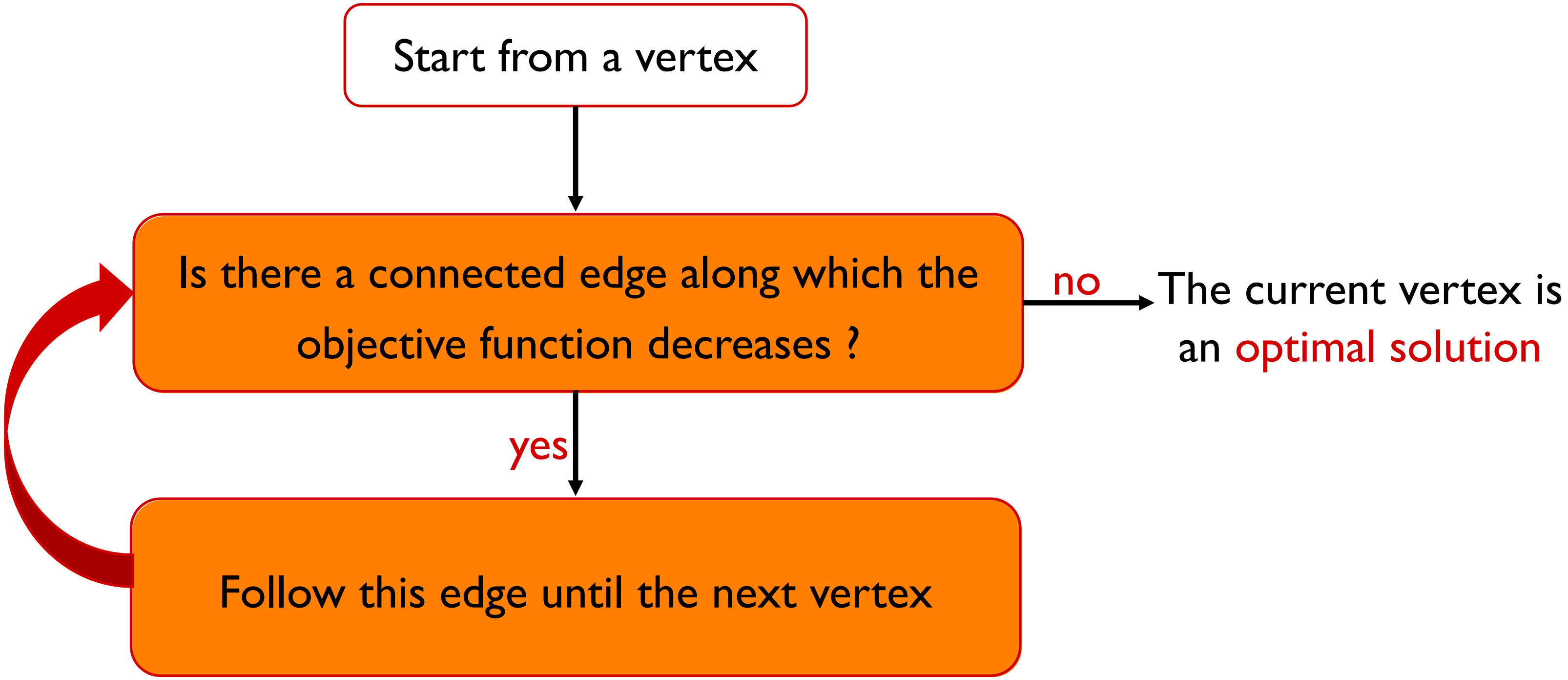
# Simplex method

## Simplex algorithm

- The simplex method goes **intelligently** through the vertices



# Simplex method

## Simplex algorithm

$Minimize \quad u_1^+ + u_1^- + u_2^+ + u_2^-$

| | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | |
|---|---|---|---|---|---|---|---|
| $u_1^-$ | 1 | 0 | -1 | 1 | 0 | 0 | 4 |
| $u_2^+$ | 1 | 0 | 0 | 0 | 1 | -1 | 4 |
| $x_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 2 |
| | $-2$ | 0 | 2 | 0 | 0 | 2 | $-8$ |

$$\boldsymbol{B} \quad \begin{matrix} u_1^- & u_2^+ & x_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{\boldsymbol{A_j^T}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{B^{-1}}} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{c_B}} = 2$$

**Reduced cost** is how the objective will change when moving along an edge direction

- $\bar{c_j} = c_j - A_J^T B^{-1} c_B$
  - ✓ $c_j$: coefficient of variable $j$ in the objective function
  - ✓ $c_B^T$: vector of the coefficients of the basic variables in the objective function
  - ✓ $B^{-1}$: inverse of the basis matrix
  - ✓ $A_j$ : column of the A matrix corresponding to the variable $j$

# Simplex method

## Simplex algorithm

$$Minimize \quad u_1^+ + u_1^- + u_2^+ + u_2^-$$

| | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | $b_j$ | |
|---|---|---|---|---|---|---|---|---|
| $u_1^-$ | 1 | 0 | -1 | 1 | 0 | 0 | 4 | $4/1 = 4$ |
| $u_2^+$ | 1 | 0 | 0 | 0 | 1 | -1 | 4 | $4/1 = 4$ |
| $x_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 2 | $2/1 = 2$ |
| | $-2$ | 0 | 2 | 0 | 0 | 2 | $-8$ | |

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{columns } u_1^-,\ u_2^+,\ x_2)$$

$$A_j^T \, B^{-1} \, c_B: \quad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2$$

- If the BFS is not optimal, then some reduced cost is negative
- How to move to a better BFS?
  - ✓ One non basic variable enters the basis:
    - ➢ A variable with a negative reduced cost
  - ✓ One basic variable leaves the basis:
    - ➢ A variable that minimizes the ratio $\frac{b_j}{a_{ij}}$, where $j$ is the column of the entering variable

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# Simplex method

## Simplex algorithm

*Minimize* $u_1^+ + u_1^- + u_2^+ + u_2^-$

| | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | |
|---|---|---|---|---|---|---|---|
| $u_1^-$ | 1 | 0 | -1 | 1 | 0 | 0 | 4 |
| $u_2^+$ | 1 | 0 | 0 | 0 | 1 | -1 | 4 |
| $x_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 2 |
| | $-2$ | 0 | 2 | 0 | 0 | 2 | -8 |

| | $x_2$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | |
|---|---|---|---|---|---|---|---|
| $u_1^-$ | | | | | | | |
| $u_2^+$ | | | | | | | |
| $x_1$ | | | | | | | |
| | | | | | | | |

- Need to update the tableau with the pivot operation:
  - ✓ The pivot column $p$ is the column of the entering variable
  - ✓ The pivot row $q$ is the row of the leaving variable
  - ✓ The pivot $T(q,p)$ is the element at the intersection of the pivot row and the pivot column

# Simplex method

## Simplex algorithm

*Minimize* $u_1^+ + u_1^- + u_2^+ + u_2^-$

| | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | |
|---|---|---|---|---|---|---|---|
| $u_1^-$ | 1 | 0 | -1 | 1 | 0 | 0 | 4 |
| $u_2^+$ | 1 | 0 | 0 | 0 | 1 | -1 | 4 |
| $x_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 2 |
| | $-2$ | 0 | 2 | 0 | 0 | 2 | $-8$ |

| | $x_2$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | |
|---|---|---|---|---|---|---|---|
| $u_1^-$ | 0 | -1 | -1 | 1 | 0 | 0 | 2 |
| $u_2^+$ | 0 | -1 | 0 | 0 | 1 | -1 | 2 |
| $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 2 |
| | 0 | 2 | 2 | 0 | 0 | 2 | -4 |

- Update the pivot row $q$ as follows:
  - ✓ $T(q,k) := \frac{T(q,k)}{T(q,p)}$
- Update the other rows as follows:
  - ✓ $T(i,k) := T(i,k) - \frac{T(i,p)T(q,k)}{T(q,p)}$
- What are the reduced costs associated with this new BFS?

All reduced costs are $\geq 0$

Optimal solution

$\boldsymbol{x_1 = 2} \quad \boldsymbol{u_1^+ = 0} \quad \boldsymbol{u_2^+ = 2}$

$\boldsymbol{x_2 = 0} \quad \boldsymbol{u_1^- = 2} \quad \boldsymbol{u_2^- = 0}$

# Simplex method

## Back to our case study

Suppose 2 new containers are expected to arrive for storage in the next planning period of a terminal. Suppose there are only 2 blocks in the terminal, each with 20 storage spaces. For the moment, there are 6 containers in block 1 and 12 containers in block 2.

$N = 2$

$\mathrm{B} = 2$

$\mathrm{A} = 20$

$a_1 = 6$

$a_2 = 12$

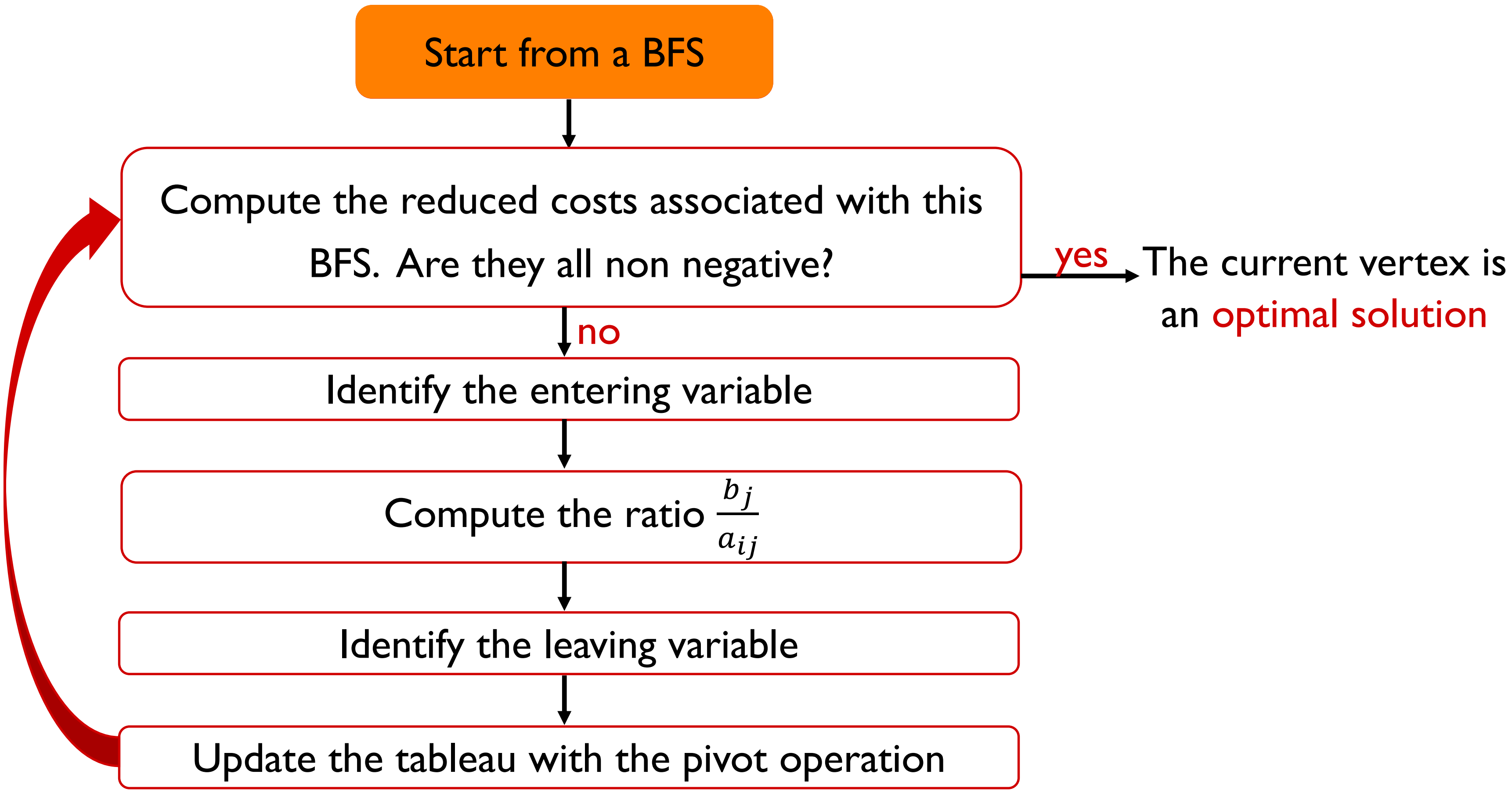
$F = \frac{6+12+2}{2\times 20} = 0.5$

- $\boldsymbol{x_1 = 2, x_2 = 0, u_1^+ = 0, u_1^- = 2, u_2^+ = 2, u_2^- = 0}$
- **The two 2 containers are assigned to block 1**

$$\boldsymbol{f_1 = \frac{6+2}{20} = 0.4, \quad f_2 = \frac{12+0}{20} = 0.6}$$

# Simplex method

## Simplex algorithm



# The real data…

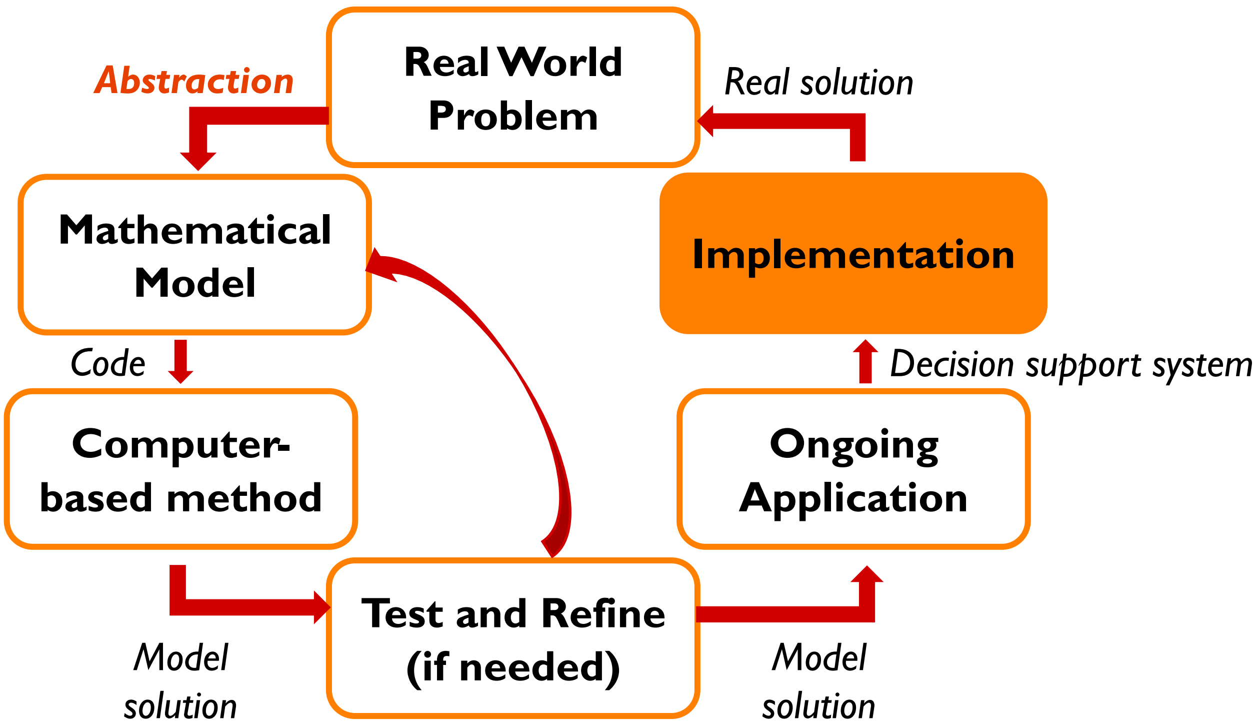
Consider a container terminal with a storage yard consisting of **100 blocks**, each with storage space to hold **600 containers**, numbered serially 1 to 100. The initial number of containers in block $i$ is:

| $i$ | $a_i$ | $i$ | $a_i$ | $i$ | $a_i$ | $i$ | $a_i$ | $i$ | $a_i$ | $i$ | $a_i$ | $i$ | $a_i$ | $i$ | $a_i$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 320 | 14 | 220 | 27 | 100 | 40 | 119 | 53 | 181 | 66 | 336 | 79 | 155 | 92 | 182 |
| 2 | 157 | 15 | 372 | 28 | 183 | 41 | 43 | 54 | 233 | 67 | 411 | 80 | 360 | 93 | 96 |
| 3 | 213 | 16 | 101 | 29 | 99 | 42 | 71 | 55 | 414 | 68 | 280 | 81 | 360 | 94 | 301 |
| 4 | 96 | 17 | 212 | 30 | 505 | 43 | 219 | 56 | 30 | 69 | 115 | 82 | 290 | 95 | 121 |
| 5 | 413 | 18 | 251 | 31 | 99 | 44 | 363 | 57 | 333 | 70 | 200 | 83 | 350 | 96 | 278 |
| 6 | 312 | 19 | 86 | 32 | 254 | 45 | 98 | 58 | 427 | 71 | 117 | 84 | 157 | 97 | 372 |
| 7 | 333 | 20 | 79 | 33 | 330 | 46 | 500 | 59 | 251 | 72 | 284 | 85 | 116 | 98 | 119 |
| 8 | 472 | 21 | 295 | 34 | 279 | 47 | 413 | 60 | 83 | 73 | 263 | 86 | 141 | 99 | 282 |
| 9 | 171 | 22 | 138 | 35 | 300 | 48 | 259 | 61 | 144 | 74 | 477 | 87 | 82 | 100 | 310 |
| 10 | 222 | 23 | 343 | 36 | 150 | 49 | 182 | 62 | 404 | 75 | 431 | 88 | 116 | | |
| 11 | 439 | 24 | 281 | 37 | 340 | 50 | 391 | 63 | 76 | 76 | 297 | 89 | 99 | | |
| 12 | 212 | 25 | 372 | 38 | 221 | 51 | 360 | 64 | 84 | 77 | 380 | 90 | 78 | | |
| 13 | 190 | 26 | 450 | 39 | 79 | 52 | 447 | 65 | 196 | 78 | 327 | 91 | 220 | | |

**Simplex solver**

The terminal estimates that in this period **15166 new containers** will be unloaded from docked vessels and dispatched to the storage yard for storage.

# 6-steps O.R. Modeling Approach



# HIT Performance Benefits

- Significant improvements in performance: a **16 % decrease in congestion**
- General improvement caused by the implementation of **decision-support systems:**
  - 30 % improvement in the GCR at HIT,
  - Average vessel turnaround time decreased from over 13 h to 9 h
  - Average number of ITs deployed/QC decreased from 8 to 4
  - HIT's annual throughput has gone from about 4 million TEUs in 1995 to about 6 million in 2002

# Simplex method

## How to find an initial basic feasible solution (BFS)?

$$
\begin{aligned}
&\textit{Minimize} \quad && u_1^+ + u_1^- + u_2^+ + u_2^- \\
&\textit{subject to} && \\
& && x_1 - u_1^+ + u_1^- = 4 \\
& && -x_2 + u_2^+ - u_2^- = 2 \\
& && x_1 + x_2 = 2 \\
& && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{aligned}
$$

- Is there a feasible solution at all? (the problem might be infeasible)
- If so, how to find it?

**Not always obvious…**

# Simplex method

## Find an initial basic feasible solution (BFS) – the auxiliary problem

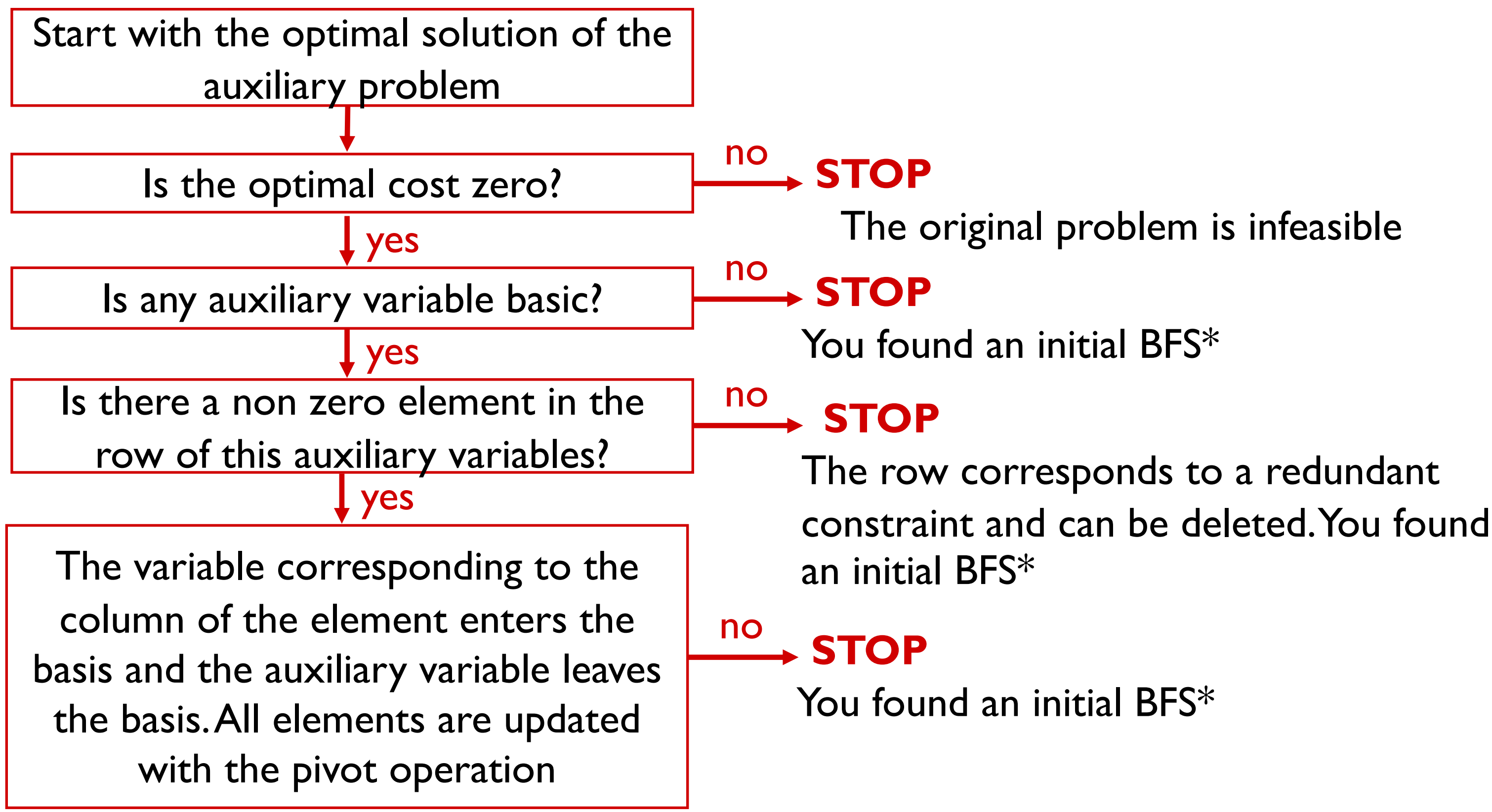
$$\begin{aligned} &\textit{Minimize} && u_1^+ + u_1^- + u_2^+ + u_2^- \\ &\textit{subject to} && x_1 - u_1^+ + u_1^- = 4 \\ & && -x_2 + u_2^+ - u_2^- = 2 \\ & && x_1 + x_2 = 2 \\ & && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0 \end{aligned}$$

$$\begin{aligned} &\textit{Minimize} && a_1 + a_2 + a_3 \\ &\textit{subject to} && x_1 - u_1^+ + u_1^- + a_1 = 4 \\ & && -x_2 + u_2^+ - u_2^- + a_2 = 2 \\ & && x_1 + x_2 + a_3 = 2 \\ & && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^-, a_1, a_2, a_3 \geq 0 \end{aligned}$$

- Introduce one auxiliary variable by constraint
- Replace the cost function by the sum of these auxiliary variables
- The original problem has a feasible solution **if and only if** the optimum value of the auxiliary problem is zero. (*Can it be negative?*)
- The optimal solution of the auxiliary problem is used to construct the initial basic feasible solution of the original problem

# Simplex method

## Find an initial basic feasible solution (BFS) – the auxiliary problem




*compute the associated reduced costs and solve the initial problem with the simplex algorithm

# Simplex method

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$\boldsymbol{F = \frac{6+12+2}{2\times 20} = 0.5}$

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\end{aligned}
$$

$$
\begin{aligned}
&\textit{Minimize} && u_1^+ + u_1^- + u_2^+ + u_2^- \\
&\textit{subject to} && \\
& && x_1 - u_1^+ + u_1^- = 4 \\
& && -x_2 + u_2^+ - u_2^- = 2 \\
& && x_1 + x_2 = 2 \\
& && x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{aligned}
$$

$\boldsymbol{m = 3 \ basic \ variables}$

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## Find an initial solution

$$\begin{array}{ll} \textit{Minimize} & a_1 + a_2 + a_3 \\ \textit{subject to} & \end{array}$$

$$x_1 - u_1^+ + u_1^- + a_1 = 4$$
$$-x_2 + u_2^+ - u_2^- + a_2 = 2$$
$$x_1 + x_2 + a_3 = 2$$
$$x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0$$

$n = 9$ variables

$m = 3$ constraints

3 basic variables

| | | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | $a_1$ | $a_2$ | $a_3$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $a_1$ | 1 | 0 | -1 | 1 | 0 | 0 | 1 | 0 | 0 | 4 /1 | |
| 1 | $a_2$ | 0 | -1 | 0 | 0 | 1 | -1 | 0 | 1 | 0 | 2 /0 | |
| 1 | $a_3$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 /1 | ← |
| $\bar{c_j} = c_j - A_j^T B^{-1} c_B$ | | -2 | 0 | 1 | -1 | -1 | 1 | 0 | 0 | 0 | -8 | $a_3$ leaves |

$x_1$ enters ↑

Are all reduced costs $\geq 0$?

$$x_1 : \bar{c_j} = 0 - \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -2$$

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# Simplex method

## Find an initial solution

| Pivot column ($p$) | | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | $a_1$ | $a_2$ | $a_3$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $a_1$ | 1 | 0 | -1 | 1 | 0 | 0 | 1 | 0 | 0 | 4 |
| | $a_2$ | 0 | -1 | 0 | 0 | 1 | -1 | 0 | 1 | 0 | 2 |
| Pivot $T(q,p)$ | $a_3$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\bar{c_j} = c_j - A_j^T B^{-1} c_B$ | | -2 | 0 | 1 | -1 | -1 | 1 | 0 | 0 | 0 | -8 |

Pivot row ($q$)

$a_3$ leaves

$x_1$ enters

$$Pivot\ row: \ T(q,k) := \frac{T(q,k)}{T(q,p)} \qquad Other\ rows: \ T(i,k) := T(i,k) - \frac{T(i,p)T(q,k)}{T(q,p)}$$

| | | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | $a_1$ | $a_2$ | $a_3$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $a_1$ | 0 | -1 | -1 | 1 | 0 | 0 | 1 | 0 | -1 | 2 |
| 1 | $a_2$ | 0 | -1 | 0 | 0 | 1 | -1 | 0 | 1 | 0 | 2 |
| 0 | $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\bar{c_j} = c_j - c_B^T B^{-1} A_j$ | | 0 | 2 | 1 | -1 | -1 | 1 | 0 | 0 | 2 | -4 |

$a_1$ leaves

$u_1^-$ enters

# Simplex method

**Find an initial solution** Pivot column($p$) Pivot $T(q,p)$

| | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | $a_1$ | $a_2$ | $a_3$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_1$ | 0 | -1 | -1 | 1 | 0 | 0 | 1 | 0 | -1 | 2 |
| $a_2$ | 0 | -1 | 0 | 0 | 1 | -1 | 0 | 1 | 0 | 2 |
| $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\bar{c_j} = c_j - A_j^T B^{-1} c_B$ | 0 | 2 | 1 | -1 | -1 | 1 | 0 | 0 | 2 | -4 |

Pivot row ($q$)

$a_1$ leaves

$u_1^-$ enters

$$Pivot\ row:\ T(q,k) := \frac{T(q,k)}{T(q,p)} \qquad Other\ rows:\ T(i,k) := T(i,k) - \frac{T(i,p)T(q,k)}{T(q,p)}$$

| | | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | $a_1$ | $a_2$ | $a_3$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $u_1^-$ | 0 | -1 | -1 | 1 | 0 | 0 | 1 | 0 | -1 | 2 |
| 1 | $a_2$ | 0 | -1 | 0 | 0 | 1 | -1 | 0 | 1 | 0 | 2 |
| 0 | $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| | $\bar{c_j} = c_j - c_B^T B^{-1} A_j$ | 0 | 1 | 0 | 0 | -1 | 1 | 1 | 0 | 1 | -2 |

$a_2$ leaves

$u_2^+$ enters

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# Simplex method

**Find an initial solution**

Pivot column($p$) Pivot $T(q,p)$

| | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | $a_1$ | $a_2$ | $a_3$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $u_1^-$ | 0 | -1 | -1 | 1 | 0 | 0 | 1 | 0 | -1 | 2 |
| $a_2$ | 0 | -1 | 0 | 0 | 1 | -1 | 0 | 1 | 0 | 2 |
| $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\bar{c_j} = c_j - A_j^T B^{-1} c_B$ | 0 | 1 | 0 | 0 | -1 | 1 | 1 | 0 | 1 | -2 |

Pivot row ($q$)

$a_2$ leaves

$u_2^+$ enters

$$Pivot\ row:\ T(q,k) := \frac{T(q,k)}{T(q,p)} \qquad Other\ rows:\ T(i,k) := T(i,k) - \frac{T(i,p)T(q,k)}{T(q,p)}$$

| | | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | $a_1$ | $a_2$ | $a_3$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $u_1^-$ | 0 | -1 | -1 | 1 | 0 | 0 | 1 | 0 | -1 | 2 |
| 0 | $u_2^+$ | 0 | -1 | 0 | 0 | 1 | -1 | 0 | 1 | 0 | 2 |
| 0 | $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| | $\bar{c_j} = c_j - c_B^T B^{-1} A_j$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |

Optimal solution

# Simplex method

## Find optimal solution

| | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | $a_1$ | $a_2$ | $a_3$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $u_1^-$ | 0 | -1 | -1 | 1 | 0 | 0 | 1 | 0 | -1 | 2 |
| $u_2^+$ | 0 | -1 | 0 | 0 | 1 | -1 | 0 | 1 | 0 | 2 |
| $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\bar{c_j} = c_j - A_j^T B^{-1} c_B$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |

Auxiliary variables are non basic

$$\begin{aligned} &Minimize\ u_1^+ + u_1^- + u_2^+ + u_2^- \\ &\text{s.t.}\quad x_1 - u_1^+ + u_1^- = 4 \\ &\quad -x_2 + u_2^+ - u_2^- = 2 \\ &\quad x_1 + x_2 = 2 \\ &\quad x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0 \end{aligned}$$

| | $x_1$ | $x_2$ | $u_1^+$ | $u_1^-$ | $u_2^+$ | $u_2^-$ | |
|---|---|---|---|---|---|---|---|
| 1 $u_1^-$ | 0 | -1 | -1 | 1 | 0 | 0 | 2 |
| 1 $u_2^+$ | 0 | -1 | 0 | 0 | 1 | -1 | 2 |
| 0 $x_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 2 |
| | 0 | 2 | 2 | 0 | 0 | 2 | -4 |

$$\mathbf{x_1 = 2, x_2 = 0, u_1^+ = 0, u_1^- = 2, u_2^+ = 2, u_2^- = 0}$$

# Linear programming

$\boldsymbol{N = 2}$
$\mathbf{B = 2}$
$\mathbf{A = 20}$
$\boldsymbol{a_1 = 6}$
$\boldsymbol{a_2 = 12}$
$\boldsymbol{F = \frac{6+12+2}{2\times 20} = 0.5}$

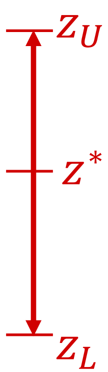
$$
\begin{array}{ll}
\textit{Minimize} & u_1^+ + u_1^- + u_2^+ + u_2^- \\
\textit{subject to} & \\
& x_1 - u_1^+ + u_1^- = 4 \\
& -x_2 + u_2^+ - u_2^- = 2 \\
& x_1 + x_2 = 2 \\
& x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{array}
$$

- $(x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^-) = (2, 0, 0, 2, 2, 0)$
- Can we test if $(x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^-) = (2, 0, 0, 2, 2, 0)$ is a feasible solution of the problem?
- Can we prove that $(x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^-) = (2, 0, 0, 2, 2, 0)$ is an optimal solution without solving the problem?
- Can we find lower and upper bounds on the optimal solution?

# Bounds

- Upper bound
  - The objective value for any feasible solution is a upper bound on $z^*$

- Lower bound




$$
\begin{aligned}
&\textit{Minimize} \quad z_L \leq u_1^+ + u_1^- + u_2^+ + u_2^- \leq z_U \\
&\textit{subject to} \\
&\quad x_1 - u_1^+ + u_1^- = 4 \\
&\quad -x_2 + u_2^+ - u_2^- = 2 \\
&\quad x_1 + x_2 = 2 \\
&\quad x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0
\end{aligned}
$$

**How can we find a lower bound ?**

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# Constraint relaxation

- General idea: incorporate constraints in the objective function

$$\begin{array}{ll} \textit{Minimize} & u_1^+ + u_1^- + u_2^+ + u_2^- \\ \textit{subject to} & \\ & x_1 - u_1^+ + u_1^- = 4 \\ & -x_2 + u_2^+ - u_2^- = 2 \\ & x_1 + x_2 = 2 \\ & x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0 \end{array}$$

$$z^* = 4$$

$$\begin{array}{ll} \textit{Minimize} & u_1^+ + u_1^- + u_2^+ + u_2^- \\ & +\lambda_1(x_1 - u_1^+ + u_1^- - 4) \\ & +\lambda_2(-x_2 + u_2^+ - u_2^- - 2) \\ & +\lambda_3(x_1 + x_2 - 2) \\ \textit{subject to} & \\ & x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- \geq 0 \end{array}$$

# Lagrangian function

$$\begin{array}{ll} Minimize & f(x) \\ subject\ to & \\ & h(x) = 0 \\ & g(x) \leq 0 \end{array}$$

- Consider the vectors $\lambda \in R^m$, and $\mu \in R^p, \mu > 0$
- The function

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \mu^T g(x)$$
$$= f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{p} \mu_j g_j(x)$$

is called *Lagrangian* or the *Lagrangian* function

# Dual function

$$L(x,\lambda,\mu) = f(x) + \lambda^T h(x) + \mu^T g(x)$$
$$= f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{p} \mu_j g_j(x)$$

- We can minimize the Lagrangian function for each fixed value of the parameters $\lambda$ and $\mu$
- The function that associates a set of parameters to the optimal value of the associated problem is called a *dual* function

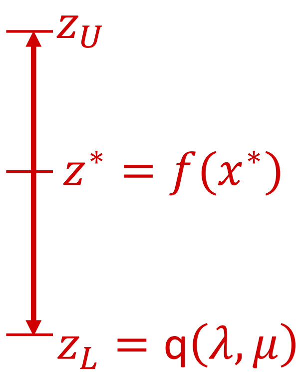
$$\mathrm{q}(\lambda,\mu) = \min_{x \in R^n} L(x,\lambda,\mu)$$

- The parameters $\lambda, \mu$ are called the dual variables, and the variables $x$ are called primal variables

# Bound from dual function

- Let $x^*$ be a feasible solution of the initial problem, and let $q(\lambda, \mu)$ be the dual function to the same problem. Consider $\lambda \in R^m$, and $\mu \in R^p, \mu \geq 0$, then

$$q(\lambda, \mu) \leq f(x^*)$$




$$q(\lambda, \mu) = \min_{x \in R^n} L(x, \lambda, \mu)$$

$$q(\lambda, \mu) \leq L(x^*, \lambda, \mu)$$

$$q(\lambda, \mu) \leq f(x^*) + \sum_{i=1}^{m} \lambda_i h_i(x^*) + \sum_{j=1}^{p} \mu_j g_j(x^*)$$

$h(x^*) = 0$

$$q(\lambda, \mu) \leq f(x^*) + \sum_{j=1}^{p} \mu_j g_j(x^*)$$

$g(x^*) \leq 0, \mu \geq 0$

$$q(\lambda, \mu) \leq f(x^*)$$

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# Duality theory

# Duality theory

- In the dual problem, the goal is to optimize the dual function, ensuring that the considered parameters $\lambda$ and $\mu \geq 0$, do not generate an unbounded problem

$$
\text{(P)} \quad \begin{array}{ll} \textit{Minimize} & f(x) \\ \textit{subject to} & \\ & h(x) = 0 \\ & g(x) \leq 0 \end{array}
$$

$$
\text{(D)} \quad \begin{array}{ll} \textit{Maximize} & \mathrm{q}(\lambda, \mu) \\ \textit{subject to} & \\ & \mu \geq 0 \\ & (\lambda, \mu) \in X_q \end{array}
$$

with $X_q = \{\lambda, \mu \,|\mathrm{q}(\lambda, \mu) > -\infty\}$

# An example

(P) $\quad Minimize \quad 2x_1 + x_2$

$$x_1 + x_2 = 1$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

$Minimize \quad 2x_1 + x_2$

$$1 - x_1 - x_2 = 0 \quad h_1(x) \longrightarrow \lambda$$
$$-x_1 \leq 0 \quad g_1(x) \longrightarrow \mu_1$$
$$-x_2 \leq 0 \quad g_2(x) \longrightarrow \mu_2$$

$$\begin{aligned} L(x_1, x_2, \lambda, \mu_1, \mu_2) &= 2x_1 + x_2 + \lambda(1 - x_1 - x_2) - \mu_1 x_1 - \mu_2 x_2 \\ &= (2 - \lambda - \mu_1)x_1 + (1 - \lambda - \mu_2)x_2 + \lambda \end{aligned}$$

# An example

$$L(x_1, x_2, \lambda, \mu_1, \mu_2) = (2 - \lambda - \mu_1)x_1 + (1 - \lambda - \mu_2)x_2 + \lambda$$

- In order for the dual function to be bounded, the coefficients of $x_1$ and $x_2$ have to be zero

$$2 - \lambda - \mu_1 = 0 \longrightarrow 2 - \lambda = \mu_1$$
$$1 - \lambda - \mu_2 = 0 \longrightarrow 1 - \lambda = \mu_2$$

- Since $\mu_1 \geq 0$, we need $\lambda \leq 2$
- Since $\mu_2 \geq 0$, we need $\lambda \leq 1$

$$X_q = \{\lambda, \mu_1, \mu_2 \mid \lambda \leq 1, \mu_1 \geq 0, \mu_2 \geq 0\}$$

- The dual function becomes

$$q(\lambda, \mu_1, \mu_2) = \lambda$$

# An example

(P) *Minimize* $2x_1 + x_2$

$$x_1 + x_2 = 1$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

(D) *Maximize* $\lambda$

$$\lambda \leq 1$$
$$\mu_1 \geq 0$$
$$\mu_2 \geq 0$$

# The dual problem – summary for LP

**Primal (P)**

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to $Ax \geq b$

$x \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, b $\in \mathbb{R}^m$, c $\in \mathbb{R}^n$

$n$ variables

$m$ constraints

**Dual (D)**

$$\max_{y \in \mathbb{R}^m} b^T y$$

subject to $A^T y \leq c$

$y \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, b $\in \mathbb{R}^m$, c $\in \mathbb{R}^n$

$m$ variables

$n$ constraints

| Primal constraint | Dual variable | Primal variable | Dual constraint |
|---|---|---|---|
| $=$ | $free$ | $\geq 0$ | $\leq$ |
| $\leq$ | $\leq 0$ | $\leq 0$ | $\geq$ |
| $\geq$ | $\geq 0$ | $free$ | $=$ |

# Duality theory

**Primal (P)**

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

subject to $Ax \geq b$

$x \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, $\mathrm{b} \in \mathbb{R}^m$, $\mathrm{c} \in \mathbb{R}^n$

**Dual (D)**

$$\max_{y \in \mathbb{R}^m} \; b^T y$$

subject to $A^T y \leq c$

$y \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, $\mathrm{b} \in \mathbb{R}^m$, $\mathrm{c} \in \mathbb{R}^n$

**Weak duality**

Let $x^*$ be the optimal solution to the primal problem and let $y^*$ be the optimal solution to the associated dual problem. Then $c^T x^* \geq b^T y^*$

- The dual function provides lower bounds on the optimal value of the problem

# Duality theory

**Primal (P)**

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

subject to $Ax \geq b$

$x \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, b $\in \mathbb{R}^m$, c $\in \mathbb{R}^n$

**Dual (D)**

$$\max_{y \in \mathbb{R}^m} \; b^T y$$

subject to $A^T y \leq c$

$y \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, b $\in \mathbb{R}^m$, c $\in \mathbb{R}^n$

**Strong duality**

Consider a linear optimization problem and its dual. If one problem has an optimal solution, so does the other one, and the optimal value of their objective functions are the same.

- It can be shown that **strong duality always holds for LPs**

# Duality theory

**Primal (P)**

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

subject to $Ax \geq b$

$x \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, $\mathrm{b} \in \mathbb{R}^m$, $\mathrm{c} \in \mathbb{R}^n$

**Dual (D)**

$$\max_{y \in \mathbb{R}^m} \; b^T y$$

subject to $A^T y \leq c$

$y \geq 0,$

where $A \in \mathbb{R}^{m \times n}$, $\mathrm{b} \in \mathbb{R}^m$, $\mathrm{c} \in \mathbb{R}^n$

| | | **Dual** | | |
|---|---|---|---|---|
| | | Optimal Solution | Unbounded | Infeasible |
| **Primal** | Optimal Solution | | | |
| | Unbounded | | | |
| | Infeasible | | | |

# Complementary slackness

**Primal (P)**

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

subject to $\quad Ax \geq b$

$\quad x \geq 0,$

where $\; A \in \mathbb{R}^{m \times n}$, $\mathrm{b} \in \mathbb{R}^m$, $\mathrm{c} \in \mathbb{R}^n$

**Dual (D)**

$$\max_{y \in \mathbb{R}^m} \; b^T y$$

subject to $\quad A^T y \leq c$

$\quad y \geq 0,$

where $\; A \in \mathbb{R}^{m \times n}$, $\mathrm{b} \in \mathbb{R}^m$, $\mathrm{c} \in \mathbb{R}^n$

- Strong duality implies that assuming a solution $x^*$ for (P) and a solution $y^*$ for (D):

  - If $x_j^* > 0$, then the $j$th constraint in (D) is binding
  - If the $j$th constraint in (D) is not binding, then $x_j^* = 0$
  - If $y_i^* > 0$, then the $i$th constraint in (P) is binding
  - If the $i$th constraint in (P) is not binding, then $y_i^* = 0$

**Complementary slackness conditions**

# Duality theory

Minimize $u_1^+ + u_1^- + u_2^+ + u_2^-$

Subject to

$$
\begin{aligned}
x_1 - u_1^+ + u_1^- &= 4 \\
-x_2 + u_2^+ - u_2^- &= 2 \\
x_1 + x_2 &= 2 \\
x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^- &\geq 0
\end{aligned}
$$

Maximize $4\,y_1 + 2\,y_2 + 2\,y_3$

Subject to

$$
\begin{aligned}
y_1 + y_3 &\leq 0 \\
-y_2 + y_3 &\leq 0 \\
-y_1 &\leq 1 \\
y_1 &\leq 1 \\
y_2 &\leq 1 \\
-y_2 &\leq 1
\end{aligned}
$$

- Is $(x_1, x_2, u_1^+, u_1^-, u_2^+, u_2^-) = (2, 0, 0, 2, 2, 0)$ an optimal solution?
- The constraints $(1)$, $(4)$, and $(5)$ in (D) are binding:

  $(4)\ y_1 = 1 \qquad (5)\ y_2 = 1 \qquad (1)\ y_1 + y_3 = 0 \longleftrightarrow y_3 = -1$

- Is $(y_1, y_2, y_3) = (1,1,-1)$ a feasible solution for (P)?
- Are both objective function values equal ?

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# Economic interpretation of duality

- Is there any special meaning of the dual variables?
- What is a dual problem trying to do?
- What's the role of the complementary slackness conditions in decision making?

# Economic interpretation of duality

- Any optimization problem can be seen from two points of view:

  1. From the viewpoint of the one solving the problem

  2. From the viewpoint of the one who defines the rules of the game

*There are those who are subject to constraints and others who impose them*

# Economic interpretation of duality

- The transportation problem:
  - A manufacturer makes a single product in factories at $m$ locations, and wishes to ship them to $n$ distribution centers.
  - Each factory $i$ makes $s_i$ units of this product, and each distribution center $j$ has a demand $d_j$ for the product, with $\sum_i s_i \geq \sum_j d_j$ .
  - Further, there is a cost $c_{ij}$ for shipping each unit of the product from factory $i$ to distribution center $j$.
  - The manufacturer wishes to determine a shipping schedule that ships from available supply to satisfy demand and has minimum total shipping cost

# Economic interpretation of duality

- Mathematical model:

  - Let $x_{ij}$ represent the amount shipped from factory $i$ to distribution center $j$

$$\text{Minimize} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{Subject to}$$

$$\sum_{j=1}^{n} x_{ij} \leq s_i, \qquad i = 1, \dots m$$

$$\sum_{i=1}^{m} x_{ij} \geq d_j, \qquad j = 1, \dots n$$

$$x_{ij} \geq 0, \qquad i = 1, \dots m, j = 1, \dots n$$

# Economic interpretation of duality

- Mathematical model:
  - Let write the LP in canonical min form

Minimize
$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to
$$-\sum_{j=1}^{n} x_{ij} \geq -s_i, \qquad i = 1, \ldots m \qquad p_i$$
$$\sum_{i=1}^{m} x_{ij} \geq d_j, \qquad j = 1, \ldots n \qquad q_i$$
$$x_{ij} \geq 0, \qquad i = 1, \ldots m, j = 1, \ldots n$$

# Economic interpretation of duality

- Mathematical model:
  - The dual is

Maximize
$$\sum_{j=1}^{n} d_j q_j - \sum_{i=1}^{m} s_i p_i$$

Subject to
$$q_j - p_i \leq c_{ij}, \quad i = 1, \dots m, j = 1, \dots n$$

$$p_i, q_j \geq 0, \quad i = 1, \dots m, j = 1, \dots n$$

# Economic interpretation of duality

- Interpretation:
  - The marketplace will naturally make transportation arrangements by decoupling the manufacturer from the distribution centers, and having an independent trucking company serve as the middleman between these centers
  - The trucker will buy all of the product made at each manufacturing center $i$ for price $p_i$, and then will sell all of the product demanded by distribution center $j$ for price $q_j$

# Economic interpretation of duality

- Interpretation:
  - The manufacturers and distribution centers are willing to accept this arrangement so long as the net loss of value along each route (buy-back price – selling price) does not exceed the cost of shipping along that particular route
  - The Trucker's objective in setting prices is to maximize profits while staying competitive with current transportation costs
  - The **profit** the trucker gets from taking over the transportation portion of the process **is exactly the same** as the **cost** the manufacturer would incur by doing it in-house.

# Main references


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