
Logit with multiple alternatives

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Logit Model

For all $i \in \mathcal{C}_n$,

$$U_{in} = V_{in} + \varepsilon_{in}$$

- What is \mathcal{C}_n ?
- What is V_{in} ?
- What is ε_{in} ?

Choice set

Universal choice set

- All potential alternatives for the population
- Restricted to relevant alternatives

Mode choice:

driving alone	sharing a ride	taxi
motorcycle	bicycle	walking
transit bus	rail rapid transit	horse

Choice set

Individual's choice set

- No driver license
- No auto available
- Awareness of transit services
- Transit services unreachable
- Walking not an option for long distance

Choice set

Individual's choice set

Choice set generation is tricky

- How to model “awareness”?
- What does “long distance” exactly mean?
- What does “unreachable” exactly mean?

We assume here deterministic rules

Logit model

Main assumption: ε_{in} are

- extreme value $EV(0, \mu)$,
- independent and
- identically distributed.

Comments:

- Independence: across i and n .
- Identical distribution: same scale parameter μ across i and n .

$$P_n(i|\mathcal{C}_n) = \frac{e^{\mu V_{in}}}{\sum_{j \in \mathcal{C}_n} e^{\mu V_{jn}}}$$

Systematic part of the utility function

$$V_{in} = V(z_{in}, S_n)$$

where

- z_{in} is a vector of attributes of alternative i for individual n
- S_n is a vector of socio-economic characteristics of n

Outline:

- Functional form: linear utility
- Explanatory variables: What is exactly contained in z_{in} and S_n ?
- Functional form: capturing nonlinearities
- Interactions

Functional form: linear utility

Notation:

$$x_{in} = (z_{in}, S_n)$$

In general, linear-in-parameters utility functions are used

$$V_{in} = V(z_{in}, S_n) = V(x_{in}) = \sum_p \beta_p (x_{in})_p$$

Not as restrictive as it may seem

Explanatory variables: alternatives attributes

- Numerical and continuous
- $(z_{in})_p \in \mathbb{R}, \forall i, n, p$
- Associated with a specific unit

Examples:

- Auto in-vehicle time (in min.)
- Transit in-vehicle time (in min.)
- Auto out-of-pocket cost (in cents)
- Transit fare (in cents)
- Walking time to the bus stop (in min.)

Straightforward modeling

Explanatory variables: alternatives attributes

- V_{in} is unitless
- Therefore, β depends on the unit of the associated attribute
- Example: consider two specifications

$$V_{in} = \beta_1 TT_{in} + \dots$$

$$V_{in} = \beta'_1 TT'_{in} + \dots$$

- If TT_{in} is a number of minutes, the unit of β_1 is 1/min
- If TT'_{in} is a number of hours, the unit of β'_1 is 1/hour
- Both models are equivalent, but the estimated value of the coefficient will be different

$$\beta_1 TT_{in} = \beta'_1 TT'_{in} \implies \frac{TT_{in}}{TT'_{in}} = \frac{\beta'_1}{\beta_1} = 60$$

Explanatory variables: alternatives attributes

Generic and alternative specific parameters

$$V_{\text{auto}} = \beta_1 \text{TT}_{\text{auto}}$$

$$V_{\text{bus}} = \beta_1 \text{TT}_{\text{bus}}$$

or

$$V_{\text{auto}} = \beta_1 \text{TT}_{\text{auto}}$$

$$V_{\text{bus}} = \beta_2 \text{TT}_{\text{bus}}$$

Modeling assumption: a minute has/has not the same marginal utility whether it is incurred on the auto or bus mode

Explanatory variables: socio-eco. characteristics

- Numerical and continuous
- $(S_n)_p \in \mathbb{R}, \forall n, p$
- Associated with a specific unit

Examples:

- Annual income (in KCHF)
- Age (in years)

Warning: S_n do not depend on i

Explanatory variables: socio-eco. characteristics

They cannot appear in all utility functions

$$\left. \begin{array}{l} V_1 = \beta_1 x_{11} + \beta_2 \text{income} \\ V_2 = \beta_1 x_{21} + \beta_2 \text{income} \\ V_3 = \beta_1 x_{31} + \beta_2 \text{income} \end{array} \right\} \iff \left\{ \begin{array}{l} V'_1 = \beta_1 x_{11} \\ V'_2 = \beta_1 x_{21} \\ V'_3 = \beta_1 x_{31} \end{array} \right.$$

In general: alternative specific characteristics

$$\begin{array}{l} V_1 = \beta_1 x_{11} + \beta_2 \text{income} + \beta_4 \text{age} \\ V_2 = \beta_1 x_{21} + \beta_3 \text{income} + \beta_5 \text{age} \\ V_3 = \beta_1 x_{31} \end{array}$$

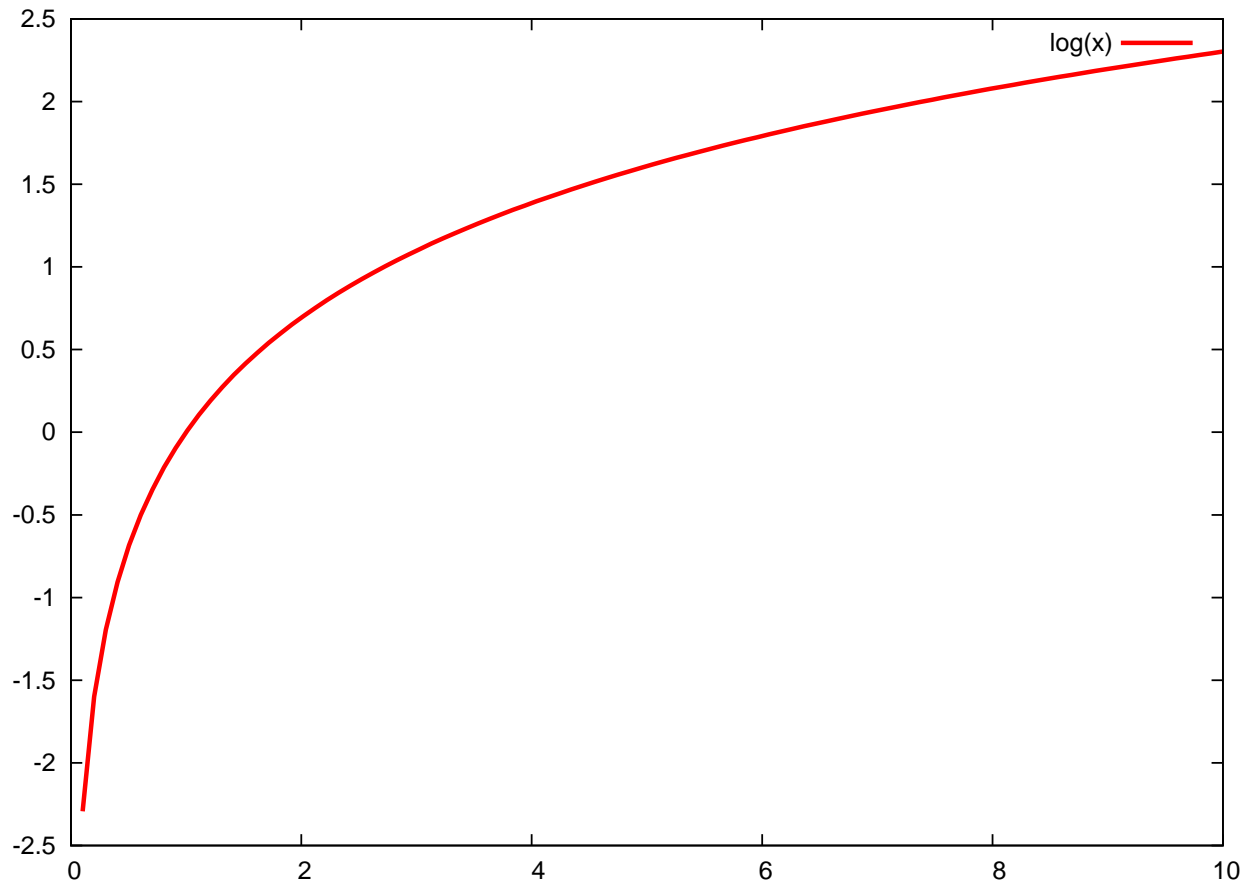
Functional form: dealing with nonlinearities

- Nonlinear transformations of the independent variables
- Discrete and qualitative variables
- Continuous variables
 - Categories
 - Splines
 - Box-Cox
 - Power series

Nonlinear transformations of the variables

- Compare a trip of 5 min with a trip of 10 min
- Compare a trip of 120 min with a trip of 125 min

Nonlinear transformations of the variables



Nonlinear transformations of the variables

Instead of

$$V_i = \beta \text{time}_i$$

one can use

$$V_i = \beta \ln(\text{time}_i)$$

It is still a linear-in-parameters form

Nonlinear transformations of the variables

Another example: **disposable income**

$$\max(\text{household income}(\$/\text{year}) - s \times \text{nbr of persons}, 0)$$

where s is the subsistence budget per person

Data can be preprocessed to account for nonlinearities

$$V_{in} = V(h(z_{in}, S_n)) = \sum_k \beta_k (h(z_{in}, S_n))_k$$

is linear-in-parameter, even with h nonlinear.

Discrete variables

- Mainly used to capture qualitative attributes
 - Level of comfort for the train
 - Reliability of the bus
 - Color
 - Shape
 - etc...
- or characteristics
 - Sex
 - Education
 - Professional status
 - etc.

Discrete variables

Procedure for model specification:

- Identify all possible levels of the attribute: **Very comfortable, Comfortable, Rather comfortable, Not comfortable.**
- Select a base case: **very comfortable**
- Define numerical attributes
- Adopt a coding convention

Discrete variables

Numerical attributes

Introduce a 0/1 attribute for all levels except the base case

- z_c for *comfortable*
- z_{rc} for *rather comfortable*
- z_{nc} for *not comfortable*

Discrete variables

Coding convention

	z_c	z_{rc}	z_{nc}
very comfortable	0	0	0
comfortable	1	0	0
rather comfortable	0	1	0
not comfortable	0	0	1

If a qualitative attribute has n levels, we introduce $n - 1$ variables (0/1) in the model

Discrete variables

Comparing two ways of coding:

	z_{vc}	z_c	z_{rc}	z_{nc}
very comfortable	1	0	0	0
comfortable	0	1	0	0
rather comfortable	0	0	1	0
not comfortable	0	0	0	1

$$V_{in} = \dots + \beta_{vc} z_{ivc} + \beta_c z_{ic} + \beta_{rc} z_{irc} + \beta_{nc} z_{inc} \quad \beta_{vc} = 0$$

$$V'_{in} = \dots + \beta'_{vc} z_{ivc} + \beta'_c z_{ic} + \beta'_{rc} z_{irc} + \beta'_{nc} z_{inc} \quad \beta'_c = 0$$

Linear-in-parameter specification

Let's add a constant to all β 's

Discrete variables

$$V_{in} = \dots + \beta_{vc} z_{ivc} + \beta_c z_{ic} + \beta_{rc} z_{irc} + \beta_{nc} z_{inc} \quad \beta_{vc} = 0$$

$$V'_{in} = \dots + \beta'_{vc} z_{ivc} + \beta'_c z_{ic} + \beta'_{rc} z_{irc} + \beta'_{nc} z_{inc} \quad \beta'_c = 0$$

$$\begin{aligned} V_{in} &= \dots + (\beta_{vc} + K) z_{ivc} + (\beta_c + K) z_{ic} + (\beta_{rc} + K) z_{irc} + (\beta_{nc} + K) z_{inc} \\ &= \dots + \beta_{vc} z_{ivc} + \beta_c z_{ic} + \beta_{rc} z_{irc} + \beta_{nc} z_{inc} + K(z_{ivc} + z_{ic} + z_{irc} + z_{inc}) \\ &= \dots + \beta_{vc} z_{ivc} + \beta_c z_{ic} + \beta_{rc} z_{irc} + \beta_{nc} z_{inc} + K \end{aligned}$$

- $K = -\beta_{vc}$: very comfortable as the base case
- $K = -\beta_c$: comfortable as the base case
- $K = -\beta_{rc}$: rather comfortable as the base case
- $K = -\beta_{nc}$: not comfortable as the base case

Discrete variables

Example of estimation with Biogeme:

	Model 1	Model 2
ASC	0.574	0.574
BETA_VC	0.000	0.918
BETA_C	-0.919	0.000
BETA_RC	-1.015	-0.096
BETA_NC	-2.128	-1.210

Continuous variables: categories

- Assumption: sensitivity to travel time varies with travel time
- Using β_{TT} is not appropriate anymore
- Categories are defined: travel time in minutes
[0–90[, [90–180[, [180–270[, [270– [
- Solutions:
 - Categories with constants (inferior solution)
 - Piecewise linear specification (spline)

Continuous variables: categories

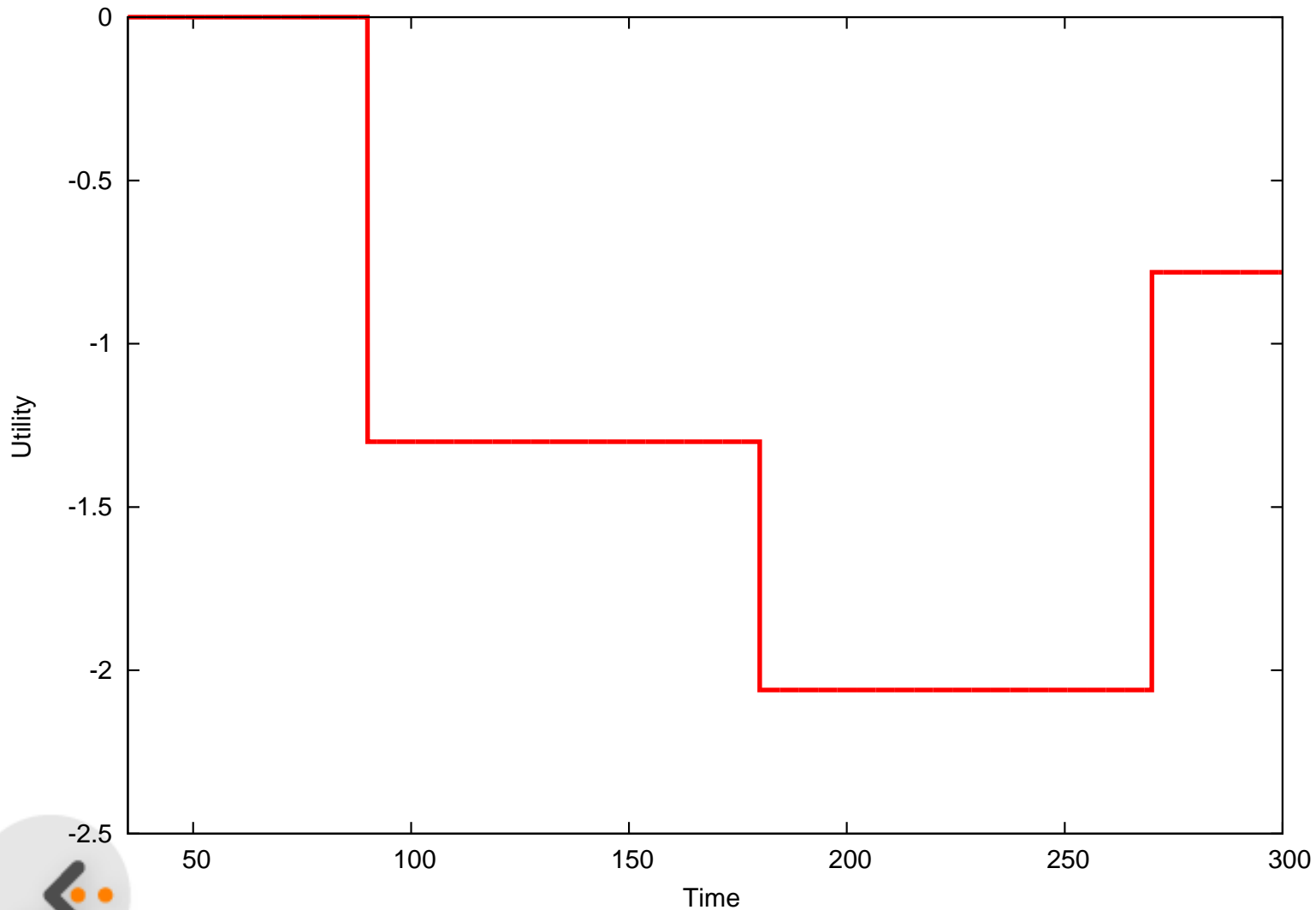
Categories with constants

- Same specification as for discrete variables

$$V_i = \beta_{T1}x_{T1} + \beta_{T2}x_{T2} + \beta_{T3}x_{T3} + \beta_{T4}x_{T4} + \dots$$

- with
 - $x_{T1} = 1$ if $TT_i \in [0-90[$, 0 otherwise
 - $x_{T2} = 1$ if $TT_i \in [90-180[$, 0 otherwise
 - $x_{T3} = 1$ if $TT_i \in [180-270[$, 0 otherwise
 - $x_{T4} = 1$ if $TT_i \in [270-[,$ 0 otherwise
- One β must be normalized to 0.

Continuous variables: categories



Continuous variables: categories

Drawbacks

- No sensitivity to travel time within the intervals
- Discontinuous utility function (jumps)
- Need for many small intervals
- Results may vary significantly with the definition of the intervals

Appropriate when

- Categories have been used in the survey (income, age)
- Definition of categories is natural (weekday)

Continuous variables: categories

Piecewise linear specification (spline)

- Capture the sensitivity within the intervals
- Enforce continuity of the utility function

Piecewise linear specification

- Specification:

$$V_i = \beta_{T1}x_{T1} + \beta_{T2}x_{T2} + \beta_{T3}x_{T3} + \beta_{T4}x_{T4} + \dots$$

where

$$x_{T1} = \begin{cases} t & \text{if } t < 90 \\ 90 & \text{otherwise} \end{cases} \quad x_{T2} = \begin{cases} 0 & \text{if } t < 90 \\ t - 90 & \text{if } 90 \leq t < 180 \\ 90 & \text{otherwise} \end{cases}$$
$$x_{T3} = \begin{cases} 0 & \text{if } t < 180 \\ t - 180 & \text{if } 180 \leq t < 270 \\ 90 & \text{otherwise} \end{cases} \quad x_{T4} = \begin{cases} 0 & \text{if } t < 270 \\ t - 270 & \text{otherwise} \end{cases}$$

Piecewise linear specification

Note: coding in Biogeme for interval [a:a+b[

$$x_{Ti} = \begin{cases} 0 & \text{if } t < a \\ t - a & \text{if } a \leq t < a + b \\ b & \text{otherwise} \end{cases} \quad x_{Ti} = \max(0, \min(t - a, b))$$

$$x_{T1} = \min(t, 90)$$

$$x_{T2} = \max(0, \min(t - 90, 90))$$

$$x_{T3} = \max(0, \min(t - 180, 90))$$

$$x_{T4} = \max(0, t - 270)$$

$$\text{TRAIN_TT1} = \min(\text{TRAIN_TT}, 90)$$

$$\text{TRAIN_TT2} = \max(0, \min(\text{TRAIN_TT} - 90, 90))$$

$$\text{TRAIN_TT3} = \max(0, \min(\text{TRAIN_TT} - 180, 90))$$

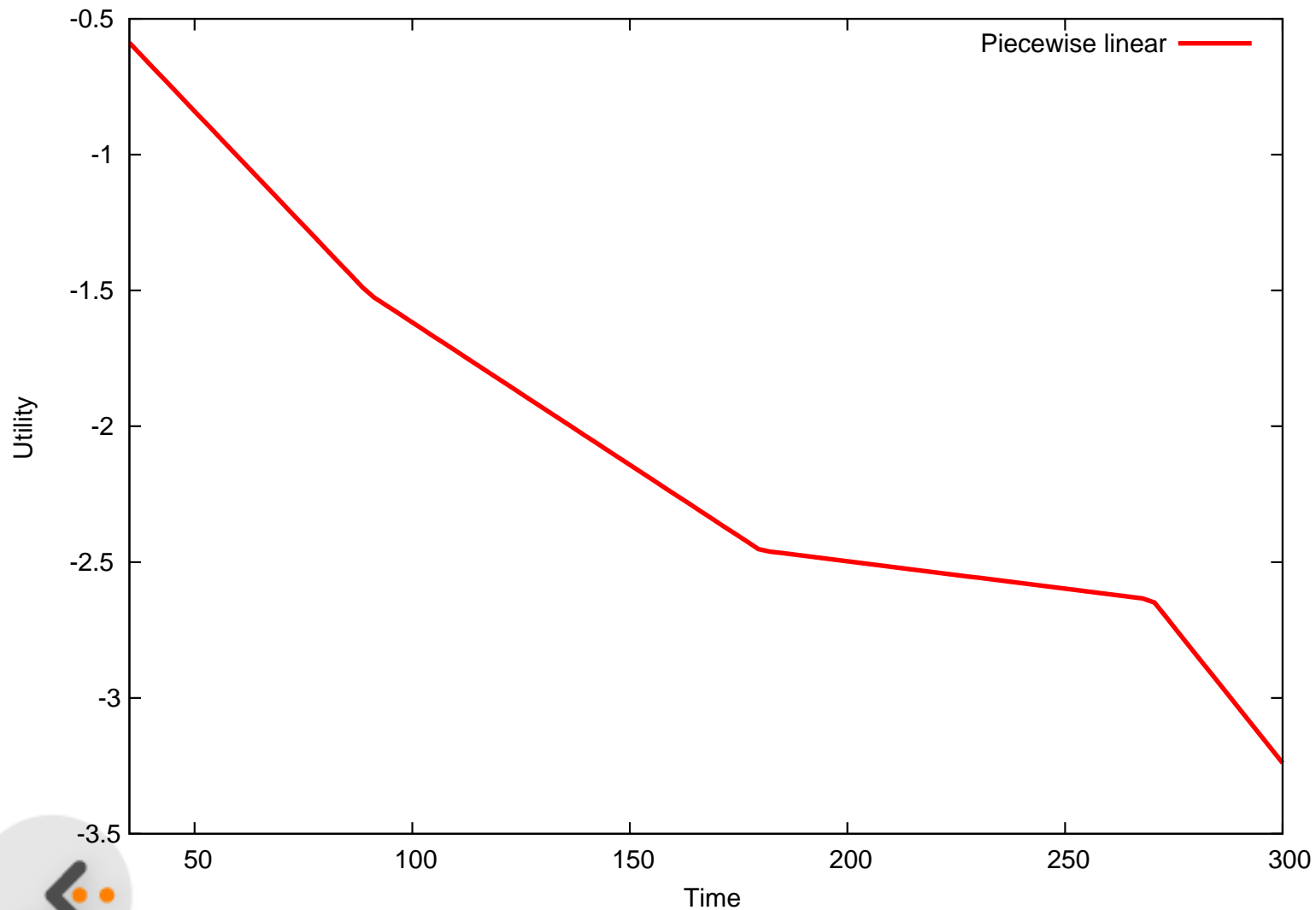
$$\text{TRAIN_TT4} = \max(0, \text{TRAIN_TT} - 270)$$

Piecewise linear specification

Examples:

t	TT1	TT2	TT3	TT4
40	40	0	0	0
100	90	10	0	0
200	90	90	20	0
300	90	90	90	30

Piecewise linear specification



Box-Cox transforms

Box and Cox, *J. of the Royal Statistical Society* (1964)

$$V_i = \beta x_i(\lambda) + \dots$$

where

$$x_i(\lambda) = \begin{cases} \frac{x_i^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln x_i & \text{if } \lambda = 0. \end{cases}$$

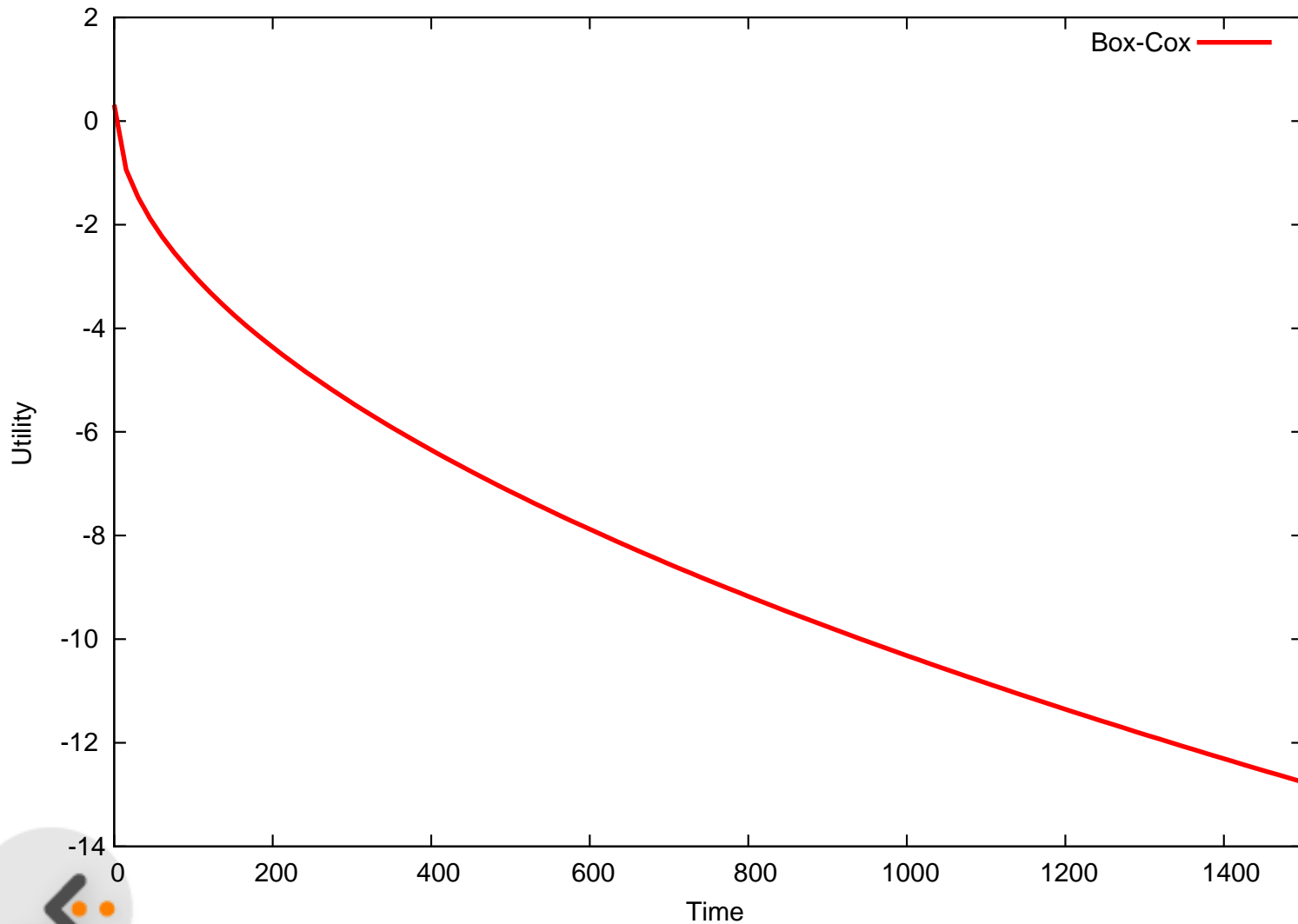
where $x_i > 0$.

Box-Cox transforms

If $x_i \leq 0$, let α such that $x_i + \alpha > 0$ and

$$x_i(\lambda, \alpha) = \begin{cases} \frac{(x_i + \alpha)^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln(x_i + \alpha) & \text{if } \lambda = 0. \end{cases}$$

Box-Cox transforms



Box-Cox transforms

Other power transforms are possible:

- Manly, *Biometrics* (1971)

$$x_i(\lambda) = \begin{cases} \frac{e^{x_i^\lambda} - 1}{\lambda} & \text{if } \lambda \neq 0 \\ x_i & \text{if } \lambda = 0. \end{cases}$$

- John and Draper, *Applied Statistics* (1980)

$$x_i(\lambda) = \begin{cases} \text{sign}(x_i) \frac{(|x_i| + 1)^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \text{sign}(x_i) \ln(|x_i| + 1) & \text{if } \lambda = 0. \end{cases}$$

Box-Cox transforms

Other power transforms are possible:

- Yeo and Johnson, *Biometrika* (2000)

$$x_i(\lambda) = \begin{cases} \frac{(x_i + 1)^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0, x_i \geq 0; \\ \ln(x_i + 1) & \text{if } \lambda = 0, x_i \geq 0; \\ \frac{(1 - x_i)^{2-\lambda} - 1}{\lambda - 2} & \text{if } \lambda \neq 2, x_i < 0; \\ -\ln(1 - x_i) & \text{if } \lambda = 2, x_i < 0. \end{cases}$$

Power series

$$V_i = \beta_1 T + \beta_2 T^2 + \beta_3 T^3 + \dots$$

- In practice, these terms can be very correlated
- Difficult to interpret
- Risk of over fitting

Interactions

- All individuals in a population are not alike
- Socio-economic characteristics define segments in the population
- How to capture heterogeneity?
 - Interactions of attributes and characteristics
 - Discrete segmentation
 - Continuous segmentation

Interactions of attributes and characteristics

Combination of attributes:

- **cost / income**
- **fare / disposable income**
- out-of-vehicle time / distance

WARNING: correlation of attributes may produce degeneracy in the model

Example: speed and travel time if distance is constant

Interactions: discrete segmentation

- The population is divided into a finite number of segments
- Each individual belongs to exactly one segment
- Example: gender (M,F) and house location (metro, suburb, perimeter areas)
- 6 segments

$$\beta_{M,m}TT_{M,m} + \beta_{M,s}TT_{M,s} + \beta_{M,p}TT_{M,p} + \\ \beta_{F,m}TT_{F,m} + \beta_{F,s}TT_{F,s} + \beta_{F,p}TT_{F,p} +$$

- $TT_i = TT$ if indiv. belongs to segment i , and 0 otherwise

Interactions: continuous segmentation

- Taste parameter varies with a continuous socio-economic characteristics
- Example: the cost parameter varies with income

$$\beta_{\text{cost}} = \hat{\beta}_{\text{cost}} \left(\frac{\text{inc}}{\text{inc}_{\text{ref}}} \right)^{\lambda} \quad \text{with } \lambda = \frac{\partial \beta_{\text{cost}}}{\partial \text{inc}} \frac{\text{inc}}{\beta_{\text{cost}}}$$

- Warning: λ must be estimated and utility is not linear-in-parameters anymore
- Reference value is arbitrary
- Several characteristics can be combined:

$$\beta_{\text{cost}} = \hat{\beta}_{\text{cost}} \left(\frac{\text{inc}}{\text{inc}_{\text{ref}}} \right)^{\lambda_1} \left(\frac{\text{age}}{\text{age}_{\text{ref}}} \right)^{\lambda_2}$$

Heteroscedasticity

- Logit is homoscedastic
- ε_{in} i.i.d. across both i and n .
- Assume there are two different groups such that

$$\begin{aligned}U_{in_1} &= V_{in_1} + \varepsilon_{in_1} \\U_{in_2} &= V_{in_2} + \varepsilon_{in_2}\end{aligned}$$

and $\text{Var}(\varepsilon_{in_2}) = \alpha^2 \text{Var}(\varepsilon_{in_1})$

- Then we prefer the model

$$\begin{aligned}\alpha U_{in_1} &= \alpha V_{in_1} + \alpha \varepsilon_{in_1} = \alpha V_{in_1} + \varepsilon'_{in_1} \\U_{in_2} &= V_{in_2} + \varepsilon_{in_2} = V_{in_2} + \varepsilon'_{in_2}\end{aligned}$$

- where ε'_{in_1} and ε'_{in_2} are i.i.d.

Heteroscedasticity

- If V_{in_1} is linear-in-parameters, that is

$$V_{in_1} = \sum_j \beta_j x_{jin_1}$$

then

$$\alpha V_{in_1} = \sum_j \alpha \beta_j x_{jin_1}$$

is nonlinear.

Derivation of the logit model

Reminder: binary case

- $\mathcal{C}_n = \{i, j\}$
- $U_{in} = V_{in} + \varepsilon_{in}$
- $\varepsilon_{in} \sim \text{EV}(0, \mu)$
- Probability

$$P(i|\mathcal{C}_n = \{i, j\}) = \frac{e^{\mu V_{in}}}{e^{\mu V_{in}} + e^{\mu V_{jn}}}$$

Derivation of the logit model

- $\mathcal{C}_n = \{1, \dots, J_n\}$
- $U_{in} = V_{in} + \varepsilon_{in}$
- $\varepsilon_{in} \sim \text{EV}(0, \mu)$
- ε_{in} i.i.d.
- Probability

$$P(i|\mathcal{C}_n) = P(V_{in} + \varepsilon_{in} \geq \max_{j=1, \dots, J_n} V_{jn} + \varepsilon_{jn})$$

- Assume without loss of generality (wlog) that $i = 1$

$$P(1|\mathcal{C}_n) = P(V_{1n} + \varepsilon_{1n} \geq \max_{j=2, \dots, J_n} V_{jn} + \varepsilon_{jn})$$

Derivation of the logit model

- Define a composite alternative: “anything but one”
- Associated utility:

$$U^* = \max_{j=2, \dots, J_n} (V_{jn} + \varepsilon_{jn})$$

- From a property of the EV distribution

$$U^* \sim \text{EV} \left(\frac{1}{\mu} \ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}, \mu \right)$$

Derivation of the logit model

- From another property of the EV distribution

$$U^* = V^* + \varepsilon^*$$

where

$$V^* = \frac{1}{\mu} \ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}$$

and

$$\varepsilon^* \sim \text{EV}(0, \mu)$$

Derivation of the logit model

- Therefore

$$\begin{aligned}P(1|\mathcal{C}_n) &= P(V_{1n} + \varepsilon_{1n} \geq \max_{j=2,\dots,J_n} V_{jn} + \varepsilon_{jn}) \\ &= P(V_{1n} + \varepsilon_{1n} \geq V^* + \varepsilon^*)\end{aligned}$$

- This is a binary choice model

$$P(1|\mathcal{C}_n) = \frac{e^{\mu V_{1n}}}{e^{\mu V_{1n}} + e^{\mu V^*}}$$

where

$$V^* = \frac{1}{\mu} \ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}$$

Derivation of the logit model

- We have $e^{\mu V^*} = e^{\ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}} = \sum_{j=2}^{J_n} e^{\mu V_{jn}}$
- and

$$\begin{aligned} P(1|C_n) &= \frac{e^{\mu V_{1n}}}{e^{\mu V_{1n}} + e^{\mu V^*}} \\ &= \frac{e^{\mu V_{1n}}}{e^{\mu V_{1n}} + \sum_{j=2}^{J_n} e^{\mu V_{jn}}} \\ &= \frac{e^{\mu V_{1n}}}{\sum_{j=1}^{J_n} e^{\mu V_{jn}}} \end{aligned}$$

Derivation of the logit model

- The scale parameter μ is not identifiable: $\mu = 1$.
- Warning: not identifiable \neq not existing
- $\mu \rightarrow 0$, that is variance goes to infinity

$$\lim_{\mu \rightarrow 0} P(i|C_n) = \frac{1}{J_n} \quad \forall i \in C_n$$

- $\mu \rightarrow +\infty$, that is variance goes to zero

$$\begin{aligned} \lim_{\mu \rightarrow \infty} P(i|C_n) &= \lim_{\mu \rightarrow \infty} \frac{1}{1 + \sum_{j \neq i} e^{\mu(V_{jn} - V_{in})}} \\ &= \begin{cases} 1 & \text{if } V_{in} > \max_{j \neq i} V_{jn} \\ 0 & \text{if } V_{in} < \max_{j \neq i} V_{jn} \end{cases} \end{aligned}$$

Derivation of the logit model

- $\mu \rightarrow +\infty$, that is variance goes to zero (ctd.)
- What if there are ties?
- $V_{in} = \max_{j \in \mathcal{C}_n} V_{jn}$, $i = 1, \dots, J_n^*$
- Then

$$P(i|\mathcal{C}_n) = \frac{1}{J_n^*} \quad i = 1, \dots, J_n^*$$

and

$$P(i|\mathcal{C}_n) = 0 \quad i = J_n^* + 1, \dots, J_n$$

A case study

- Choice of residential telephone services
- Household survey conducted in Pennsylvania, USA, 1984
- Revealed preferences
- 434 observations

A case study

Telephone services and availability

	metro, suburban		
	& some	other	
	perimeter	perimeter	non-metro
	areas	areas	areas
Budget Measured	yes	yes	yes
Standard Measured	yes	yes	yes
Local Flat	yes	yes	yes
Extended Area Flat	no	yes	no
Metro Area Flat	yes	yes	no

A case study

Universal choice set

$$\mathcal{C} = \{BM, SM, LF, EF, MF\}$$

Specific choice sets

- Metro, suburban & some perimeter areas: $\{BM, SM, LF, MF\}$
- Other perimeter areas: \mathcal{C}
- Non-metro areas: $\{BM, SM, LF\}$

A case study

Specification table

	β_1	β_2	β_3	β_4	β_5
BM	0	0	0	0	$\ln(\text{cost}(\text{BM}))$
SM	1	0	0	0	$\ln(\text{cost}(\text{SM}))$
LF	0	1	0	0	$\ln(\text{cost}(\text{LF}))$
EF	0	0	1	0	$\ln(\text{cost}(\text{EF}))$
MF	0	0	0	1	$\ln(\text{cost}(\text{MF}))$

A case study

$$V_{\text{BM}} = \beta_5 \ln(\text{cost}_{\text{BM}})$$

$$V_{\text{SM}} = \beta_1 + \beta_5 \ln(\text{cost}_{\text{SM}})$$

$$V_{\text{LF}} = \beta_2 + \beta_5 \ln(\text{cost}_{\text{LF}})$$

$$V_{\text{EF}} = \beta_3 + \beta_5 \ln(\text{cost}_{\text{EF}})$$

$$V_{\text{MF}} = \beta_4 + \beta_5 \ln(\text{cost}_{\text{MF}})$$

A case study

Specification table II

	β_1	β_2	β_3	β_4	β_5	β_6	β_7
BM	0	0	0	0	$\ln(\text{cost}(\text{BM}))$	users	0
SM	1	0	0	0	$\ln(\text{cost}(\text{SM}))$	users	0
LF	0	1	0	0	$\ln(\text{cost}(\text{LF}))$	0	1 if metro/suburb
EF	0	0	1	0	$\ln(\text{cost}(\text{EF}))$	0	0
MF	0	0	0	1	$\ln(\text{cost}(\text{MF}))$	0	0

A case study

$$V_{\text{BM}} = \beta_5 \ln(\text{cost}_{\text{BM}}) + \beta_6 \text{users}$$

$$V_{\text{SM}} = \beta_1 + \beta_5 \ln(\text{cost}_{\text{SM}}) + \beta_6 \text{users}$$

$$V_{\text{LF}} = \beta_2 + \beta_5 \ln(\text{cost}_{\text{LF}}) + \beta_7 \text{MS}$$

$$V_{\text{EF}} = \beta_3 + \beta_5 \ln(\text{cost}_{\text{EF}})$$

$$V_{\text{MF}} = \beta_4 + \beta_5 \ln(\text{cost}_{\text{MF}})$$

Maximum likelihood estimation

Logit Model:

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}}}{\sum_{j \in \mathcal{C}_n} e^{V_{jn}}}$$

Log-likelihood of a sample:

$$\mathcal{L}(\beta_1, \dots, \beta_K) = \sum_{n=1}^N \left(\sum_{j=1}^J y_{jn} \ln P_n(j|\mathcal{C}_n) \right)$$

where $y_{jn} = 1$ if ind. n has chosen alt. j , 0 otherwise

Maximum likelihood estimation

$$\begin{aligned}\ln P_n(i|\mathcal{C}_n) &= \ln \frac{e^{V_{in}}}{\sum_{j \in \mathcal{C}_n} e^{V_{jn}}} \\ &= V_{in} - \ln(\sum_{j \in \mathcal{C}_n} e^{V_{jn}})\end{aligned}$$

Log-likelihood of a sample:

$$\mathcal{L}(\beta_1, \dots, \beta_K) = \sum_{n=1}^N \sum_{i=1}^J y_{in} \left(V_{in} - \ln \sum_{j \in \mathcal{C}_n} e^{V_{jn}} \right)$$

Maximum likelihood estimation

The maximum likelihood estimation problem:

$$\max_{\beta \in \mathbb{R}^K} \mathcal{L}(\beta)$$

Maximization of a concave function with K variables
Nonlinear programming

Maximum likelihood estimation

Numerical issue:

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}}}{\sum_{j \in \mathcal{C}_n} e^{V_{jn}}}$$

Largest value that can be stored in a computer $\approx 10^{308}$, that is

$$e^{709.783}$$

It is equivalent to compute

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}-V_{in}}}{\sum_{j \in \mathcal{C}_n} e^{V_{jn}-V_{in}}} = \frac{1}{\sum_{j \in \mathcal{C}_n} e^{V_{jn}-V_{in}}}$$

Simple models

Null model

$$U_i = \varepsilon_i \quad \forall i$$

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}}}{\sum_{j \in \mathcal{C}_n} e^{V_{jn}}} = \frac{e^0}{\sum_{j \in \mathcal{C}_n} e^0} = \frac{1}{\#\mathcal{C}_n}$$

$$\mathcal{L} = \sum_n \ln \frac{1}{\#\mathcal{C}_n} = - \sum_n \ln(\#\mathcal{C}_n)$$

Simple models

Constants only [Assume $C_n = C, \forall n$]

$$U_i = c_i + \varepsilon_i \quad \forall i$$

In the sample of size n , there are n_i persons choosing alt. i .

$$\ln P(i) = c_i - \ln\left(\sum_j e^{c_j}\right)$$

If C_n is the same for all people choosing i , the log-likelihood for this part of the sample is

$$\mathcal{L}_i = n_i c_i - n_i \ln\left(\sum_j e^{c_j}\right)$$

Simple models

Constants only

The total log-likelihood is

$$\mathcal{L} = \sum_j n_j c_j - n \ln\left(\sum_j e^{c_j}\right)$$

At the maximum, the derivatives must be zero

$$\frac{\partial \mathcal{L}}{\partial c_1} = n_1 - n \frac{e^{c_1}}{\sum_j e^{c_j}} = n_1 - nP(1) = 0.$$

Simple models

Constants only

Therefore,

$$P(1) = \frac{n_1}{n}$$

If all alternatives are always available, a model with only Alternative Specific Constants reproduces exactly the market shares in the sample

Back to the case study

Alt.	n_i	n_i/n	c_i	e^{c_i}	P(i)
BM	73	0.168	0.247	1.281	0.168
SM	123	0.283	0.769	2.158	0.283
LF	178	0.410	1.139	3.123	0.410
EF	3	0.007	-2.944	0.053	0.007
MF	57	0.131	0.000	1.000	0.131
	434	1.000			

Null-model: $\mathcal{L} = -434 \ln(5) = -698.496$

Warning: these results have been obtained assuming that all alternatives are always available