# Decision Aid Methodologies In Transportation
# Lecture 1: Polyhedra and Simplex method


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# Outline

*Teaching syllabuses*

# Topics that will cover

- Introduction to optimization
  1. Polyhedron theory
  2. Linear programming (Simplex method, Duality theory, Column generation)
  3. Integer programming (Cutting plane, Branch and Bound)
  4. Network problem
  5. Approximation methods and heuristics
- Optimization in airline transport
- Optimization in maritime transport
- Optimization in railway transport

# Software will be used in laboratories

IBM ILOG CPLEX Optimization Studio. You can download the trail version, free for 90 days. However, it has problem size limitation. The latest version is V12.5 (platform: Windows, Mac OS & Linux). Download **HERE**. Note that you need to create an IBM account in order to download.

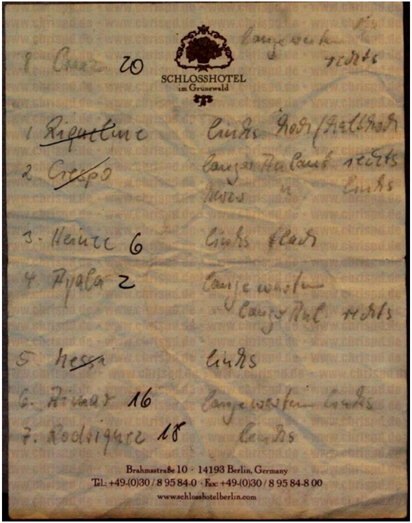
*Decision aid tool and mathematical model*

# Do you still remember?

- 2006 FIFA World Cup (Germany)
- Quarter final match: Germany VS Argentina
- In the penalty shoot-out, Jens Lehmann (goalkeeper) guessed the directions correctly 4 out of 4 and saved two goals!
- The secret:

# Decision aid "tool" — the note

The legendary note was sold for ONE MILLION EUROs!



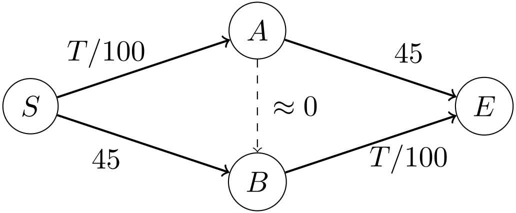
# The steps to build an optimization mathematical model

- Understand the system (problem analysis)
- Establish objective(s)
- Identify the key decision variables
- Spot all the boundary conditions (constraints)
- Model the problem
- Verify the model
- Solve the problem
- Evaluate the results

# The importance of system understanding

### Braess's paradox

building a new road doesn't always lead to shorter traveling time experienced by individual driver.




Suppose all the drivers (4000) are **selfish**, before $A$ and $B$ are connected, half of the travelers will take $SAE$ and the other half take $SBE$ with 65 minutes of traveling time for each driver. After the road $AB$ is built, all the drivers will take route $SABE$ with 80 minutes of traveling time for each driver.

# The importance of system understanding

### The case of Seikan Tunnel

Seikan tunnel connects Honshu and Hokkaido in Japan and is both the **longest and the deepest** operational rail tunnel in the world.

# The importance of system understanding

### Why built Seikan tunnel?

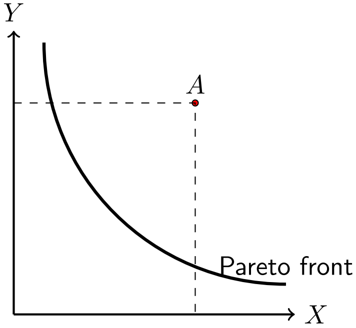
"A booming economy saw traffic levels on the JNR-operated Seikan. Ferry doubled to 4,040,000 persons/year from 1955 to 1965, and cargo levels rose 1.7 times to 6,240,000 tonnes/year."

### After operation in 1988

"However, for passenger transport, **90%** of people use air due to the speed and cost. For example, to travel between Tokyo and Sapporo by train takes more than ten hours and thirty minutes, with several transfers. By air, the journey is three hours and thirty minutes, including airport access times. Deregulation and competition in Japanese domestic air travel has brought down prices on the Tokyo-Sapporo route, making rail more expensive in comparison."

# Establish objective(s)

- The objective should be clearly established for the problem.
- In practice, one problem with multiple objectives are very common. Usually, there are two ways to handle multi-objective problems:
  1. Introduce the weights and transform the multi-objective problem to single objective problem.
  2. Use the concept of Pareto optimality



# Model the problem

If possible, **always** try to build a linear model, i.e., the objective function and all the constraints only contain linear terms of decision variables. For example,

$$\min: \quad x_1^2 + x_2 \tag{1}$$

$$s.t.$$

$$x_1^2 + 2x_2 \leq 1 \tag{2}$$

$$x_1, x_2 \geq 0 \tag{3}$$

is not a good formulation. Instead, if we substitute $x_1^2$ by $y_1$, the resulting formulation is much better since it is a linear one.

# An example for practice

- A shipping company plans to acquire an aircraft and is designing a customized interior to carry thermally insulated and normal products
- Temperature controlled products are sold in the market for a profit of CHF 7 per unit, while normal ones yield a profit of CHF 5 per unit
- Temperature controlled products require 2 kWh electric power and 2 $m^3$ space for carrying one unit
- Normal products requires 1 kWh power and 4 $m^3$ space per unit
- Total power and space availability are 1000 kWh and 2400 $m^3$
- Assuming that aircraft will always fly full capacity, how many units of temperature controlled and normal products should it be designed for?

# An example for practice

- Objective: To maximize the profit for the shipping company
- Decision variables: $x_1$, units of temperature controlled products; $x_2$, units of normal products
- Mathematical model

$$\max: \quad 7x_1 + 5x_2 \tag{4}$$

$$s.t.$$

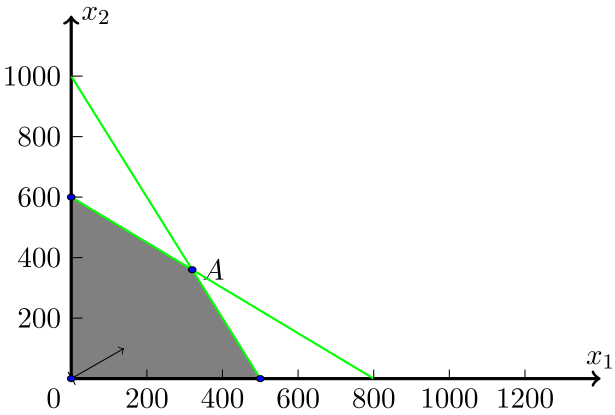
$$2x_1 + x_2 \leq 1000 \tag{5}$$

$$3x_1 + 4x_2 \leq 2400 \tag{6}$$

$$x_1, x_2 \geq 0 \tag{7}$$

In this formulation, (5) reflects the power availability constraint; (6) is the space constraint; and (7) makes sure the results are non-negative.

# An example for practice




Geometric representation of the constraints (the gray part is called the feasible domain). By the theory of linear programming, we know that the optimal solution for this example is "corner point" $A$.

# *Basic polyhedron and convex theory*

# Some notations

In this lecture, matrix $\mathbf{A}$ has $m$ rows and $n$ columns ($\mathbf{A}'$ is the transpose of $\mathbf{A}$; $\mathbf{A}^{-1}$ is the inverse of of $\mathbf{A}$ if it is a square matrix). We use $\mathbf{A}_j$ to denote its $j$th column, that is, $\mathbf{A}_j = (a_{1j}, a_{2j}, \cdots, a_{mj})$. We also use $\mathbf{a}_i$ to denote the vector formed by the entries of the $i$th row, that is, $\mathbf{a}_i = (a_{i1}, a_{i2}, \cdots, a_{in})$. Note that all the vectors are **column** vector.

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} - & \mathbf{a}_1' & - \\ & \vdots & \\ - & \mathbf{a}_m' & - \end{bmatrix}$$

# Polyhedra

### Polyhedron

A polyhedron is a set that can be described in the form $\{\mathbf{x} \in R^n \,|\, \mathbf{Ax} \geq \mathbf{b}\}$, where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b}$ is a vector in $R^m$.

We call $\{\mathbf{x} \in R^n \,|\, \mathbf{Ax} \geq \mathbf{b}\}$ the *general form* of a polyhedron; $\{\mathbf{x} \in R^n \,|\, \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ the *standard form* of a polyhedron.

### Hyperplane and halfspace

Let $\mathbf{a}$ be a nonzero vector in $R^n$ and let $b$ be a scalar. Then the set $\{\mathbf{x} \in R^n \,|\, \mathbf{a}'\mathbf{x} = b\}$ is called a hyperplane and the set $\{\mathbf{x} \in R^n \,|\, \mathbf{a}'\mathbf{x} \geq b\}$ is called a halfspace.

Note that both hyperplane and halfspace are polyhedra and a hyperplane is the boundary of a corresponding halfspace.

# Convex set

### Convex set

A set $S \subset R^n$ is convex if for any $\mathbf{x}, \mathbf{y} \in S$, and any $\lambda \in [0,1]$, we have $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S$.

### Convex combination & convex hull

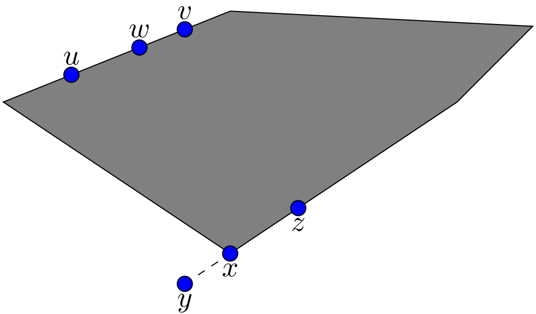
Let $\mathbf{x}^1, \cdots, \mathbf{x}^k$ be vectors in $R^n$ and let $\lambda_1, \cdots, \lambda_k$ be nonnegative scalars whose sum is unity. The vector $\sum_{i=1}^k \lambda_i \mathbf{x}^i$ is said to be a convex combination of the vector $\mathbf{x}^1, \cdots, \mathbf{x}^k$; The convex hull of the vector $\mathbf{x}^1, \cdots, \mathbf{x}^k$ is the set of all convex combination of these vectors.

# Extreme point of a polyhedron

### Extreme point

Let $P$ be a polyhedron. A vector $\mathbf{x} \in P$ is an extreme point of $P$ if we cannot find two vectors $\mathbf{y}, \mathbf{z} \in P$, both different from $\mathbf{x}$, and a scalar $\lambda \in [0, 1]$, such that $\mathbf{x} = \lambda \mathbf{y} + (1 - \lambda)\mathbf{z}$.

Geometrically speaking, an extreme point of a polyhedron is the "corner point" of the polyhedron.



# Basic solution and basic feasible solution associated with a polyhedron

### Basic solution & basic feasible solution

Consider a polyhedron $P$ defined by linear equality and inequality constraints, and let $\mathbf{x}^*$ be an element of $R^n$. The vector $\mathbf{x}^*$ is a basic solution if:

1. All equality constraints are active;
2. Out of the constraints that are active at $\mathbf{x}^*$, there are $n$ of them that are linearly independent.

If $\mathbf{x}^*$ is a basic solution that satisfies all of the constraints, we say that it is a basic feasible solution.

For example, if $P = \{x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\}$, then $(1, 0)$ and $(0, 1)$ are basic (feasible) solutions of $P$.

# Extreme point=basic feasible solution

Extreme point and basic feasible solution associated with a polyhedron are equivalent. That is, if $\mathbf{x} \in P$ is an extreme point of $P$ then $\mathbf{x}$ is also a basic feasible solution of $P$, vice versa. For a polyhedron, extreme point is the geometric interpretation of basic feasible solution and basic feasible solution is the algebraic representation of extreme point.

# Recession cone of a polyhedron

Consider a nonempty polyhedron

$$P = \{\mathbf{x} \in R^n \,|\, \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$$

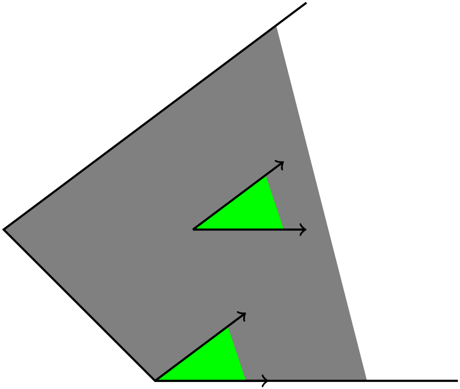
and let us fix some $\mathbf{y} \in P$. We define the **recession cone** at $\mathbf{y}$ as the set of all directions $\mathbf{d}$ along which we can move **indefinitely away from** $\mathbf{y}$, without leaving the set $P$. That is, the recession cone of $P$ is the set

$$\{\mathbf{d} \in R^n \,|\, \mathbf{A}(\mathbf{y} + \lambda\mathbf{d}) \geq \mathbf{b}, \forall\lambda \geq 0\}$$

or more concise

$$\{\mathbf{d} \in R^n \,|\, \mathbf{A}\mathbf{d} \geq \mathbf{0}\}$$

# Recession cone of a polyhedron



A polyhedron and its recession cone

# Extreme rays of a polyhedron

Intuitively, extreme rays of a polyhedron are the directions associated with "**edges**" of the polyhedron that **extend to infinity**.

## Extreme ray

- A nonzero element $\mathbf{x}$ of a recession cone $C \subset R^n$ is called an extreme ray if there are $n-1$ linearly independent constraints that are active at $\mathbf{x}$.
- An extreme ray of the recession cone associated with a nonempty polyhedron $P$ is also called an extreme ray of $P$.

# Why to study the extreme points & extreme rays of a polyhedron?

Consider a linear programming problem

$$\begin{aligned} \min: \quad & \mathbf{c}'\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{aligned}$$

Let $\{\mathbf{x}^1, \cdots, \mathbf{x}^p\}$ be the extreme points of the polyhedron $P = \{\mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ and $\{\mathbf{w}^1, \cdots, \mathbf{w}^q\}$ be the set of extreme rays of $P$. The we have the following theorem:

**Optimality of extreme points**

- If $\mathbf{c}'\mathbf{w}^i \geq 0, \forall 1 \leq i \leq q$, there exists an extreme point in $\{\mathbf{x}^1, \cdots, \mathbf{x}^p\}$ which is optimal;
- Otherwise (i.e., there exists $i$, $\mathbf{c}'\mathbf{w}^i < 0$), the optimal value is equal to $-\infty$.

# Why to study the extreme points & extreme rays of a polyhedron?

Besides, we have the following famous theorom:

**Resolution theorem**

Let $\{\mathbf{x}^1, \cdots, \mathbf{x}^p\}$ be the extreme points of the polyhedron $P = \{\mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ and $\{\mathbf{w}^1, \cdots, \mathbf{w}^q\}$ be the set of extreme rays of $P$. The polyhedron $P$ can also be expressed as

$$P = \left\{ \sum_{i=1}^{p} \lambda_i \mathbf{x}^i + \sum_{j=1}^{q} \theta_j \mathbf{w}^j \,|\, \lambda_i \geq 0, \theta_j \geq 0, \sum_{i=1}^{p} \lambda_i = 1 \right\}$$

Therefore, a polyhedron can be expressed as (1) intersection of finite number of halfspaces (2) the sum of convex combination of extreme points and nonnegative combination (also call conic combination) of extreme rays.

# *Linear programming and Simplex method*

# General form to standard form

How to transform an LP in general form to standard form?

1. $\mathbf{a}'\mathbf{x} \leq b$: Introduce a slack variable $s \geq 0$ and $\mathbf{a}'\mathbf{x} + s = b$.
2. $\mathbf{a}'\mathbf{x} \geq b$: Introduce a surpass variable $p \geq 0$ and $\mathbf{a}'\mathbf{x} - p = b$.
3. $x_i$ is a free variable: Introduce two new variables $y, z \geq 0$ and $x_i = y - z$.

# Why standard form?

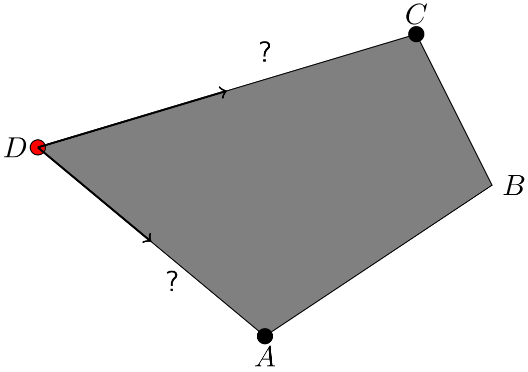
$$\begin{aligned} \min: \quad & \mathbf{c}'\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Without loss of generality, we can assume that matrix $\mathbf{A}$ has $m$ linearly independent rows and $n \gg m$ (in reality, this is usually the case). Question: how to obtain a basic solution quickly?

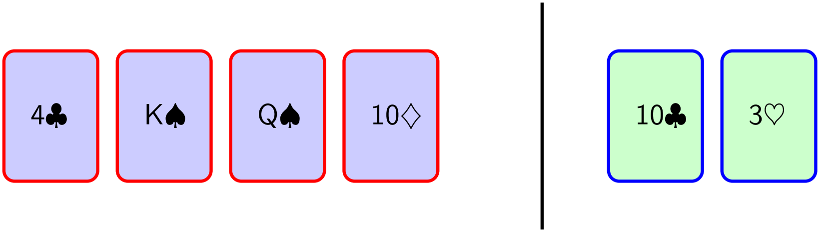
$$\mathbf{A} = \begin{bmatrix} | & | & | & & | \\ \color{red}{\mathbf{A}_1} & \mathbf{A}_2 & \color{red}{\mathbf{A}_3} & \cdots & \color{red}{\mathbf{A}_n} \\ | & | & | & & | \end{bmatrix}$$

$\mathbf{A}_1$, $\mathbf{A}_3$, $\cdots$, $\mathbf{A}_n$ are linearly independent and we call them **the base** and their associated $m$ variables $(x_1, x_3, \cdots, x_n)$ **basic variables**.

# The basic idea of Simplex method



# The basic idea of Simplex method




The objective of this card game is to increase the sum of card numbers in your hand (at most 4 cards). Which two cards you will switch?

# Simplex method in math form

Let $\mathbf{x}$ be a basic feasible solution to the standard form problem, let $B(1), B(2), \cdots, B(m)$ be the indices of the basic variables, and let $\mathbf{B} = [\mathbf{A}_{B(1)} \cdots \mathbf{A}_{B(m)}]$ be the corresponding basis matrix. In particular, we have $x_i = 0$ for every nonbasic variable, while the vector $\mathbf{x}_B = (x_{B(1)}, \cdots, x_{B(m)})$ of basic variables is given by

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$

We consider the possibility of moving away from $\mathbf{x}$, to a new vector $\mathbf{x} + \theta\mathbf{d}$ ($\theta > 0$). Here we choose a special direction $\mathbf{d}$: select a nonbasic variable $x_j$ (which is initially at zero level) and try to increase it to a positive value while keep the remaining nonbasic variables at zero. Algebraically, $d_j = 1$, and $d_i = 0$ for every nonbasic index $i$ other than $j$. At the same time, the vector $\mathbf{x}_B$ of basic variables changes to $\mathbf{x}_B + \theta\mathbf{d}_B$, where $\mathbf{d}_B = (d_{B(1)}, d_{B(2)}, \ldots, d_{B(m)})$ is the vector with those components of $\mathbf{d}$ that correspond to the basic variables.

# Simplex method in math form

Since $\mathbf{x} + \theta\mathbf{d}$ should be a feasible solution, we have $\mathbf{A}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{A}\mathbf{x} + \theta\mathbf{A}\mathbf{d} = \mathbf{b}$. That is $\mathbf{A}\mathbf{d} = \mathbf{0}$. Thus, we have:

$$\mathbf{0} = \mathbf{A}\mathbf{d} = \sum_{i=1}^{n} \mathbf{A}_i d_i = \sum_{i=1}^{m} \mathbf{A}_{B(i)} d_{B(i)} + \mathbf{A}_j = \mathbf{B}\mathbf{d}_B + \mathbf{A}_j$$

Since the basis matrix $\mathbf{B}$ is invertible, we have

$$\mathbf{d}_B = -\mathbf{B}^{-1}\mathbf{A}_j$$

The obtained direction $\mathbf{d}$ is called the $j$th basic direction.
**Question**: how many basic directions are there?

# Simplex method in math form

What would be the effects on the cost function if we move along a basic direction? If $\mathbf{d}$ is the $j$th basic direction, then the rate $\mathbf{c}'\mathbf{d}$ of cost change along the direction $\mathbf{d}$ is given by $\mathbf{c}'_B\mathbf{d}_B + c_j$, where $\mathbf{c}_B = (c_{B(1)}, \ldots, c_{B(m)})$. This is the same as

$$c_j - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{A}_j$$

The above value is defined as the **reduced cost** $\bar{c}_j$ of the variable $x_j$.

Reduced costs for basic variables?!

The reduced cost for any basic variable is always 0.

# The importance of reduced costs

Consider a basic feasible solution $\mathbf{x}$ associated with a basis matrix $\mathbf{B}$, and let $\overline{\mathbf{c}}$ be the corresponding vector of reduced costs. If $\overline{\mathbf{c}} \geq \mathbf{0}$, then $\mathbf{x}$ is optimal.

**Proof:**

Assume that $\overline{\mathbf{c}} \geq 0$ and let $\mathbf{y}$ be an arbitrary feasible solution, and define $\mathbf{d} = \mathbf{y} - \mathbf{x}$. Feasibility implies that $\mathbf{Ax} = \mathbf{Ay} = \mathbf{b}$ and, therefore, $\mathbf{Ad} = \mathbf{0}$. The latter equality can be rewritten in the form $\mathbf{Bd}_B + \sum_{i \in N} \mathbf{A}_i d_i = \mathbf{0}$, where $N$ is the set of indices corresponding to the nonbasic variables under the given basis. Since $\mathbf{B}$ is invertible, we have $\mathbf{d}_B = -\sum_{i \in N} \mathbf{B}^{-1}\mathbf{A}_i d_i$ and

$$\mathbf{c}'\mathbf{d} = \mathbf{c}'_B\mathbf{d}_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{A}_i) d_i = \sum_{i \in N} \overline{c}_i d_i$$

Since, $\forall i \in N$, $x_i = 0$, $y_i \geq 0$. Thus, $d_i \geq 0$ and $\overline{c}_i d_i \geq 0$ and $\mathbf{c}'\mathbf{d} = \mathbf{c}'(\mathbf{y} - \mathbf{x}) \geq 0$. $\mathbf{x}$ is optimal.

# The pivot process

At a basic feasible solution $\mathbf{x}$ and we have computed the reduced cost vector $\overline{\mathbf{c}}$. If $\overline{\mathbf{c}} \geq 0$, we can stop since $\mathbf{x}$ is optimal. If $\overline{c}_j < 0$, then the $j$th basic direction $\mathbf{d}$ is a **feasible and profitable** direction. While moving along $\mathbf{d}$, the nonbasic variable $x_j$ becomes positive and all other nonbasic variables remain at 0 (i.e., $x_j$ is brought into the basis). Since costs decrease along the direction $\mathbf{d}$, it is desirable to move as far as possible (Greedy!).

$$\theta^* = \max\{\theta \geq 0 | \mathbf{x} + \theta\mathbf{d} \geq \mathbf{0}\}$$

If $d_i < 0$ for some $i$, the largest possible value of $\theta$ is

$$\theta^* = \min_{\{i|d_i<0\}} \left(\frac{-x_i}{d_i}\right)$$

If $\theta$ is chosen as $\theta^*$, then at least one of the basic variables, say, $x_i$ will become 0 (i.e., $x_i$ leaves the basis).

# Simplex method essential steps

1. Start with a basis consisting of the basic columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$, and an associated basic feasible solution $\mathbf{x}$.
2. Compute the reduced costs $\bar{c}_j = c_j - \mathbf{c}'_B \mathbf{B}^{-1} \mathbf{A}_j$ for all nonbasic indices $j$. If they are all nonnegative, the current basic feasible solution is optimal, and the algorithm terminates; else, choose some $j$ for which $\bar{c}_j < 0$.
3. Calculate $\theta^* = \min_{\{i|d_i<0\}} \left( \frac{-x_i}{d_i} \right)$ and determine the variable to leave the basis.

# An example for practice

$$\begin{aligned} \max: \quad & 7x_1 + 5x_2 \\ s.t. \quad & 2x_1 + x_2 \leq 1000 \\ & 3x_1 + 4x_2 \leq 2400 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min: \quad & -7x_1 - 5x_2 \\ s.t. \quad & 2x_1 + x_2 + x_3 = 1000 \\ & 3x_1 + 4x_2 + x_4 = 2400 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$\mathbf{A} = \left[ \begin{array}{cccc} 2 & 1 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right]$$

$$\mathbf{c} = \left[ \begin{array}{cccc} -7 & -5 & 0 & 0 \end{array} \right]$$

# An example for practice

1. $\mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,$\mathbf{x}_1 = [0, 0, 1000, 2400]$,$\bar{\mathbf{c}}_1 = \mathbf{c} - [0, 0]\mathbf{B}_1^{-1}\mathbf{A} = [-\mathbf{7}, -5, 0, 0]$, $\mathbf{d}_1 = [1, 0, -2, -3]$, $\theta_1^* = \min\{1000/2, 2400/3\}$, $x_3$ will be out of basis.
2. $\mathbf{B}_2 = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$,$\mathbf{x}_2 = [500, 0, 0, 900]$,$\bar{\mathbf{c}}_2 = \mathbf{c} - [-7, 1]\mathbf{B}_2^{-1}\mathbf{A} = [0, -\mathbf{1.5}, 3.5, 0]$, $\mathbf{d}_2 = [-0.5, 1, 0, -2.5]$, $\theta_2^* = \min\{500/0.5, 900/2.5\}$, $x_4$ will be out of basis.
3. $\mathbf{B}_3 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$,$\mathbf{x}_3 = [320, 360, 0, 0]$,$\bar{\mathbf{c}}_3 = \mathbf{c} - [-7, -5]\mathbf{B}_3^{-1}\mathbf{A} = [0, 0, 2.6, 0.6]$, $\mathbf{x}_3$ is optimal.

# Tabular implementation of Simplex

| $-\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{b}$ | $\mathbf{c}' - \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{A}$ |
|---|---|
| $\mathbf{B}^{-1}\mathbf{b}$ | $\mathbf{B}^{-1}\mathbf{A}$ |

In more detail,

| $-\mathbf{c}'_B\mathbf{x}_B$ | $\bar{c}_1$ | $\cdots$ | $\bar{c}_n$ |
|---|---|---|---|
| $x_{B(1)}$ | $\vert$ | | $\vert$ |
| $\vdots$ | $\mathbf{B}^{-1}\mathbf{A}_1$ | $\cdots$ | $\mathbf{B}^{-1}\mathbf{A}_n$ |
| $x_{B(m)}$ | $\vert$ | | $\vert$ |

# Tabular implementation of Simplex

$$
\begin{array}{llllll}
\min & -7x_1 & -5x_2 & & & \\
s.t. & 2x_1 & +x_2 & +x_3 & & = 1000 \\
 & 3x_1 & +4x_2 & & +x_4 & = 2400 \\
 & x_1, & x_2, & x_3, & x_4 & \geq 0
\end{array}
$$

It is easy to observe that initially, we can choose $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In the tabular form:

| | | | | | |
|---|---|---|---|---|---|
| | 0 | -7 | -5 | 0 | 0 |
| $x_3 =$ | 1000 | 2 | 1 | 1 | 0 |
| $x_4 =$ | 2400 | 3 | 4 | 0 | 1 |

# Tabular implementation of Simplex

| | | $\overline{c}_1$ | $\overline{c}_2$ | $\overline{c}_3$ | $\overline{c}_4$ | |
|---|---|---|---|---|---|---|
| | 0 | $-7^*$ | -5 | 0 | 0 | |
| $x_3^* =$ | 1000 | $2^*$ | 1 | 1 | 0 | $\frac{1000}{2} = 500$ |
| $x_4 =$ | 2400 | 3 | 4 | 0 | 1 | $\frac{2400}{3} = 800$ |

# Tabular implementation of Simplex

| | $\overline{c}_1$ | $\overline{c}_2$ | $\overline{c}_3$ | $\overline{c}_4$ |
|---|---|---|---|---|
| 0 | $-7^*$ | -5 | 0 | 0 |
| 500 | 1 | 0.5 | 0.5 | 0 |
| | | | | |
| 2400 | 3 | 4 | 0 | 1 |

# Tabular implementation of Simplex

| | $\overline{c}_1$ | $\overline{c}_2$ | $\overline{c}_3$ | $\overline{c}_4$ |
|---|---|---|---|---|
| 0 | $-7^*$ | -5 | 0 | 0 |
| 500 | 1 | 0.5 | 0.5 | 0 |
| 900 | 0 | 2.5 | -1.5 | 1 |

# Tabular implementation of Simplex

| | $\overline{c}_1$ | $\overline{c}_2$ | $\overline{c}_3$ | $\overline{c}_4$ |
|---|---|---|---|---|
| 3500 | 0 | -1.5 | 3.5 | 0 |
| 500 | 1 | 0.5 | 0.5 | 0 |
| 900 | 0 | 2.5 | -1.5 | 1 |

# Tabular implementation of Simplex

| | | $\overline{c}_1$ | $\overline{c}_2$ | $\overline{c}_3$ | $\overline{c}_4$ |
|---|---|---|---|---|---|
| | 3500 | 0 | -1.5 | 3.5 | 0 |
| $x_1 =$ | 500 | 1 | 0.5 | 0.5 | 0 |
| $x_4 =$ | 900 | 0 | 2.5 | -1.5 | 1 |

# Tabular implementation of Simplex

| | | $\overline{c}_1$ | $\overline{c}_2$ | $\overline{c}_3$ | $\overline{c}_4$ | |
|---|---|---|---|---|---|---|
| | 3500 | 0 | $-1.5^*$ | 3.5 | 0 | |
| $x_1 =$ | 500 | 1 | 0.5 | 0.5 | 0 | $\frac{500}{0.5} = 1000$ |
| $x_4^* =$ | 900 | 0 | $2.5^*$ | -1.5 | 1 | $\frac{900}{2.5} = 360$ |

# Tabular implementation of Simplex

| | $\overline{c}_1$ | $\overline{c}_2$ | $\overline{c}_3$ | $\overline{c}_4$ |
|---|---|---|---|---|
| 3500 | 0 | -1.5 | 3.5 | 0 |
| 500 | 1 | 0.5 | 0.5 | 0 |
| 360 | 0 | 1 | -0.6 | 0.4 |

# Tabular implementation of Simplex

| | $\overline{c}_1$ | $\overline{c}_2$ | $\overline{c}_3$ | $\overline{c}_4$ |
|---|---|---|---|---|
| 3500 | 0 | -1.5 | 3.5 | 0 |
| 320 | 1 | 0 | 0.3 | -0.2 |
| 360 | 0 | 1 | -0.6 | 0.4 |

# Tabular implementation of Simplex

| | $\overline{c}_1$ | $\overline{c}_2$ | $\overline{c}_3$ | $\overline{c}_4$ |
|---|---|---|---|---|
| 4040 | 0 | 0 | 2.6 | 0.6 |
| 320 | 1 | 0 | 0.3 | -0.2 |
| 360 | 0 | 1 | -0.6 | 0.4 |

# Tabular implementation of Simplex

| | | $\bar{c}_1$ | $\bar{c}_2$ | $\bar{c}_3$ | $\bar{c}_4$ |
|---|---|---|---|---|---|
| | 4040 | 0 | 0 | 2.6 | 0.6 |
| $x_1 =$ | 320 | 1 | 0 | 0.3 | -0.2 |
| $x_2 =$ | 360 | 0 | 1 | -0.6 | 0.4 |