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## Specification Testing: Switzerland mode choice (OPTIMA)

The topic of this case study is the testing of different hypotheses regarding both model specifications and structures. The objectives can be summarized as follows:

- Illustration of the market segmentation concept and related testing.
- Testing of non-nested hypotheses using the composite model test.
- Testing of non-linear specifications using the piecewise linear approximation, the power series expansion and the Box-Cox transformation methods.

### Market Segmentation

Files to use with BIOGEME:

Model files: MNL\_optima\_age\_higher\_40.mod, MNL\_optima\_age\_less\_40.mod, MNL\_optima\_socio\_economic.mod

Data file: optimaTOT3\_valid.dat

We start by defining a new variable called `age` and categorising it into two groups: adults and elderly. We have a total of 1906 observations in our sample. For the group of adults (`age < 40`) the total number of observations is 570. For the group of elderly (`age ≥ 40`) the number of observations is 1336. We would like to test if there are taste variations across market segments. We therefore estimate separate models for each group as well as a model estimated on the complete data set. We reuse a model from the Multinomial Logit case study (MNL\_optima\_socio\_economic.mod) which we have already estimated on the complete data set ( $\mathcal{L}(\hat{\beta}) = -1265.113$ ). The results from the model estimated on the adults group are presented in Table 1 and the results from the model based on elderly group in Table 2.

We perform a likelihood ratio test of taste variation or market segmentation.

Null hypothesis:  $H_0 : \beta_{\text{adult}} = \beta_{\text{elderly}}$

Reject  $H_0$  if the following inequality applies:

$$-2 \left( \mathcal{L}_N(\hat{\beta}) - \sum_{g \in G} \mathcal{L}_{N_g}(\hat{\beta}^g) \right) > \chi_{((1-\alpha), \text{df})}$$

Table 1: MNL for the adult group

Parameter number	Description	Coeff. estimate	Robust	t-stat	p-value
			Asympt. std. error		
1	$ASC_{CAR}$	0.109	0.224	0.49	0.63
2	$ASC_{MD}$	-0.0152	0.545	-0.03	0.98
3	$\beta_{DIST}$	-0.287	0.0923	-3.11	0.00
4	$\beta_{LANGUAGE}$	0.957	0.230	4.16	0.00
5	$\beta_{TIME\_CAR}$	-0.0432	0.00873	-4.95	0.00
6	$\beta_{TIME\_PT}$	-0.0207	0.00463	-4.46	0.00

**Summary statistics**

Number of observations = 570

$$\mathcal{L}(0) = -626.209$$

$$\mathcal{L}(c) = -431.731$$

$$\mathcal{L}(\hat{\beta}) = -349.263$$

$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 553.893$$

$$\rho^2 = 0.442$$

$$\hat{\rho}^2 = 0.433$$

where  $N$  represents the full sample size,  $G$  is the set of all groups,  $df = \left(\sum_g K_g\right) - K$  are the degrees of freedom,  $K$  is the number of parameters and  $K_g$  is the number of parameters in the segment  $g$  (adult=6, elderly=6). The first term is the final log likelihood the restricted model. The second term is the sum of the final log likelihood terms for each segment. Putting the value into the formula we have:

$$-2(-1265.113 + 349.263 + 909.006) = 13.688.$$

The critical value for  $\chi_{(0.95,6)} = 12.59$ . As  $13.688 > 12.59$  and we can therefore reject the null hypothesis at a 95% level of confidence. This suggests further exploration of this differences. We shall keep in mind that this is a rejection of the joint hypothesis, which implies that the two models differ but we have not established yet on which particular coefficients.

**Test of Non-Nested Hypotheses**

In discrete choice analysis, we often perform tests based on so-called nested hypotheses, which means that we specify two models such that the first one (the restricted model) is a special case of the second one (the unrestricted model). For

Table 2: MNL for the elderly group

Parameter number	Description	Coeff. estimate	Robust	t-stat	p-value
			Asympt. std. error		
1	$ASC_{CAR}$	0.436	0.116	3.75	0.00
2	$ASC_{MD}$	0.262	0.353	0.74	0.46
3	$\beta_{DIST}$	-0.209	0.0596	-3.51	0.00
4	$\beta_{LANGUAGE}$	1.33	0.178	7.48	0.00
5	$\beta_{TIME\_CAR}$	-0.0303	0.00660	-4.60	0.00
6	$\beta_{TIME\_PT}$	-0.0144	0.00320	-4.50	0.00

**Summary statistics**

Number of observations = 1336

$$\mathcal{L}(0) = -1467.746$$

$$\mathcal{L}(c) = -1089.505$$

$$\mathcal{L}(\hat{\beta}) = -909.006$$

$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 1117.480$$

$$\rho^2 = 0.381$$

$$\hat{\rho}^2 = 0.377$$

this type of comparison, the classical likelihood ratio test can be applied. However, there are situations, such as non-linear specifications, in which we aim at comparing models which are not nested, i.e. one model cannot be obtained as a restricted version of the other. One way to compare two non-nested models is to build a composite model from which both models can be derived. We can thus perform two likelihood ratio tests, testing each of the restricted models against the composite model. This procedure is known as the Cox test of separate families of hypothesis.

**Cox Test**

The Cox test is described in detail in Ben-Akiva and Lerman (1985, MIT Press), pages 171-174, and in the Textbook of the course, section “Tests of Non-Nested Hypothesis”. Assume that we want to test a model  $M_1$  against another model  $M_2$  (and one model is not a restricted version of the other). We start by generating a composite model  $M_C$  such that both models  $M_1$  and  $M_2$  are restricted cases of  $M_C$ . We then test  $M_1$  against  $M_C$  and  $M_2$  against  $M_C$  using the likelihood ratio test. There are three possible outcomes of this test:

- One of the two models is rejected. Then we keep the one that is not rejected.

- Both models are rejected. Then better models should be developed. The composite model could be used as a new basis for future specifications.
- Both models are accepted. Then we choose the model with the highest  $\bar{\rho}^2$  index.

We present here the expressions of the utility functions used for three different models  $M_1$ ,  $M_2$  and  $M_C$  developed on the optima case study.

$M_1$  has the following systematic utilities:

$$\begin{aligned} V_{PT} &= \beta_{\text{TIME\_PT}} * \text{DureeTP1} + \beta_{\text{COST}} * \text{MarginalCost} \\ V_{CAR} &= \text{ASC}_{CAR} + \beta_{\text{TIME\_CAR}} * \text{DureeAuto} + \beta_{\text{COST}} * \text{CoutAutoCHF} \end{aligned}$$

where the cost related coefficients are *linear*.

The systematic utilities of  $M_2$  are expressed as follows:

$$\begin{aligned} V_{PT} &= \beta_{\text{TIME}} * \text{DureeTP1} + \beta_{\log\text{COST}} * \log(\text{MarginalCost}) \\ V_{CAR} &= \text{ASC}_{CAR} + \beta_{\text{TIME}} * \text{DureeAuto} + \beta_{\log\text{COST}} * \log(\text{CoutAutoCHF}) \end{aligned}$$

where the cost related coefficients are *logarithmic*.

We now define the composite model  $M_C$  with the following systematic utilities:

$$\begin{aligned} V_{PT} &= \beta_{\text{TIME\_PT}} * \text{DureeTP1} + \beta_{\text{COST}} * \text{MarginalCost} + \beta_{\log\text{COST}} * \log(\text{MarginalCost}) \\ V_{CAR} &= \text{ASC}_{CAR} + \beta_{\text{TIME\_CAR}} * \text{DureeAuto} + \beta_{\text{COST}} * \text{CoutAutoCHF} + \beta_{\log\text{COST}} * \log(\text{CoutAutoCHF}) \end{aligned}$$

where we have one generic cost coefficient and one generic cost logarithmic coefficient.

After estimating the three models we can apply the likelihood ratio test for  $M_1$  against  $M_C$ . In this case, the null hypothesis is:

$$H_0 : \beta_{\text{LogFare}} = 0$$

As usual,  $-2(L(M_1) - L(M_C))$  is  $\chi^2$  distributed with  $K = 1$  degrees of freedom.

If the result of the first test suggests that we can reject the null hypothesis  $H_0$ : it means the composite model is better than  $M_1$ . Then the linear model is rejected. Then we apply the same test for  $M_2$  against  $M_C$ .

In the case when both models are rejected, better models should be developed: we cannot keep the composite model with two different cost-related coefficients since it does not have a behavioral interpretation. If both models had been accepted, we would choose the one with the highest  $\bar{\rho}^2$  index.

## Tests of Non-Linear Specifications

*Files to use with Biogeme:*

*Model files:* *SpecTest\_airline\_piecewise.mod*,  
*SpecTest\_airline\_powerseries.mod*,  
*SpecTest\_airline\_boxcox.mod*

*Data file:* *airline.dat*

The models studied previously were specified with linear-in-parameter formulations of the deterministic parts of the utilities (i.e. parameters that remain constant throughout the whole range of the values of each variable). However, in some cases non-linear specifications may be more justified. In this section, we test three different non-linear specifications of the deterministic utility functions: a piecewise linear specification of the time parameter of the non-stop itinerary, a power series method and Box-Cox transformation.

### Piecewise Linear Approximation

In this first example we want to test the hypothesis that the coefficient associated with travel time for private modes assumes different values for different values of the variable itself. The full range of values for this variable is [0,494]. To test this hypothesis we first generate the following variables:

$$\begin{aligned} \text{TimeAuto}_1 &= \min\{\text{DureeAuto}, 30\} \\ \text{TimeAuto}_2 &= \max(0, \min(\text{DureeAuto} - 30, 60)) \\ \text{TimeAuto}_3 &= \max(0, \text{DureeAuto} - 90) \end{aligned}$$

Such variables can be defined in BIOGEME as follows:

```
[Expressions]
// Define here arithmetic expressions for name that are not directly
// available from the data
TimeAuto_1 = min( DureeAuto , 30)
TimeAuto_2 = max(0, min( DureeAuto - 30, 60 ) )
TimeAuto_3 = max(0, DureeAuto - 90)
```

The deterministic utility for private modes is:

$$V_{\text{CAR}} = \text{ASC}_{\text{CAR}} + \beta_{\text{TIME\_CAR}_1} * \text{TimeAuto}_1 + \beta_{\text{TIME\_CAR}_2} * \text{TimeAuto}_2 + \\ \beta_{\text{TIME\_CAR}_3} * \text{TimeAuto}_3 + \beta_{\text{LANGUAGE}} * \text{FrenchRegion}$$

Table 3: Piecewise linear model  
Robust

Parameter number	Description	Coeff. estimate	Asympt. std. error	t-stat	p-value
1	$ASC_{CAR}$	0.894	0.176	5.09	0.00
2	$ASC_{MD}$	0.528	0.388	1.36	0.17
3	$\beta_{DIST}$	-0.262	0.0617	-4.24	0.00
4	$\beta_{LANGUAGE}$	1.09	0.137	7.99	0.00
5	$\beta_{TIME\_CAR_1}$	-0.0655	0.0129	-5.09	0.00
6	$\beta_{TIME\_CAR_2}$	-0.0436	0.00577	-7.55	0.00
7	$\beta_{TIME\_CAR_3}$	-0.0252	0.00673	-3.75	0.00
8	$\beta_{TIME\_PT}$	-0.0187	0.00288	-6.49	0.00

**Summary statistics**

Number of observations = 1906  
 $\mathcal{L}(0) = -2093.955$   
 $\mathcal{L}(c) = -1524.919$   
 $\mathcal{L}(\hat{\beta}) = -1245.341$   
 $-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 1697.227$   
 $\rho^2 = 0.405$   
 $\bar{\rho}^2 = 0.401$

The estimation results for this model are reported in Table 3.

All coefficients related to the piecewise linear formulation are negative and statistically significant. When the travel time increases, the related coefficient increases in magnitude, indicating that the sensitivity for the unit of time decreases at a certain reasonable extent.

We can perform a likelihood ratio test where the restricted model is the one with linear travel time and the unrestricted model is the piecewise linear specification. The null hypothesis is given as follows:

$$H_0 : \beta_{\text{Time}_{\text{CAR}_1}} = \beta_{\text{Time}_{\text{CAR}_2}} = \beta_{\text{Time}_{\text{CAR}_3}}$$

And then we test if we can reject the null hypothesis of a linear travel time for the car alternative at a 95% level of confidence.

### The Power Series Expansion

We introduce here a power series expansion for the travel time for car alternative. Other polynomial expressions could be tried as well, but in the following example, we only specify a squared term.

$$V_{\text{CAR}} = ASC_{\text{CAR}} + \beta_{\text{TIME\_CAR\_LIN}} * \text{DureeAuto} + \beta_{\text{TIME\_CAR\_SQ}} * \text{TimeAuto}_{\text{SQ}} + \beta_{\text{LANGUAGE}} * \text{FrenchRegion},$$

in which  $\text{TimeAuto}_{\text{SQ}} = \text{DureeAuto}^2$ . We report in Table 4 the estimation results for this model. This model has in general a better goodness-of-fit than the model with linear coefficients. However, the coefficient of the squared term, though statistically significant, has a very small value. It may be noted that the coefficient of the squared term is positive while the coefficient of the linear term is negative and the coefficient of the linear term is greater than that of the squared term. However, since the squared term is very small in magnitude, the total effect is expected to remain negative in the cost range.

In order to see if the power series specification is better than the linear one, we perform a likelihood ratio test. Here, the restricted model is the one with linear travel time and the unrestricted model is the one with the power series expansion. The null hypothesis is given by

$$H_0 : \beta_{\text{TIME\_CAR\_SQ}} = 0.$$

And then we test if we can reject the null hypothesis at a 95% level of confidence.

Table 4: Power Series model  
Robust

Parameter number	Description	Coeff. estimate	Asympt. std. error	t-stat	p-value
1	$ASC_{CAR}$	0.621	0.121	5.14	0.00
2	$ASC_{MD}$	0.339	0.318	1.07	0.29
3	$\beta_{DIST}$	-0.244	0.0536	-4.55	0.00
4	$\beta_{LANGUAGE}$	1.13	0.139	8.10	0.00
5	$\beta_{TIME\_CAR\_LIN}$	-0.0508	0.00677	-7.50	0.00
6	$\beta_{TIME\_CAR\_SQ}$	6.76e-05	1.99e-05	3.40	0.00
7	$\beta_{TIME\_PT}$	-0.0180	0.00262	-6.87	0.00

### Summary statistics

Number of observations = 1906

$$\mathcal{L}(0) = -2093.955$$

$$\mathcal{L}(c) = -1524.919$$

$$\mathcal{L}(\hat{\beta}) = -1247.979$$

$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 1691.952$$

$$\rho^2 = 0.404$$

$$\bar{\rho}^2 = 0.401$$

### The Box-Cox Transformation

Files to use with BIOGEME:

Model file: MNL\_boxcox.mod

Data file: optimaTOT3\_valid.dat

In this section, we specify a Box-Cox transformation, which is a non-linear transformation of a variable that also depends on an unknown parameter  $\lambda$ .

Precisely, a Box-Cox transformation of a variable  $x$  is given as follows:

$$\frac{x^\lambda - 1}{\lambda}, \text{ where } x \geq 0.$$

We apply this transformation to the travel time variable for the car alternative. The utilities are the same as the previous models, apart from the one relative to the car alternative, which we report below:

[Utilities]

```
// Id Name Avail linear-in-parameter expression (beta1 * x1 + beta2 * x2 + ... )
```

Table 5: Box-Cox model

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	$ASC_{CAR}$	0.879	0.177	4.98	0.00
2	$ASC_{MD}$	0.516	0.386	1.34	0.18
3	$\beta_{DIST}$	-0.262	0.0626	-4.18	0.00
4	$\beta_{LANGUAGE}$	1.12	0.137	8.17	0.00
5	$\beta_{TIME\_CAR}$	-0.128	0.0350	-3.65	0.00
6	$\beta_{TIME\_PT}$	-0.0181	0.00281	-6.46	0.00
7	$\lambda$	0.708	0.0656	10.80	0.00

**Summary statistics**

Number of observations = 1906

$$\mathcal{L}(0) = -2093.955$$

$$\mathcal{L}(c) = -1524.919$$

$$\mathcal{L}(\hat{\beta}) = -1250.472$$

$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 1686.965$$

$$\rho^2 = 0.403$$

$$\bar{\rho}^2 = 0.399$$

```

0 PT one BETA_TIME_PT * DureeTP1
1 CAR one ASC_CAR * one + BETA_LANGUAGE * FrenchRegion
2 MD one ASC_MD * one + BETA_DIST * distance_km
[GeneralizedUtilities]
1 BETA_TIME_CAR * ( ( ( DureeAuto ) ^ LAMBDA - 1) / LAMBDA )

```

Let us note that in this specification, we have one more unknown parameter,  $\lambda$ . The parameter  $\lambda$  is estimated along with the other parameters (its starting value needs to be specified as value other than zero, since  $\lambda \neq 0$ ). In this case we need to introduce an extra tab for the Generalized Utilities.

Let us remark that the Box-Cox transformation reduces to a linear function as a special case when the parameter  $\lambda$  is equal to 1. The estimate of  $\lambda$  is significantly different from 1 at a 95 % level of confidence, with a t-test equal to  $-3.36$ .

We perform a likelihood ratio test between the linear model and the Box-Cox model. The null hypothesis is given by:

$$H_0 : \lambda = 1$$