# Decision Aid Methodologies In Transportation
# Lecture 2: Modeling

## **Mathematical Modeling**

## **Linear Programming**

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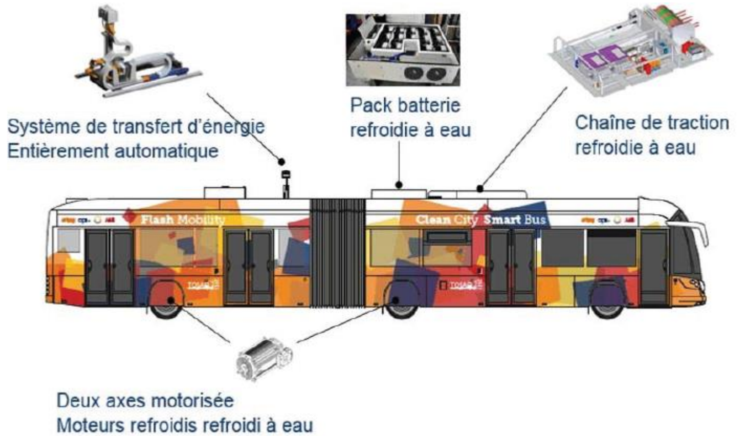
# MyTosa

## Catenary-free 100% electric urban public mass-transportation system

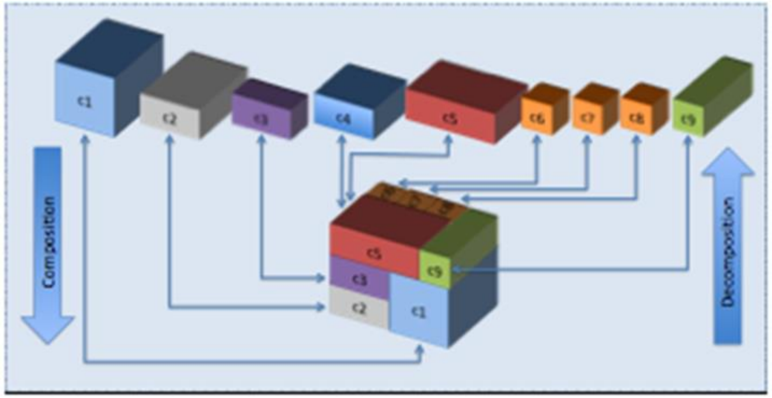
myTOSA is a simulation tool for the dimensioning, commercial promotion and case study set-up for ABB's revolutionary "catenary-free" 100% electric urban public mass-transportation system TOSA 2013. The objective of the project is to provide a simulation tool that will allow ABB to perform the proper dimensioning, promote the commercial idea and allow for specific study cases for the implementation of ABB's new public electric transportation concept, namely TOSA.




# Modulushca

## Modular logistics units in shared co-modal networks

The objective is to achieve the first genuine contribution to the development of intercontinental logistics at the European level, in close coordination with North America partners and the international Physical Internet Initiative. The goal of the project is to enable operations with developed iso-modular logistics units of size adequate for real modal and co-modal flows of fast-moving consumer goods, providing a basis for an interconnected logistics system for 2030.



# Problem definition

Special form of **mathematical programming**

Equations must be linear : Using arithmetic operation such as **addition subtraction**

- $Y = a(X) + b$
- The following terms are not linear!!
  - $Y = X^a + b\ ; XY = b\ ; \frac{X}{Y} - b = Z; Y = a|X| + b$

**Simple solution** procedures

- Linear algebra, Simplex Method

Very **powerful**

Extremely large problems

100,000 variables

1000's of constraints

Useful design information by **Sensitivity Analysis**

- Answers to "what if" questions

# Example 1

A glass company has three plants: aluminum frame and hardware, wood frame, glass and assembly. Two product with highest profit:

- Product 1: An 8-foot glass door with aluminum frame → plants 1 and 3
- Product 2: A $4 \times 6$ foot double hung wood frame window → plants 2 and 3

The benefit of selling a batch (including 20) of products 1 and 2 are \$3000 and \$5000 respectively.

Each batch of product 1 produced per week uses 1 hour of production time per week in plant 1, whereas only 4 hours per week plant 1 is available.

Each batch of product 2 produced per week uses 2 hours of production time per week in plant 2, whereas only 12 hours per week plant 2 is available.

Each batch of products 1 and 2 produced per week uses 3 and 2 hours of production time per week in plant 3 respectively, whereas only 18 hours per week are available.

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# Example 1

Formulation as a Linear Programming Problem

To formulate the mathematical (linear programming) model for this problem, let

- $x_1$ = number of batches of product 1 produced per week
- $x_2$ = number of batches of product 2 produced per week
- $Z$ = total profit per week (in thousands of dollars) from producing these two products

Thus, $x_1$ and $x_2$ are the decision variables for the model and the objective function is as follows

- $Z = 3x_1 + 5x_2$

The objective is to choose the values of $x_1$ and $x_2$ so as to maximize $Z$ subject to the restrictions imposed on their values by the limited production capacities available in the three plants.

# Example 1

| Plant | Production Time per Batch, Hours — Product 1 | Production Time per Batch, Hours — Product 2 | Production Time Available per Week, Hours |
|---|---|---|---|
| 1 | 1 | 0 | 4 |
| 2 | 0 | 2 | 12 |
| 3 | 3 | 2 | 18 |
| Profit per batch | $3,000 | $5,000 | |

# Example 1

To summarize, in the mathematical language of linear programming, the problem to choose values of $x_1$ and $x_2$ so as to

Maximize $Z = 3x_1 + 5x_2$

subject to the restrictions

$$x_1 \leq 4$$

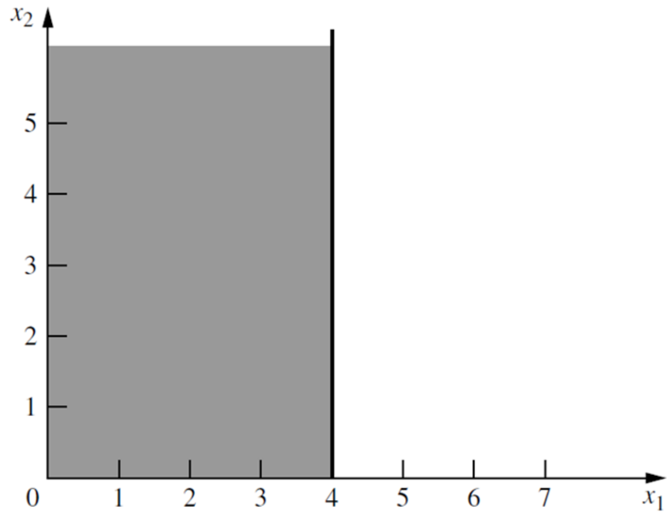
$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

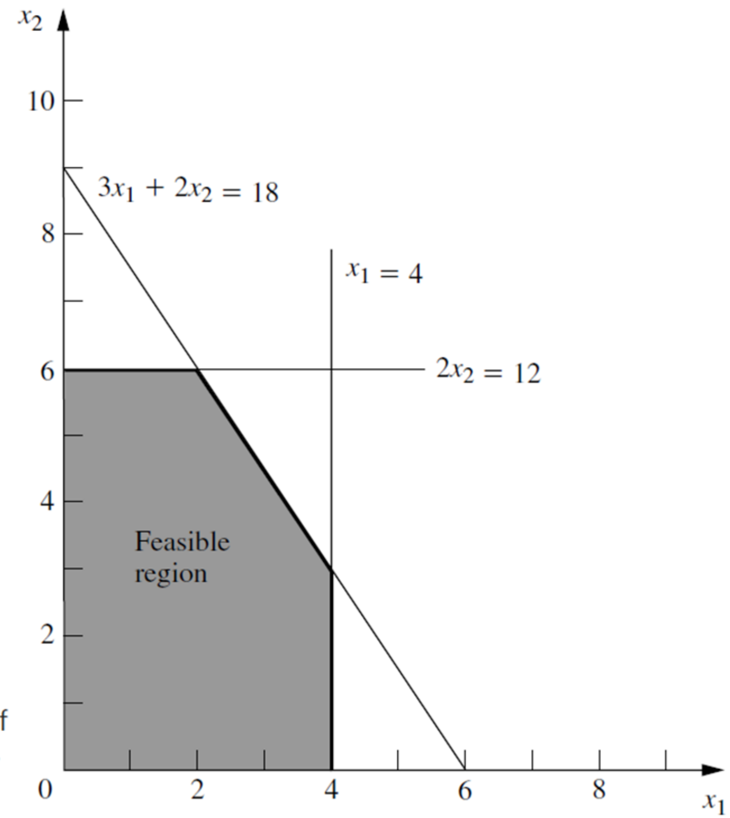
and

$$x_1 \geq 0, x_2 \geq 0$$

# Simplex Method – Graphical Solution




Shaded area shows values of $(x_1, x_2)$ allowed by $x_1 \geq 0$, $x_2 \geq 0$, $x_1 \leq 4$.



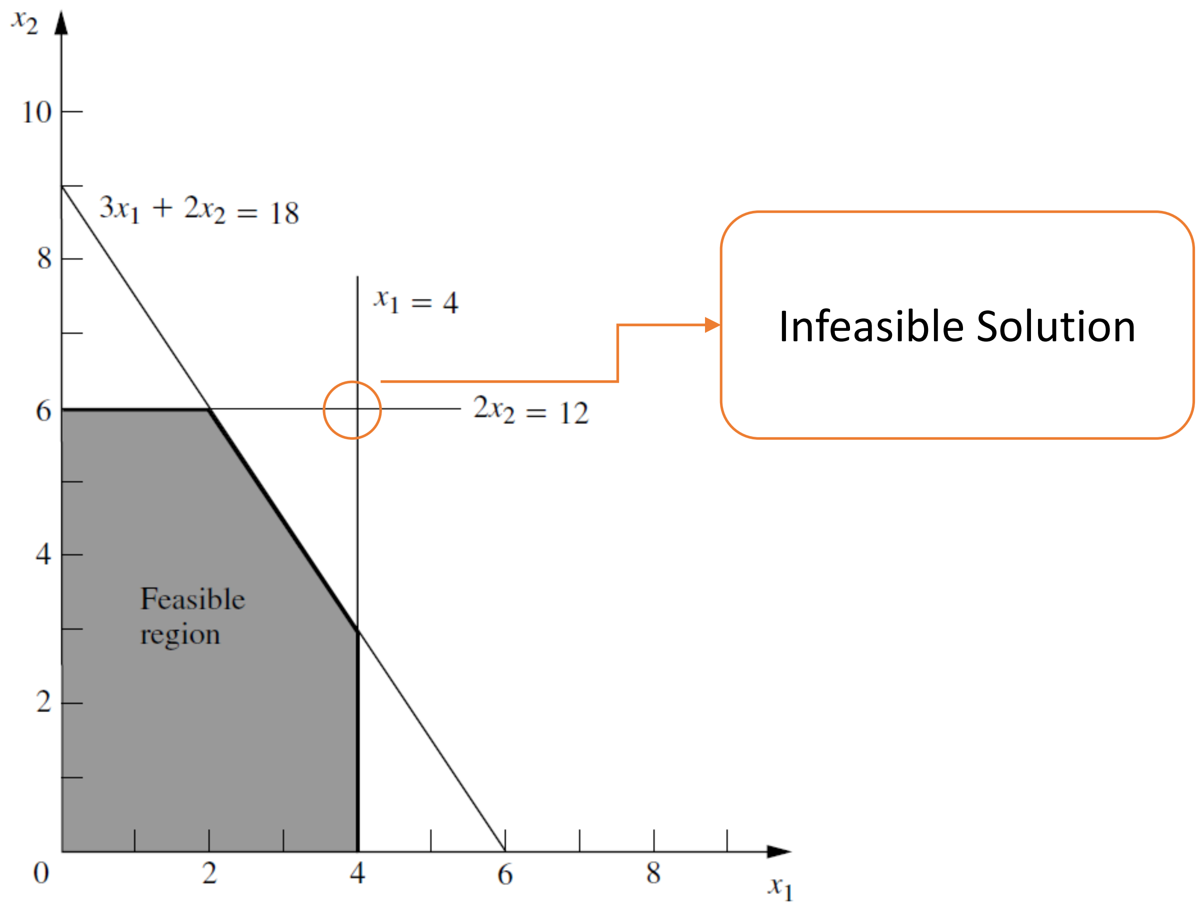

Shaded area shows the set of permissible values of $(x_1, x_2)$, called the feasible region.

# Simplex Method – Graphical Solution

## Terminology for Solutions of the Model

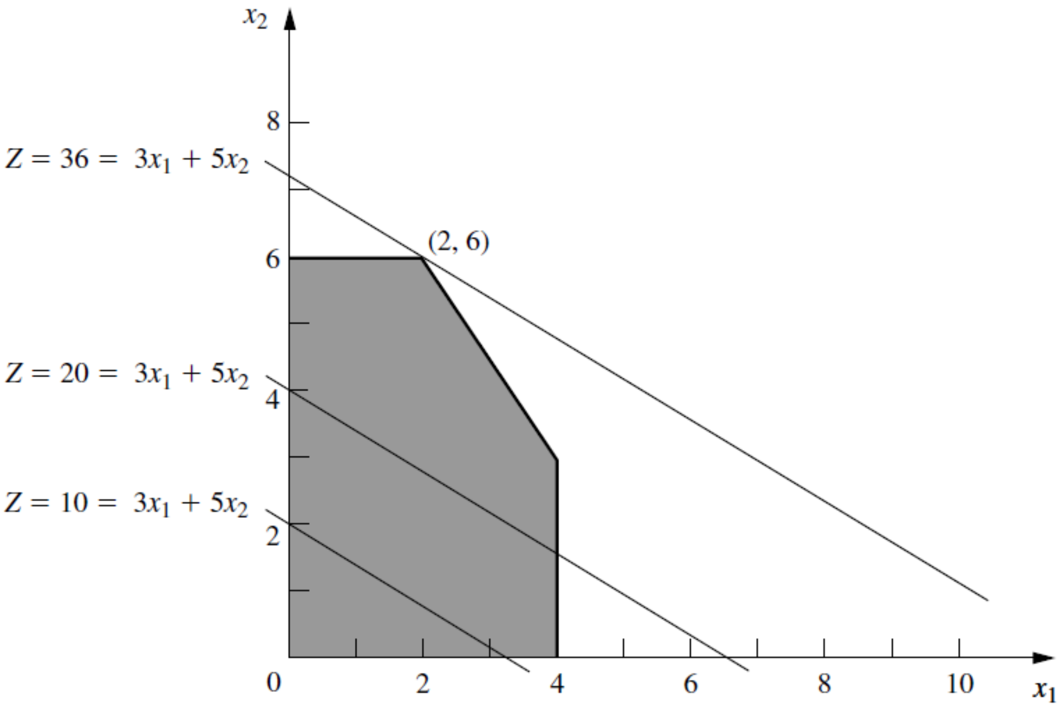
- **Feasible solution**: a solution for which all the constraints are satisfied.
- **Infeasible solution**: a solution for which at least one constraint is violated.
- **Feasible region**: the collection of all feasible solutions.
- **No feasible solutions**: It is possible for a problem to have no feasible solutions.

# Simplex Method – Graphical Solution



# Simplex Method – Graphical Solution

The value of $(x_1, x_2)$ that maximizes $3x_1 + 5x_2$ is $(2, 6)$.




**Optimal solution**: a feasible solution that has the best objective value

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# Simplex Method

**General Solution Approach (Graphical Method)**

Step 1: Find a **corner point**

An "initial feasible solution"

Step 2: **Proceed** to **improved** corner points

Step 3: **Stop** when **no further improvements** are possible

Step 4: For large problems, a variety of more sophisticated approaches are used!

**Solution Calculations**

F**ind a corner point**

It is necessary to solve system of constraint equations from linear algebra, this requires working with matrix of constraint equations, specifically, manipulating the "determinants"

Amount of effort set by number of constraints, So number of constraints defines amount of effort. This is why LP can handle many more decision variables than constraints

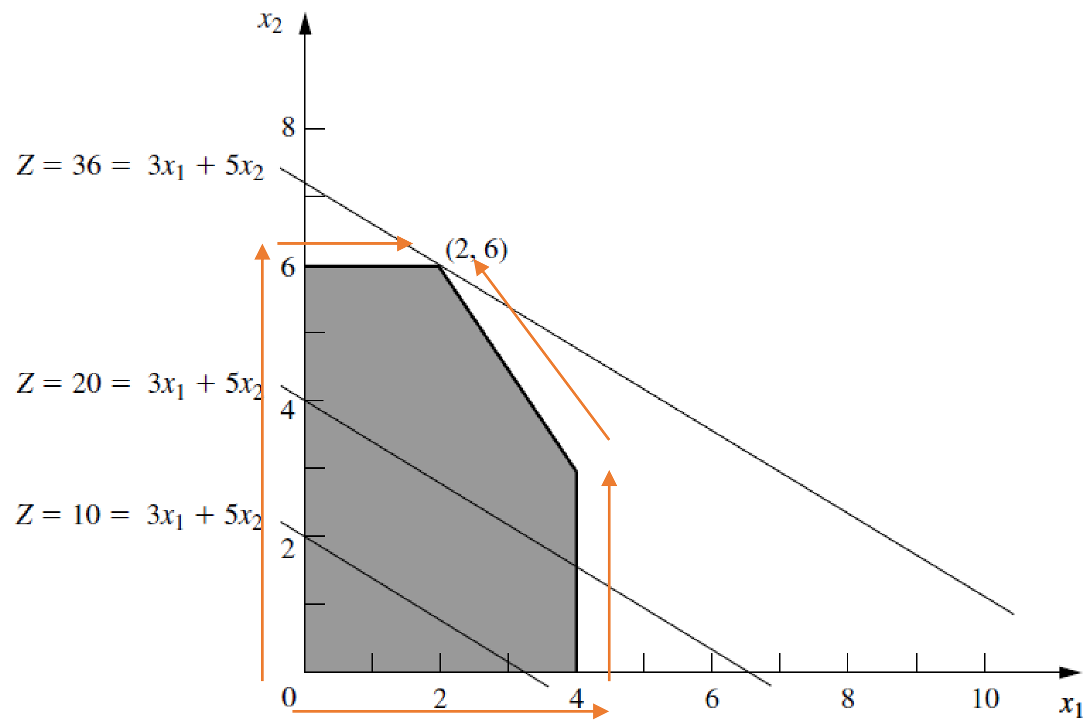
# Simplex Method

Select **improved corners**

Always goes to the <u>best corner</u>

Searches until <u>no further improvement</u> possible

The value of $(x_1, x_2)$ that maximizes $3x_1 + 5x_2$ is (2, 6).



# Simplex Method

Standard Form of LP - Three Parts

<u>Objective function</u>

maximize or minimize

$$\mathrm{Y} = \sum_{i=1}^{r} c_i \, X_i$$

$$Y = C_1X_1 + C_2X_2 + \dots + C_nX_n$$

$X_i$ known as decision variables

<u>Constraints</u>

subject to

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1nXn} = b_1$$
$$a_{21}X_1 + a_{22}X_2 + \dots + a_{2nXn} = b_2$$
$$\dots$$
$$a_{m1}X_1 + am_2X_2 + \dots + a_{mn}X_n = b_m$$

<u>Non-Negativity</u>

$x_i \geq 0$ for all $i$

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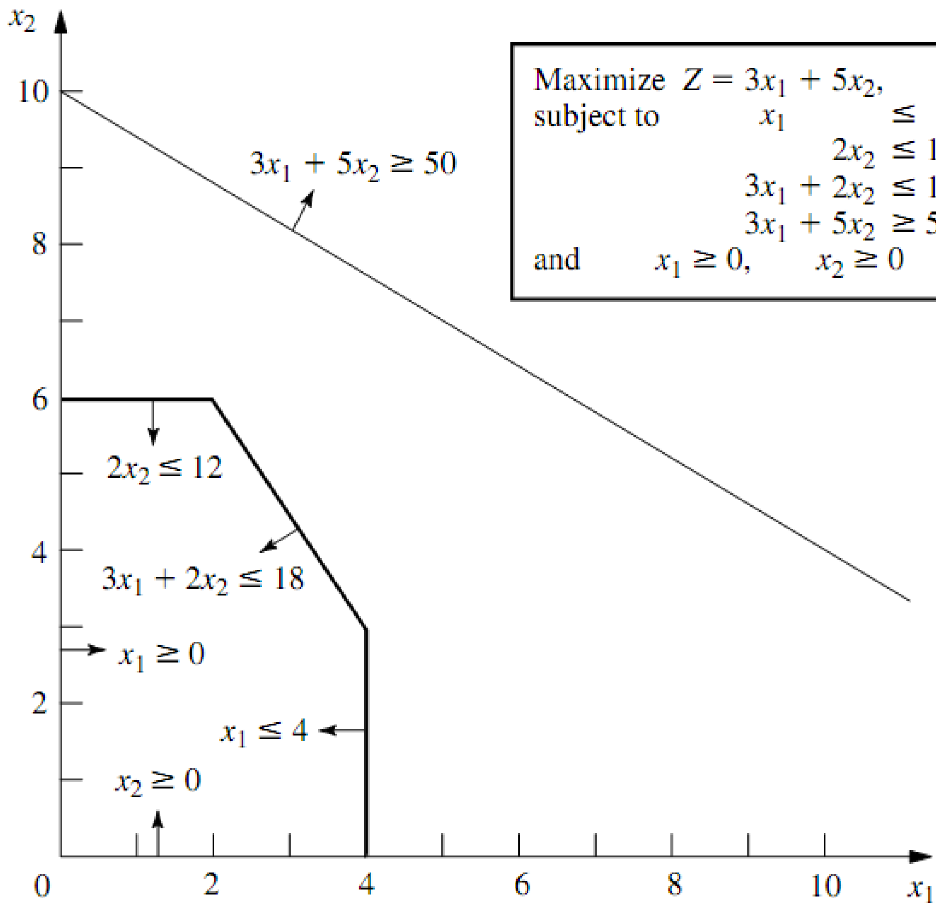
# Simplex Method

Data needed for a linear programming model involving the allocation of resources to activities

| Resource | Resource Usage per Unit of Activity | | | | Amount of Resource Available |
|---|---|---|---|---|---|
| | Activity | | | | |
| | 1 | 2 | . . . | $n$ | |
| 1 | $a_{11}$ | $a_{12}$ | . . . | $a_{1n}$ | $b_1$ |
| 2 | $a_{21}$ | $a_{22}$ | . . . | $a_{2n}$ | $b_2$ |
| . | | | | | . |
| . | . . . | . . . | . . . | . . . | . |
| . | | | | | . |
| $m$ | $a_{m1}$ | $a_{m2}$ | . . . | $a_{mn}$ | $b_m$ |
| Contribution to $Z$ per unit of activity | $c_1$ | $c_2$ | . . . | $c_n$ | |

# Simplex Method

The Wyndor Glass Co. problem would have no feasible solutions if the constraint $3x_1 + 5x_2 \geq 50$ were added to the problem.



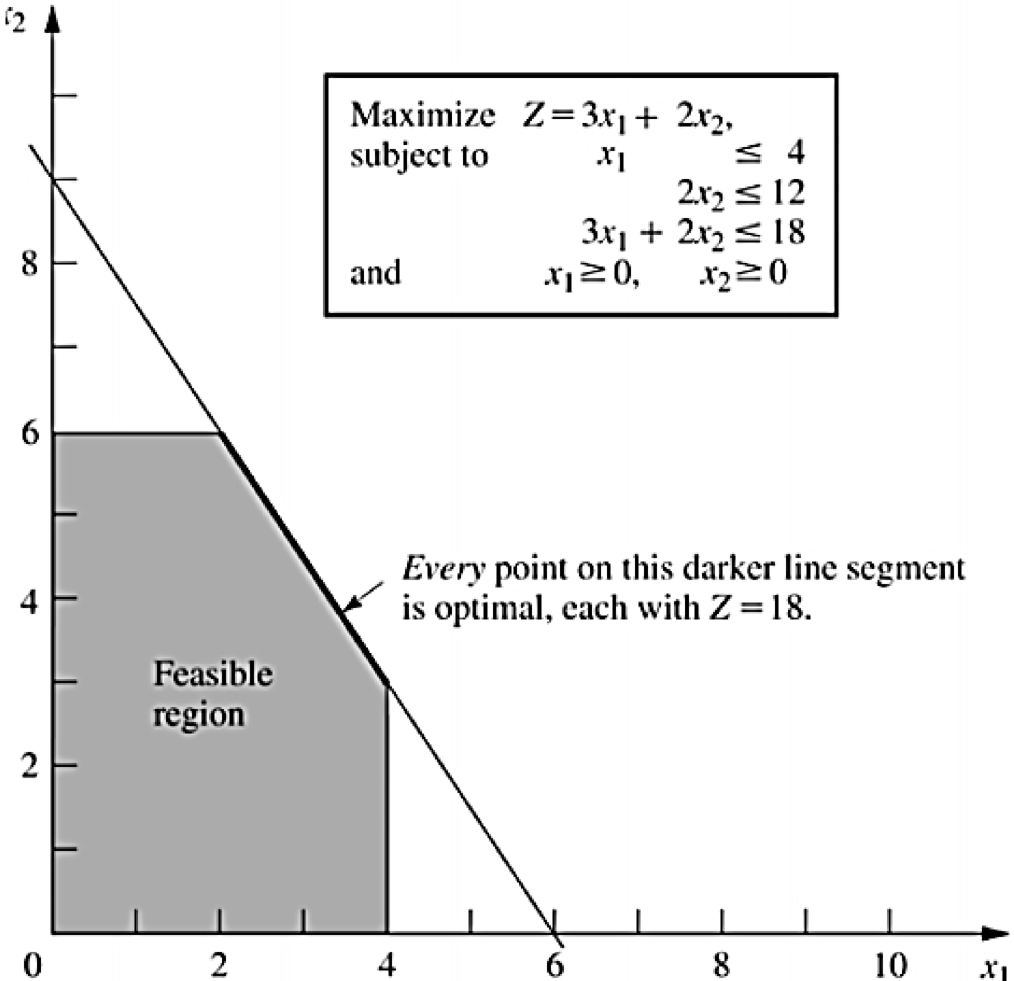

$$
\begin{aligned}
\text{Maximize } & Z = 3x_1 + 5x_2, \\
\text{subject to } & x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
& 3x_1 + 5x_2 \geq 50 \\
\text{and } & x_1 \geq 0, \quad x_2 \geq 0
\end{aligned}
$$

# Simplex Method

**Multiple optimal solutions**: Most problems will have just one optimal solution. However, it is possible to have more than one. This would occur in the example if the profit per batch produced of product 2 were changed from \$5000 to \$2000. This changes the objective function to $Z = 3x_1 + 2x_2$ so that all the points on the line segment connecting (2, 6) and (4, 3) would be optimal. As in this case, any problem having multiple optimal solutions will have an infinite number of them, each with the same optimal value of the objective function.

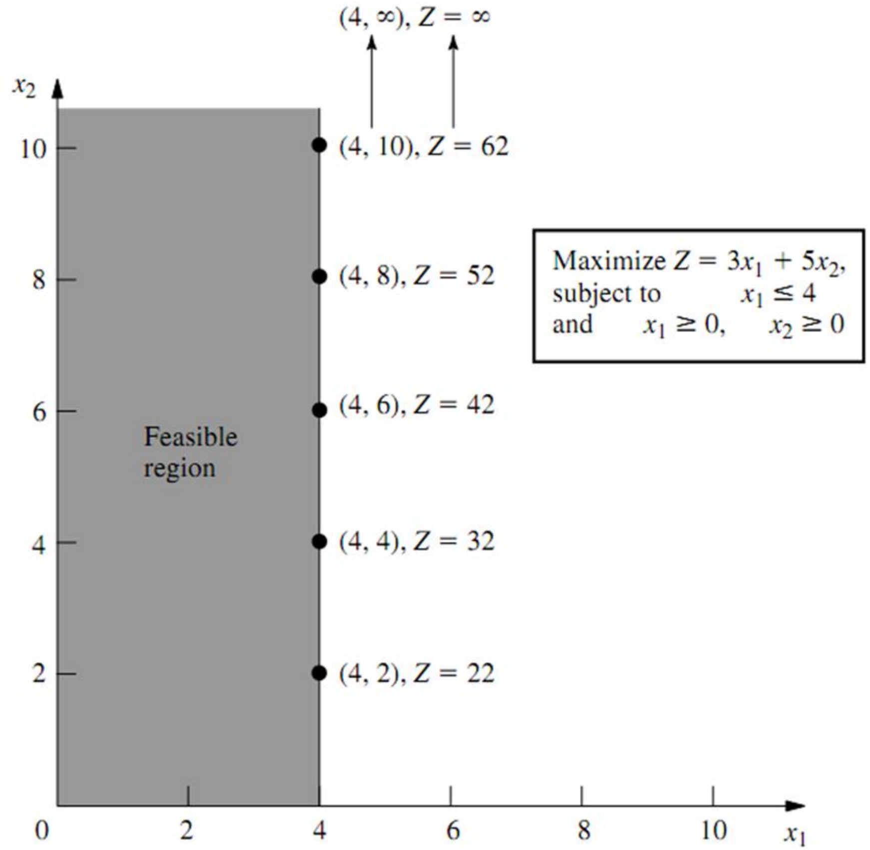
**No optimal solutions**: Another possibility is that a problem has no optimal solutions. This occurs only if (1) it has no feasible solutions or (2) the constraints do not prevent improving the value of the objective function (Z) indefinitely in the favorable direction (positive or negative).

# Simplex Method



# Simplex Method

The latter case is referred to as having an **unbounded Z**. To illustrate, this case would result if the last two functional constraints were mistakenly deleted in the example.



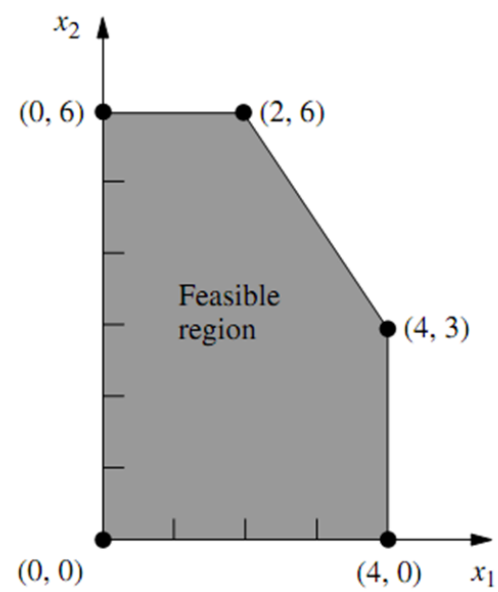

The Wyndor Glass Co. problem would have no optimal solutions if the only functional constraint were $x_1 \leq 4$, because $x_2$ then could be increased indefinitely in the feasible region without ever reaching the maximum value of $Z = 3x_1 + 5x_2$.

# Simplex Method

A **corner-point feasible (CPF) solution** is a solution that lies at a corner of the feasible region.

**Relationship between optimal solutions and CPF solutions**: Consider any linear programming problem with feasible solutions and a bounded feasible region. The problem must possess CPF solutions and at least one optimal solution. Furthermore, the best CPF solution must be an optimal solution. Thus, if a problem has exactly one optimal solution, it must be a CPF solution. If the problem has multiple optimal solutions, at least two must be CPF solutions.

The five dots are the five CPF solutions for the Wyndor Glass Co. problem.

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# Simplex Method - Tables

| Original Formulation | Standard Formulation |
|---|---|

$$\max Z = 100x_1 + 200x_2$$

$$\begin{aligned} 4x_1 + 3x_2 &\leq 240 \\ 2x_1 + x_2 &\leq 100 \\ x_2 &\leq 60 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

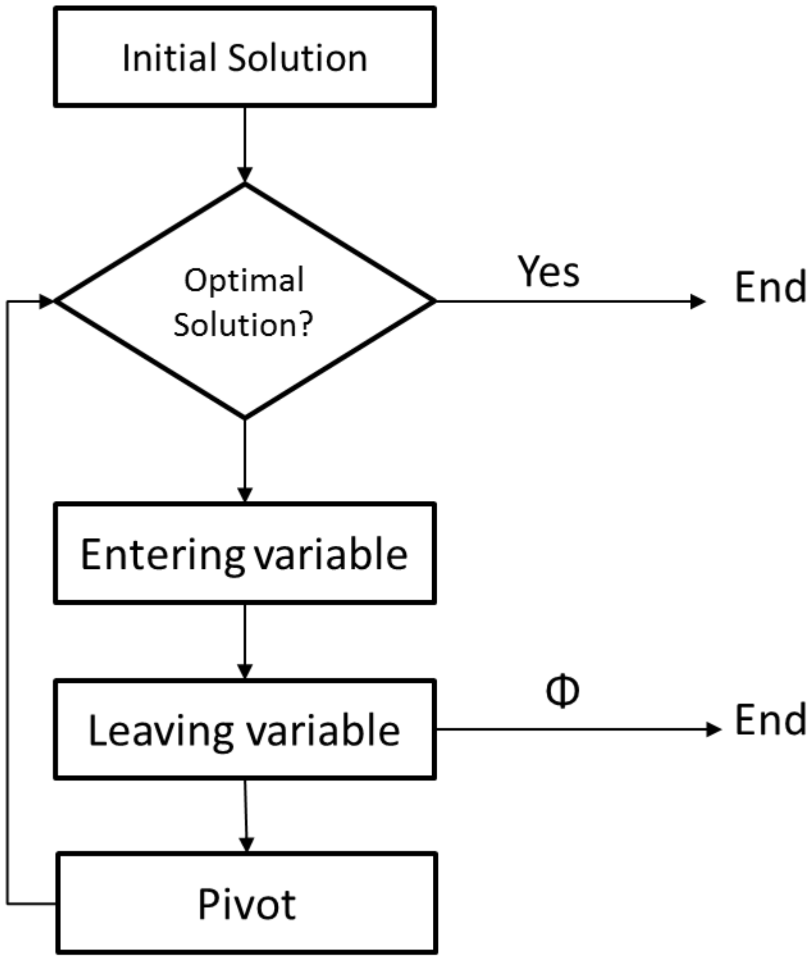
$$\max Z = 100x_1 + 200x_2$$

$$\begin{aligned} 4x_1 + 3x_2 + e_1 &= 240 \\ 2x_1 + x_2 + e_2 &= 100 \\ x_2 + e_3 &= 60 \\ x_1,\ x_2,\ e_1,\ e_2,\ e_3 &\geq 0. \end{aligned}$$

| | $c_j \longrightarrow$ | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $c_B$ | **Basics** | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | **Value** |
| 0 | $e_1$ | 4 | 3 | 1 | 0 | 0 | 240 |
| 0 | $e_2$ | 2 | 1 | 0 | 1 | 0 | 100 |
| 0 | $e_3$ | 0 | 1 | 0 | 0 | 1 | 60 |
| $\square \times \square$ | $z_j$ | 0 | 0 | 0 | 0 | 0 | |
| | $c_j - z_j$ | 100 | 200 | 0 | 0 | 0 | 0 |

- ☐ Coefficient of base variables in the objective function
- ☐ Base variables
- ☐ Reduced cost – Marginal profit

# Simplex Method - Tables



# Simplex Method - Tables

| | | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $c_j$ | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | 4 | 3 | 1 | 0 | 0 | 240 |
| 0 | $e_2$ | 2 | 1 | 0 | 1 | 0 | 100 |
| 0 | $e_3$ | 0 | 1 | 0 | 0 | 1 | 60 |
| | $z_j$ | 0 | 0 | 0 | 0 | 0 | |
| | $c_j - z_j$ | 100 | 200 | 0 | 0 | 0 | 0 |

$Z = 100X_1 + 200X_2$

If we increase $X_1$ 1 unit → the objective increase 100 units

If we increase $X_2$ 1 unit → the objective increase 200 units

In Maximization problem, the solution in simplex table is optimal if for all variables $c_j - zj \leq 0$

# Simplex Method - Tables

| $c_j$ | | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | 4 | 3 | 1 | 0 | 0 | 240 |
| 0 | $e_2$ | 2 | 1 | 0 | 1 | 0 | 100 |
| 0 | $e_3$ | 0 | 1 | 0 | 0 | 1 | 60 |
| | $z_j$ | 0 | 0 | 0 | 0 | 0 | |
| | $c_j - z_j$ | 100 | 200 | 0 | 0 | 0 | 0 |

In Maximization problem, the solution in simplex table is optimal if for all variables $c_j - zj \leq 0$

# Simplex Method - Tables

| $c_j$ | | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | 4 | 3 | 1 | 0 | 0 | 240 |
| 0 | $e_2$ | 2 | 1 | 0 | 1 | 0 | 100 |
| 0 | $e_3$ | 0 | 1 | 0 | 0 | 1 | 60 |
| | $z_j$ | 0 | 0 | 0 | 0 | 0 | |
| | $c_j - z_j$ | 100 | $\boxed{200}$ | 0 | 0 | 0 | 0 |

Among all variables with $c_j - zj \geq 0$ we choose a variable with the highest value

# Simplex Method - Tables

| $c_j$ | | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | 4 | 3 | 1 | 0 | 0 | 240 |
| 0 | $e_2$ | 2 | 1 | 0 | 1 | 0 | 100 |
| 0 | $e_3$ | 0 | 1 | 0 | 0 | 1 | 60 |
| | $z_j$ | 0 | 0 | 0 | 0 | 0 | |
| | $c_j - z_j$ | 100 | $\boxed{200}$ | 0 | 0 | 0 | 0 |

How much we can increase the value of $X_2$?

- We can increase the value till the value of other variables is non-negative

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# Simplex Method - Tables

| $c_j$ | | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | 4 | 3 | 1 | 0 | 0 | 240 |
| 0 | $e_2$ | 2 | 1 | 0 | 1 | 0 | 100 |
| 0 | $e_3$ | 0 | 1 | 0 | 0 | 1 | 60 |
| $z_j$ | | 0 | 0 | 0 | 0 | 0 | |
| $c_j - z_j$ | | 100 | $\boxed{200}$ | 0 | 0 | 0 | 0 |

$e_1 = 240 - 4x_1 - 3x_2$ ⇒ $e_1 = 240 - 3x_2 >= 0$ ⇒ $x_2 <= 240/3 = 80$

$e_2 = 100 - 2x_1 - x_2$ ⇒ $e_2 = 100 - x_2 >= 0$ ⇒ $x_2 <= 100/1 = 100$

$e_3$: 60 - $x_2$ ⇒ $e_3 = 60 - x_2 >= 0$ ⇒ $x_2 <= 60/1 = 60$

Fix

# Simplex Method - Tables

| $c_j$ | | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | 4 | 3 | 1 | 0 | 0 | 240 |
| 0 | $e_2$ | 2 | 1 | 0 | 1 | 0 | 100 |
| 0 | $e_3$ | 0 | 1 | 0 | 0 | 1 | 60 |
| $z_j$ | | 0 | 0 | 0 | 0 | 0 | |
| $c_j - z_j$ | | 100 | 200 | 0 | 0 | 0 | 0 |

If $x_j$ is entering variable, it is sufficient to divide right hand side value with $a_{ij}$ for all the constraints ( non-zero value) we choose the smallest ratio

# Simplex Method - Tables

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | 100 | 200 | 0 | 0 | 0 | |
| $c_j$ | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | 4 | 3 | 1 | 0 | 0 | 240 |
| 0 | $e_2$ | 2 | 1 | 0 | 1 | 0 | 100 |
| 0 | $e_3$ | 0 | 1 | 0 | 0 | 1 | 60 |
| | $z_j$ | 0 | 0 | 0 | 0 | 0 | |
| | $c_j - z_j$ | 100 | 200 | 0 | 0 | 0 | 0 |

# Simplex Method - Tables

Gauss-Jordan

| | | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $c_j$ | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | 4 | 3 | 1 | 0 | 0 | 240 |
| 0 | $e_2$ | 2 | 1 | 0 | 1 | 0 | 100 |
| 0 | $e_3$ | 0 | 1 | 0 | 0 | 1 | 60 |
| $z_j$ | | 0 | 0 | 0 | 0 | 0 | |
| $c_j - z_j$ | | 100 | 200 | 0 | 0 | 0 | 0 |

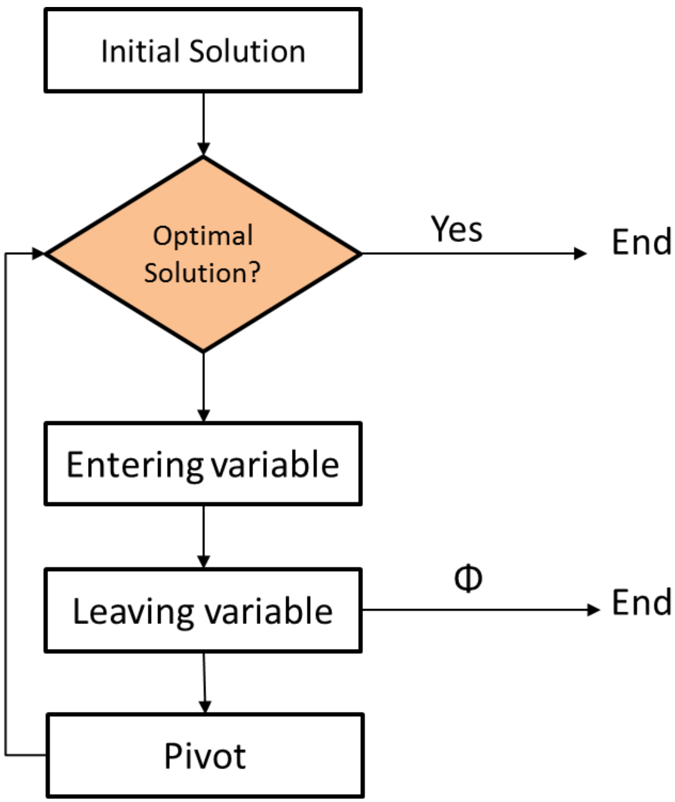
Third row times minus 1 + second row

| | | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $c_j$ | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | | | | | | |
| 0 | $e_2$ | 2 | 0 | 0 | 1 | -1 | 40 |
| 200 | $x_2$ | 0 | 1 | 0 | 0 | 1 | 60 |
| $z_j$ | | | | | | | |
| $c_j - z_j$ | | | | | | | |

Third row times minus three + first row

| | | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| $c_j$ | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | 4 | 0 | 1 | 0 | -3 | 60 |
| 0 | $e_2$ | 2 | 0 | 0 | 1 | -1 | 40 |
| 200 | $x_2$ | 0 | 1 | 0 | 0 | 1 | 60 |
| $z_j$ | | | | | | | |
| $c_j - z_j$ | | | | | | | |

# Simplex Method - Tables




| $c_j$ | | 100 | 200 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $e_1$ | $e_2$ | $e_3$ | |
| 0 | $e_1$ | 4 | 0 | 1 | 0 | -3 | 60 |
| 0 | $e_2$ | 2 | 0 | 0 | 1 | -1 | 40 |
| 200 | $x_2$ | 0 | 1 | 0 | 0 | 1 | 60 |
| | $z_j$ | 0 | 200 | 0 | 0 | 200 | |
| | $c_j - z_j$ | 100 | 0 | 0 | 0 | -200 | 12 000 |

# Simplex Method - Tables

| $c_j$ | | 100 $x_1$ | 200 $x_2$ | 0 $e_1$ | 0 $e_2$ | 0 $e_3$ | |
|---|---|---|---|---|---|---|---|
| 0 | $e_1$ | 4 | 0 | 1 | 0 | -3 | 60 |
| 0 | $e_2$ | 2 | 0 | 0 | 1 | -1 | 40 |
| 200 | $x_2$ | 0 | 1 | 0 | 0 | 1 | 60 |
| $z_j$ | | 0 | 200 | 0 | 0 | 200 | |
| $c_j - z_j$ | | 100 | 0 | 0 | 0 | -200 | 12 000 |

First row divided by 4

-1/2 first row + second row

Pivot

| $c_j$ | | 100 $x_1$ | 200 $x_2$ | 0 $e_1$ | 0 $e_2$ | 0 $e_3$ | |
|---|---|---|---|---|---|---|---|
| 100 | $x_1$ | 1 | 0 | 1/4 | 0 | -3/4 | 15 |
| 0 | $e_2$ | 0 | 0 | -1/2 | 1 | 1/2 | 10 |
| 200 | $x_2$ | 0 | 1 | 0 | 0 | 1 | 60 |
| $z_j$ | | 100 | 200 | 25 | 0 | 125 | |
| $c_j - z_j$ | | 0 | 0 | -25 | 0 | -125 | 13 500 |

Optimal solution?

# Variation of Simplex Algorithm

Big-M Method

Equivalent to two phase simplex

General idea: penalizing in the objective function

$$\max Z = 100x_1 + 200x_2 - Ma_4$$

$$\begin{aligned}
4x_1 + 3x_2 + e_1 &= 240 \\
2x_1 + x_2 + e_2 &= 100 \\
x_2 + e_3 &= 60 \\
x_1 - e_4 + a_4 &= 10 \\
x_1,\ x_2,\ e_1,\ e_2,\ e_3,\ e_4 &\geq 0
\end{aligned}$$

$$a_4 \geq 0$$

# Modeling by Graphs

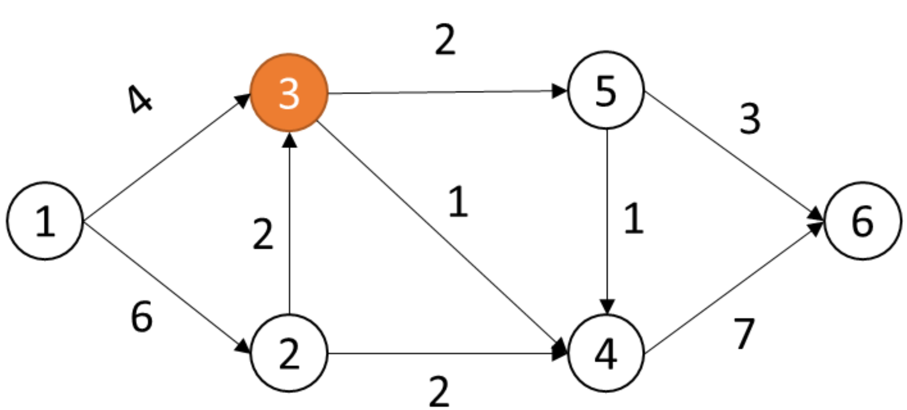
For all algorithm and notations G=(V,A) represents the graph in which V is the set of nodes and A is the set of arcs.

Number of nodes = $n$ in our example graph we have 6 nodes

Number of arcs = $m$ in our example graph we have 9 arcs

We consider $V^{+(i)}$ as the set of imediate successor of node $i$ and $V^{-(i)}$ as the set of immediate predecessor nodes.

In our example graph $V^{+(3)}$ ={5,4} and $V^{-(3)}$ ={1,2}

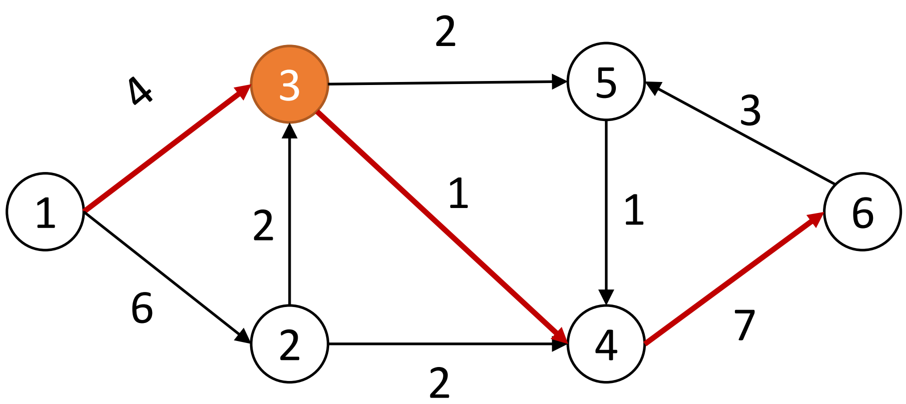


# Modeling by Graphs

A **chain** of a graph G is an alternating sequence of vertices $x_0$, $x_1$,…, $x_n$ beginning and ending with vertices in which each edge is incident with the two vertices immediately preceding and following it. if the first and the last node is the same we have the cycle.

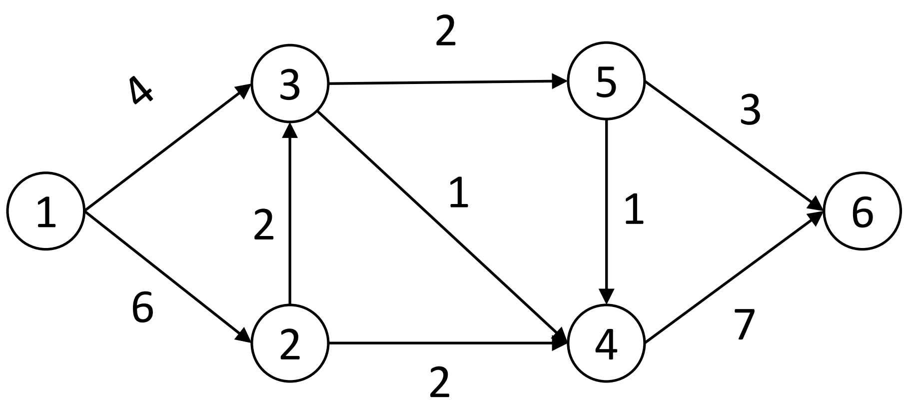
For directed graph chain → path  and cycle → directed cycle

Path={1,3,4,6}
Directed cycle={4,6,5}



# Modeling by Graphs-Min Cost Flow (shortest path)

Graph:




Mathematical Model:

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\sum_{k\in V^+(i)} x_{ik} - \sum_{k\in V^-(i)} x_{ki} = 1 \quad i = s$$

$$\sum_{k\in V^+(i)} x_{ik} - \sum_{k\in V^-(i)} x_{ki} = 0 \quad i \in V \setminus \{s,t\}$$

$$\sum_{k\in V^+(i)} x_{ik} - \sum_{k\in V^-(i)} x_{ki} = -1 \quad i = t$$

$$x_{ij} \geq 0 \quad \forall (i,j) \in A$$

TRANSP-OR

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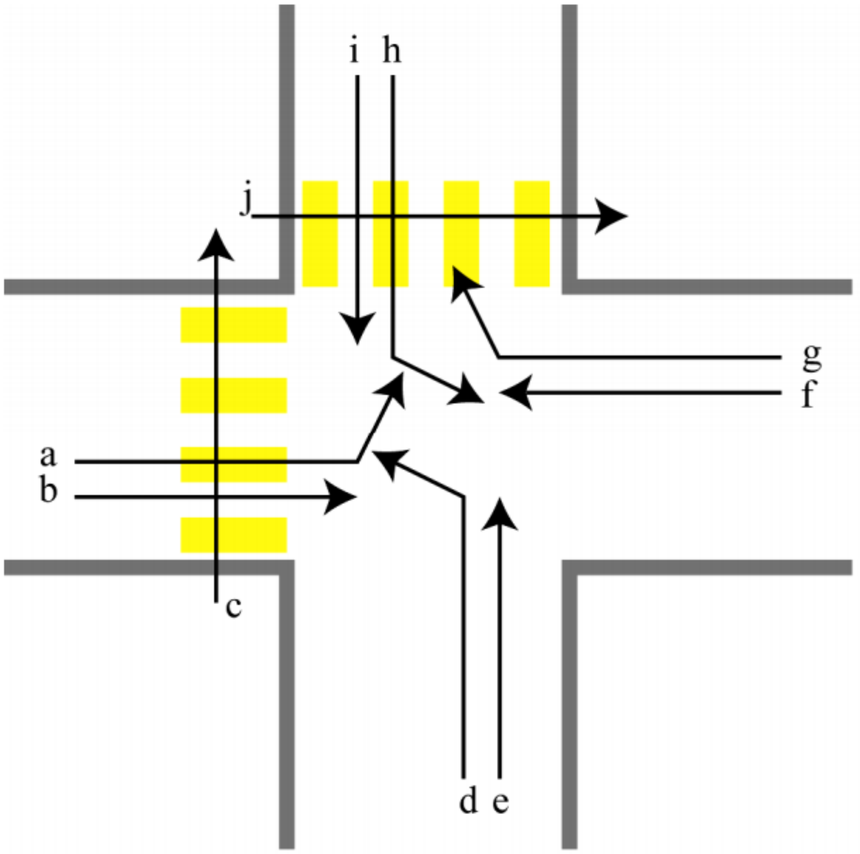
# Modeling by Graphs

**Objective:**

Maximize the green period of each light

Subject to known time of cycle

Minimum green duration for each direction

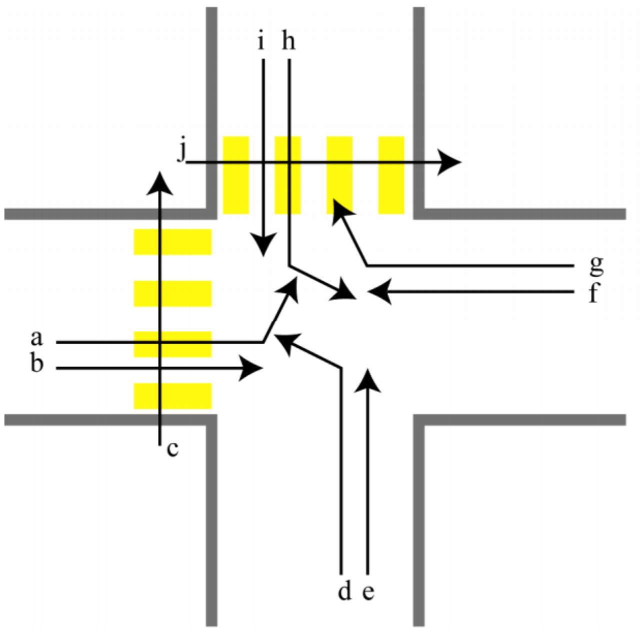


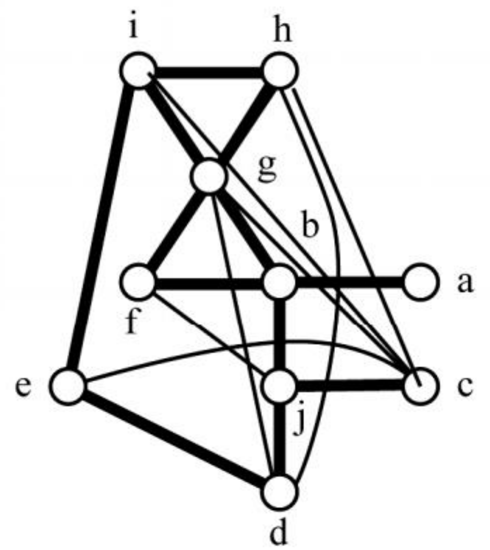
# Modeling by Graphs

We associate a node for each route

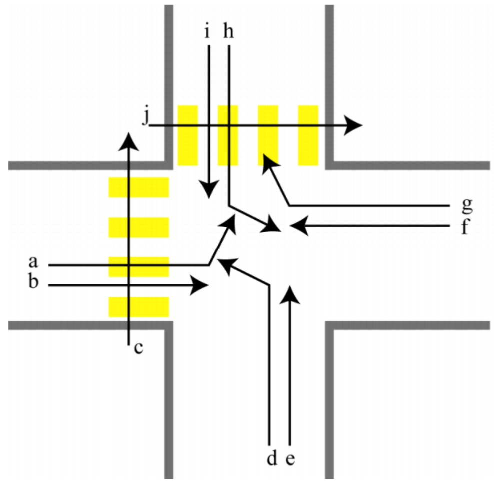
Nodes are connected by an arc if they can perform simultaneously

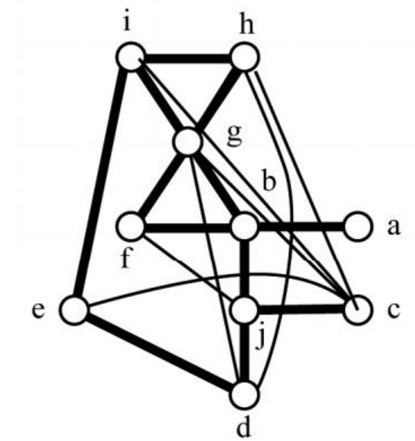
Cover nodes with maximum clique (there is at least one subgraph of at least size $m$ whose vertices are completely connected to each other)

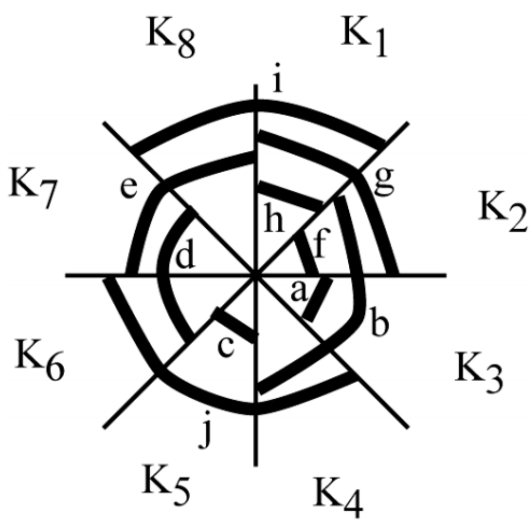


# Modeling by Graphs




$K_1=\{g,h,i\}$, $K_2=\{g,b,f\}$

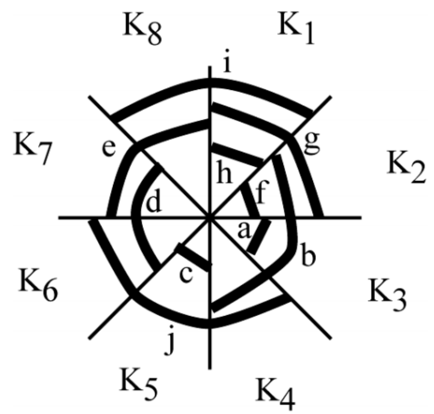
$K_3=\{a,b\}$, $K_4=\{b,j\}$

$K_5=\{c,j\}$, $K_6=\{d,j\}$

$K_7=\{d,e\}$ $K_8=\{e,i\}$

# Modeling by Graphs

- $t = \{a, b, \dots, j\}$
- $x_{K_t}$: time period during which clique $K_t$ is green
- S: minimum time of each direction staying green
- C: Total time of the cycle of repetition




$$
\begin{array}{ll}
\text{Max} & \overset{a}{x_3} + \overset{b}{(x_2+x_3+x_4)} + \overset{c}{x_5} + \overset{d}{(x_6+x_7)} + \overset{e}{(x_7+x_8)} + \overset{f}{x_2} + \overset{g}{(x_1+x_2)} + \overset{h}{x_1} + \overset{i}{(x_8+x_1)} + \overset{j}{(x_4+x_5+x_6)} \\
\text{s.c.} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = C
\end{array}
$$

$$
\begin{array}{rrl}
a & x_3 & \geq s \\
b & x_2+x_3+x_4 & \geq s \\
c & x_5 & \geq s \\
d & x_6+x_7 & \geq s \\
e & x_7+x_8 & \geq s \\
f & x_2 & \geq s \\
g & x_1+x_2 & \geq s \\
h & x_1 & \geq s \\
i & x_8+x_1 & \geq s \\
j & x_4+x_5+x_6 & \geq s
\end{array}
$$

$K_1=\{g,h,i\}$, $K_2=\{g,b,f\}$
$K_3=\{a,b\}$, $K_4=\{b,j\}$
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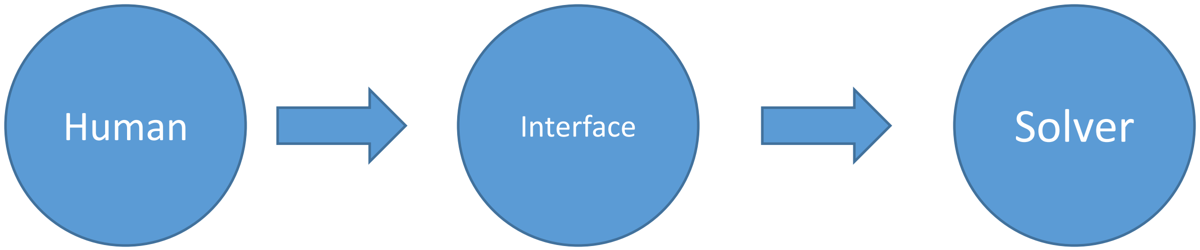
# Software and Solvers

AMPL: **A Modeling Language for Mathematical Programming**




Convert the problem to mathematical terms

Convert mathematical model to the form that is used by solver

Free student version:

- http://www.ampl.com/DOWNLOADS/index.html

Documentation

- http://www.ampl.com/BOOK/download.html

# Software and Solvers

CPLEX:

Very powerful solver can handle upto 1M variables
Primal, dula, interior point , …
Linear programming, integer programming, quadratic programming
Cost: 9600$
Highest market share

X-Press

Primal is the same as CPLEX
The other solvers are not comparable with CPLEX
Cost: 9600$

GUROBI

New solver
Less developed than CPLEX
Cost : 9600$

NEOS (Network-Enabled Optimization System) Server

free Internet-based service for solving optimization problems
You can use all the solvers free
The program must be written with AMPL or GAMS

# References

Richard de Neufville, Joel Clark and Frank R. Field, Intro. to Linear Programming, Massachusetts Institute of Technology.

Hiller, Liberman, Introduction to Operations Research, 7th edition, McGraw-Hill Companies, 2001

Laurence A. Wolsey, Integer Programming, Wiley-Interscience, 1998

Der-San Chen et al. Applied integer proragmming: Modeling and solution 2009