Decision Aid Methodologies In Transportation
Lecture 3: Duality and sensitivity analysis

Duality

Sensitivity Analysis

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Motivation for duality

Consider the following LP problem:

$$
\begin{array}{lllllll}
\min & -x_1 & -x_2 & & & \\
s.t. & 2x_1 & +x_2 & +x_3 & & = 12 \\
& x_1 & +2x_2 & & +x_4 & = 9 \\
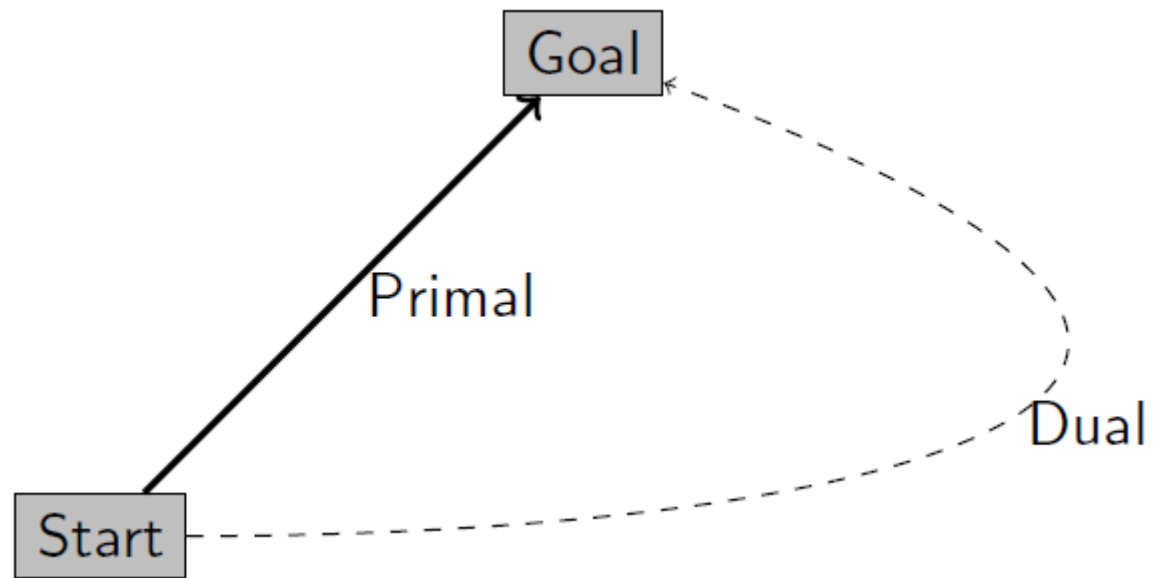& x_1, & x_2, & x_3, & x_4 & \geq 0
\end{array}
$$

It is easy to test that *(x1, x2, x3, x4)=(5, 2, 0, 0)* is a feasible solution of the LP.

Question : can we prove that *(5, 2, 0, 0)* is an optimal solution without solving the original LP?

The key is to find a suitable lower bound of the original LP. How?

Basic idea

Lagrangian multipliers to duality

Consider a linear programming problem (Primal)

$$\begin{aligned} \min : \quad & \mathbf{c}'\mathbf{x} \\ s.t. \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Let x^* be an optimal solution to this model. We introduce a relaxed problem in which the constraint $Ax=b$ is replaced by a penalty $\mathbf{p}'(b-Ax)$, where p is the lagrangian multiplier vector. We are then faced with the problem

$$\begin{aligned} \min : \quad & \mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{Ax}) \\ s.t. \quad & \mathbf{x} \geq 0 \end{aligned}$$

Lagrangian multipliers to duality

$$
\begin{aligned}
g(\mathbf{p}) &= \min_{\mathbf{x} \geq 0}[\mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x})] \\
&= \mathbf{p}'\mathbf{b} + \min_{\mathbf{x} \geq 0}(\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{x}
\end{aligned}
$$

Note that

$$
\min_{\mathbf{x} \geq 0}(\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{x} = \begin{cases} 0, & \text{if } \mathbf{c}' - \mathbf{p}'\mathbf{A} \geq 0' \\ -\infty, & \text{otherwise.} \end{cases}
$$

In maximizing *g(p)*, we only need to consider those values of **p** for which *g(p)* is not equal to − infinity. We therefore conclude that the dual problem is the same as the other linear programming model

$$
\begin{aligned}
\max: &\quad \mathbf{p}'\mathbf{b} \\
s.t. &\quad \mathbf{p}'\mathbf{A} \leq \mathbf{c}'
\end{aligned}
$$

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Shortcut to write a dual for a primal

PRIMAL	min	max	DUAL
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
variables	≥ 0 ≤ 0 free	$\leq b_i$ $\geq b_i$ $= b_i$	constraints

The dual of the dual is the primal

If we transform the dual into an equivalent minimization problem and then find its dual, we obtain a problem equivalent to the original problem.

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Shortcut to write a dual for a primal

$$
\begin{array}{ccc}
\text{min} & \Leftrightarrow & \text{max} \\
\text{membre de droit} & \Leftrightarrow & \text{objectif} \\
\text{objectif} & \Leftrightarrow & \text{membre de droit} \\
\text{contraintes} & \Leftrightarrow & \text{variables} \\
\text{variables} & \Leftrightarrow & \text{constraints} \\
[\text{coefficients}] & \Leftrightarrow & [\text{coefficients}]^t \\
\text{variable} \geq 0 & \Leftrightarrow & \text{contrainte} \leq \\
\text{variable libre} & \Leftrightarrow & \text{contrainte} = \\
\text{contrainte} \geq & \Leftrightarrow & \text{variable} \geq 0 \\
\text{contrainte} = & \Leftrightarrow & \text{variable libre}
\end{array}
$$

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An example for practice

$$
\begin{array}{llllll}
\min & x_1 & +2x_2 & +3x_3 & & \\
s.t. & x_1 & +3x_2 & & = 5 & (p_1) \\
 & 2x_1 & -x_2 & +3x_3 & \geq 6 & (p_2) \\
 & & & x_3 & \leq 4 & (p_3) \\
 & x_1 & & & \geq 0 & \\
 & & x_2 & & \leq 0 & \\
 & & & x_3 & \text{free} &
\end{array}
$$

$$
\begin{array}{lllll}
\max & 5p_1 & +6p_2 & +4p_3 & \\
s.t. & p_1 & & & \text{free} \\
 & & p_2 & & \geq 0 \\
 & & & p_3 & \leq 0 \\
 & p_1 & +2p_2 & & \leq 1 \\
 & 3p_1 & -p_2 & & \geq 2 \\
 & & 3p_2 & +p_3 & = 3
\end{array}
$$

Dual problem

$$
\begin{array}{llllll}
\max & 5x_1 & +6x_2 & +4x_3 & & \\
\text{s.t.} & x_1 & +2x_2 & & \leq 1 & (p1) \\
& 3x_1 & -x_2 & & \geq 2 & (p2) \\
& & 3x_2 & +x_3 & = 3 & (p3) \\
& & x_2 & & \geq 0 & (p4) \\
& & & x_3 & \leq 0 & (p5)
\end{array}
$$

Two rules to remember:

1) p'A=c'

2) If we a minimization problem, the lagrangian function always tries to provide a lower bound; however, if the problem is a maximization, the lagrangian function always tries to sow an upper bound.

Duality theorem

Weak duality

If x is a feasible solution to the primal problem and **p** is a feasible solution to the dual problem, then,

$$\mathbf{P'}b \leq c'x$$

Strong duality

Let x and **p** be feasible solutions to the primal and dual problem, respectively, and suppose that **p**'b=c'x. then, x and **p** are optimal solutions to the primal and the dual, respectively.

Example to solve in the class

$$\begin{aligned}
\max_{x} \quad & -x_1 - 2x_2 \\
\text{s.c.} \quad & -x_1 + x_2 && \leq 2 \\
& x_1 - x_2 && \leq 1 \\
(P) \qquad & -x_1 - x_2 && \leq -3 \\
& -2x_1 - x_2 && \leq -5 \\
& x_1 && \leq 4 \\
& x_2 && \leq 4 \\
& x && \geq 0
\end{aligned}$$

Special cases in the simplex method

1. Degeneracy

2. Unbounded solutions

3. Nonexisting (or infeasible) solutions

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Degeneracy

In the application of the feasibility condition of the simplex method, a tie for the minimum ratio may occur and can be broken arbitrarily. When this happens, at least one *basic* variable will be zero in the next iteration and the new solution is said to be degenerate.

There is nothing alarming about a degenerate solution, with the exception of a small theoretical inconvenience, called cycling or circling. The condition reveals that the model has at least one *redundant* constraint.
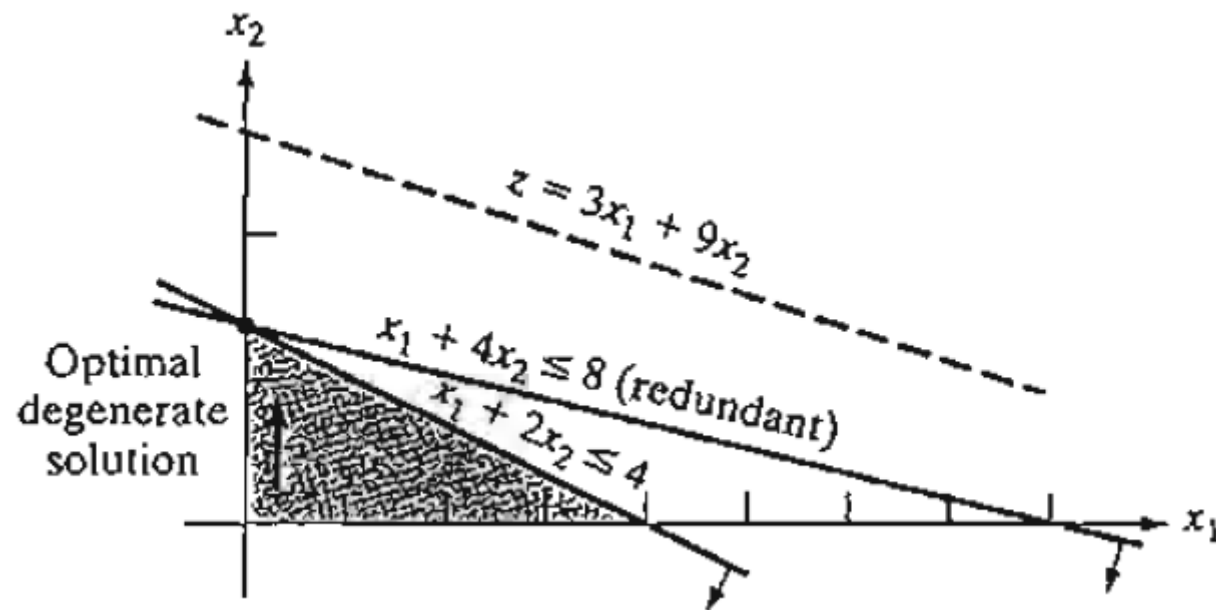
Degeneracy

$$\text{Maximize } z = 3x_1 + 9x_2$$

$$x_1 + 4x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Degeneracy

Unbounded solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraints-meaning that the solution space is *unbounded* in at least one variable. As a result, the objective value may increase (maximization case) or decrease (minimization case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded. Unboundedness points to the possibility that the model is poorly constructed.

Unbounded solution

$$\text{Maximize } z = 2x_1 + x_2$$

$$x_1 - x_2 \leq 10$$
$$2x_1 \leq 40$$
$$x_1, x_2 \geq 0$$

Infeasible solution

LP models with inconsistent constraints have no feasible solution. This situation can never occur if *all* the constraints are of the type ≤ with nonnegative right-hand sides because the slacks provide a feasible solution. For other types of constraints, we use artificial variables. Although the artificial variables are penalized in the objective function to force them to zero at the optimum, this can occur only if the model has a feasible space.

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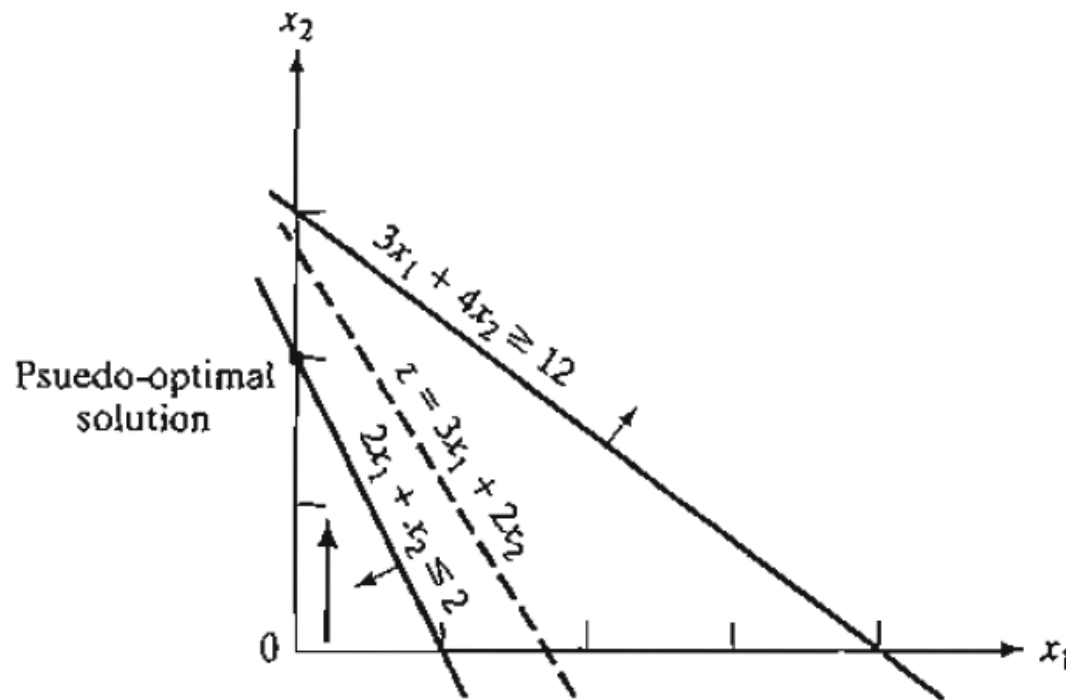FÉDÉRALE DE LAUSANNE

Infeasible solution

$$\text{Maximize } z = 3x_1 + 2x_2$$

$$2x_1 + x_2 \leq 2$$
$$3x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

Infeasible solution

Sensitivity Analysis

Why sensitivity analysis important in practice ?

The impact of
- change in passenger load factor.
- One of the stations out of order.
- Some stations to be added inside the network

on the overall cost.

Sensitivity Analysis

In LP, the parameters (input data) of the model can change within certain limits without causing the optimum solution to change. This is referred to as *sensitivity analysis.*

In LP models, the parameters are usually not exact. With sensitivity analysis, we can ascertain the impact of this uncertainty on the quality of the optimum solution. For example, for an estimated unit profit of a product, if sensitivity analysis reveals that the optimum remains the same for a ±10% change in the unit profit (i.e changing the coefficients of the objective do not change X*), we can conclude that the solution is more robust than in the case where the indifference range is only ±1 %.

> 1. Sensitivity of the optimum solution to changes in the availability of the resources (right-hand side of the constraints).

> 2. Sensitivity of the optimum solution to changes in unit profit or unit cost (coefficients of the objective function).

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Sensitivity Analysis-example

JOBCO produces two products on two machines. A unit of product 1 requires 2 hours on machine 1 and 1 hour on machine 2. For product 2, a unit requires 1 hour on machine 1 and 3 hours on machine 2. The revenues per unit of products 1 and 2 are $30 and $20, respectively. The total daily processing time available for each machine is 8 hours. Letting X_1 and X_2 represent the daily number of units of products 1 and 2, respectively, the LP model is given as
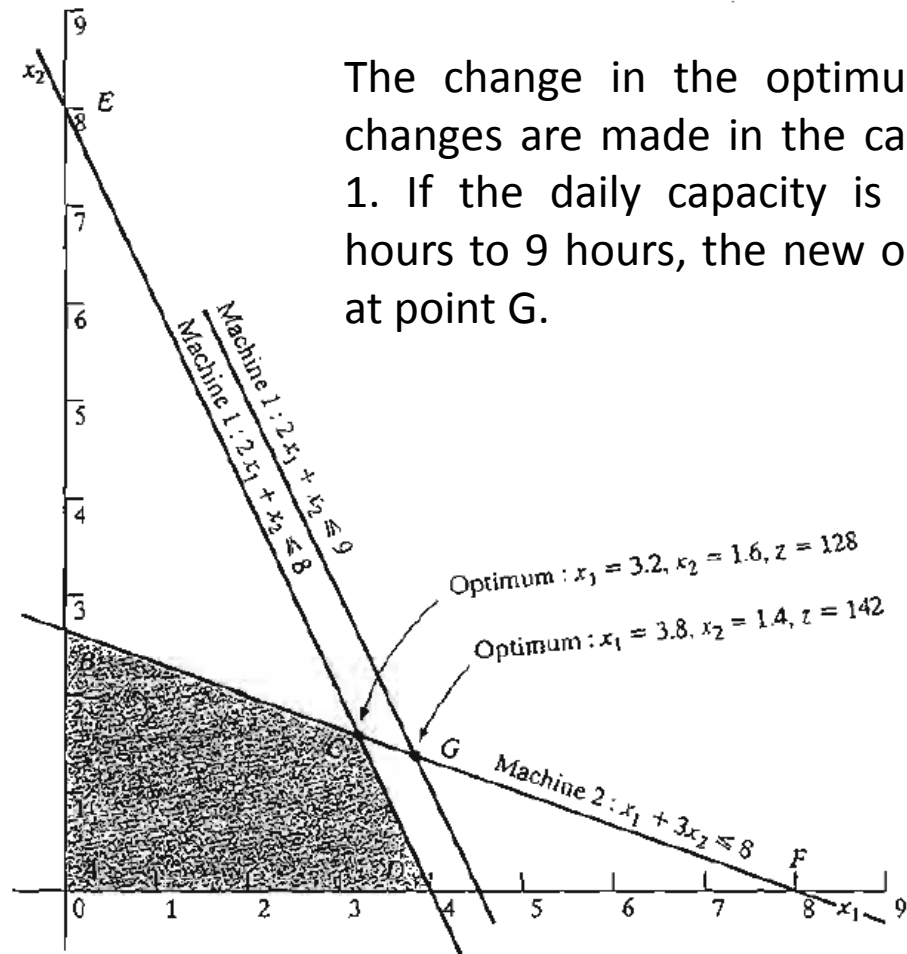
$$\text{Maximize } z = 30x_1 + 20x_2$$

$$2x_1 + x_2 \leq 8 \qquad \text{(Machine 1)}$$
$$x_1 + 3x_2 \leq 8 \qquad \text{(Machine 2)}$$
$$x_1, x_2 \geq 0$$

Sensitivity Analysis

The change in the optimum solution when changes are made in the capacity of machine 1. If the daily capacity is increased from 8 hours to 9 hours, the new optimum will occur at point G.

Sensitivity Analysis-example

The rate of change in optimum *z* resulting from changing machine 1 capacity from 8 hours to 9 hours can be computed as follows:

$$
\begin{pmatrix}
\text{Rate of revenue change} \\
\text{resulting from increasing} \\
\text{machine 1 capacity by 1 hr} \\
(\text{point } C \text{ to point } G)
\end{pmatrix} = \frac{z_G - z_C}{(\text{Capacity change})} = \frac{142 - 128}{9 - 8} \approx \$14.00/\text{hr}
$$

The computed rate provides a *direct link* between the model input (resources) and its output (total revenue) that represents the unit worth of a resource (in \$/hr)-that is, the change in the optimal objective value per unit change in the availability of the resource (machine capacity). This means that a unit increase (decrease) in machine 1 capacity will increase (decrease) revenue by \$14.00.

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Sensitivity Analysis-example

We can see that the dual price of $14.00/hr remains valid for changes (increases or decreases) in machine 1 capacity that move its constraint parallel to itself to any point on the line segment *BF*. This means that the range of applicability of the given dual price can be computed as follows:

$$\text{Minimum machine 1 capacity [at } B = (0, 2.67)] = 2 \times 0 + 1 \times 2.67 = 2.67 \text{ hr}$$

$$\text{Maximum machine 1 capacity [at } F = (8, 0)] = 2 \times 8 + 1 \times 0 = 16 \text{ hr}$$

We can thus conclude that the dual price of $14.00/hr will remain valid for the range.

$$2.67 \text{ hrs} \leq \text{Machine 1 capacity} \leq 16 \text{ hrs}$$

Changes outside this range will produce a different dual price (worth per unit).