Nested logit

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Outline

Simple choice model

Mode choice

- Two alternatives: car and bus.
- There are red buses and blue buses.
- Car and bus travel times are equal: T.
- Only travel time is considered in the utility function.

Red bus/Blue bus paradox

Model 1

$$
\begin{aligned}
U_{\text{car}} &= \beta T + \varepsilon_{\text{car}} \\
U_{\text{bus}} &= \beta T + \varepsilon_{\text{bus}}
\end{aligned}
$$

Choice probability

$$
P(\text{car}|\{\text{car}, \text{bus}\}) = P(\text{bus}|\{\text{car}, \text{bus}\}) = \frac{e^{\beta T}}{e^{\beta T} + e^{\beta T}} = \frac{1}{2}
$$

Red bus/Blue bus paradox

Model 2

$$\begin{aligned} U_{\text{car}} &= \beta T + \varepsilon_{\text{car}} \\ U_{\text{blue bus}} &= \beta T + \varepsilon_{\text{blue bus}} \\ U_{\text{red bus}} &= \beta T + \varepsilon_{\text{red bus}} \end{aligned}$$

Choice probability

$$P(\text{car}|\{\text{car}, \text{blue bus}, \text{red bus}\}) = \frac{e^{\beta T}}{e^{\beta T} + e^{\beta T} + e^{\beta T}} = \frac{1}{3}$$

$$\left.\begin{aligned} P(\text{car}|\{\text{car}, \text{blue bus}, \text{red bus}\}) \\ P(\text{blue bus}|\{\text{car}, \text{blue bus}, \text{red bus}\}) \\ P(\text{red bus}|\{\text{car}, \text{blue bus}, \text{red bus}\}) \end{aligned}\right\} = \frac{1}{3}.$$

Red bus/Blue bus paradox

Conclusion

If you paint the buses of a city red and blue, the mode share for public transportation increases from 50% to 66%.

Explaining the paradox

Model specification

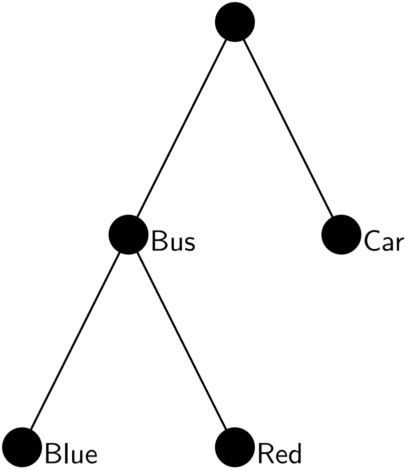
- Only travel time appears in the utility function.
- Other attributes are captured by the error term.
- Some of them are shared by $\varepsilon_{\text{blue bus}}$ and $\varepsilon_{\text{red bus}}$
  - fare
  - headway
  - comfort
  - convenience
  - etc.

Logit model

- Assumes that $\varepsilon_{\text{blue bus}}$ and $\varepsilon_{\text{red bus}}$ are independent.
- Inappropriate assumption in this case.

Outline

Capturing the correlation



Capturing the correlation

If bus is chosen then

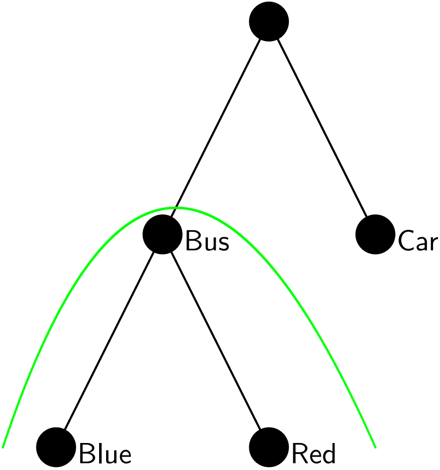
$$\begin{aligned} U_{\text{blue bus}} &= V_{\text{blue bus}} + \varepsilon_{\text{blue bus}} \\ U_{\text{red bus}} &= V_{\text{red bus}} + \varepsilon_{\text{red bus}} \end{aligned}$$

where $V_{\text{blue bus}} = V_{\text{red bus}} = \beta T$

Choice probability

$$P(\text{blue bus}|\{\text{blue bus}, \text{red bus}\}) = \frac{e^{\beta T}}{e^{\beta T} + e^{\beta T}} = \frac{1}{2}$$

Capturing the correlation



Capturing the correlation

What about the choice between bus and car?

$$
\begin{aligned}
U_{\text{car}} &= \beta T + \varepsilon_{\text{car}} \\
U_{\text{bus}} &= V_{\text{bus}} + \varepsilon_{\text{bus}}
\end{aligned}
$$

with

$$
\begin{aligned}
V_{\text{bus}} &= V_{\text{bus}}(V_{\text{blue bus}}, V_{\text{red bus}}) \\
\varepsilon_{\text{bus}} &= \,?
\end{aligned}
$$

Idea

- Use a logit model at the higher level.
- Define V_{bus} as the expected maximum utility of red bus and blue bus

Expected maximum utility

Definition

For a set of alternative $\mathcal{C}$, define

$$U_{\mathcal{C}} \equiv \max_{i \in \mathcal{C}} U_i = \max_{i \in \mathcal{C}} (V_i + \varepsilon_i)$$

and

$$U_{\mathcal{C}} \equiv V_{\mathcal{C}} + \varepsilon_{\mathcal{C}}$$

For logit

$$E[\max_{i \in \mathcal{C}} U_i] = \frac{1}{\mu} \ln \sum_{i \in \mathcal{C}} e^{\mu V_i} + \frac{\gamma}{\mu}$$

Expected maximum utility

For logit

$$U_{\mathcal{C}} = V_{\mathcal{C}} + \varepsilon_{\mathcal{C}}$$

$$V_{\mathcal{C}} = \frac{1}{\mu} \ln \sum_{i \in \mathcal{C}} e^{\mu V_i}$$

$$E[\varepsilon_{\mathcal{C}}] = \frac{\gamma}{\mu}$$

Expected maximum utility

Back to the blue/red buses

$$
\begin{aligned}
V_{\text{bus}} &= \frac{1}{\mu_b} \ln(e^{\mu_b V_{\text{blue bus}}} + e^{\mu_b V_{\text{red bus}}}) \\
&= \frac{1}{\mu_b} \ln(e^{\mu_b \beta T} + e^{\mu_b \beta T}) \\
&= \beta T + \frac{1}{\mu_b} \ln 2
\end{aligned}
$$

where μ_b is the scale parameter for the logit model associated with the choice between red bus and blue bus

Outline

Nested Logit Model

Probability model: car

$$P(\text{car}) = \frac{e^{\mu V_{\text{car}}}}{e^{\mu V_{\text{car}}} + e^{\mu V_{\text{bus}}}} = \frac{e^{\mu \beta T}}{e^{\mu \beta T} + e^{\mu \beta T + \frac{\mu}{\mu_b} \ln 2}} = \frac{1}{1 + 2^{\frac{\mu}{\mu_b}}}$$

Extreme cases

- If $\mu = \mu_b$, then P(car) $= \frac{1}{3}$ (Model 2, logit with 3 alternatives)
- If $\mu_b \to \infty$, then $\frac{\mu}{\mu_b} \to 0$, and P(car) $\to \frac{1}{2}$ (Model 1, logit with 2 alternatives)

Nested Logit Model

Probability model: bus

$$P(\text{bus}) = \frac{e^{\mu V_{\text{bus}}}}{e^{\mu V_{\text{car}}} + e^{\mu V_{\text{bus}}}} = \frac{e^{\mu\beta T + \frac{\mu}{\mu_b}\ln 2}}{e^{\mu\beta T} + e^{\mu\beta T + \frac{\mu}{\mu_b}\ln 2}} = \frac{1}{1 + 2^{-\frac{\mu}{\mu_b}}}$$

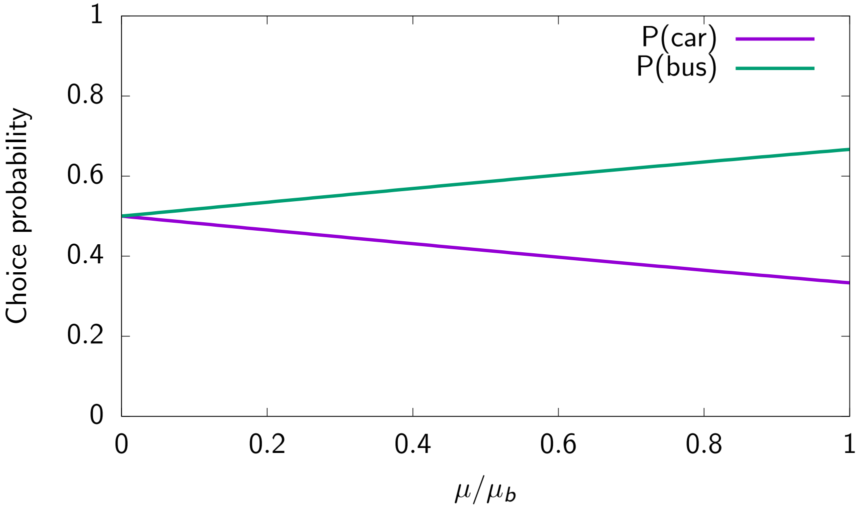
Extreme cases

- If $\mu = \mu_b$, then P(bus) $= \frac{2}{3}$ (Model 2)
- If $\mu_b \to \infty$, then $\frac{\mu}{\mu_b} \to 0$, then P(bus) $\to \frac{1}{2}$ (Model 1)

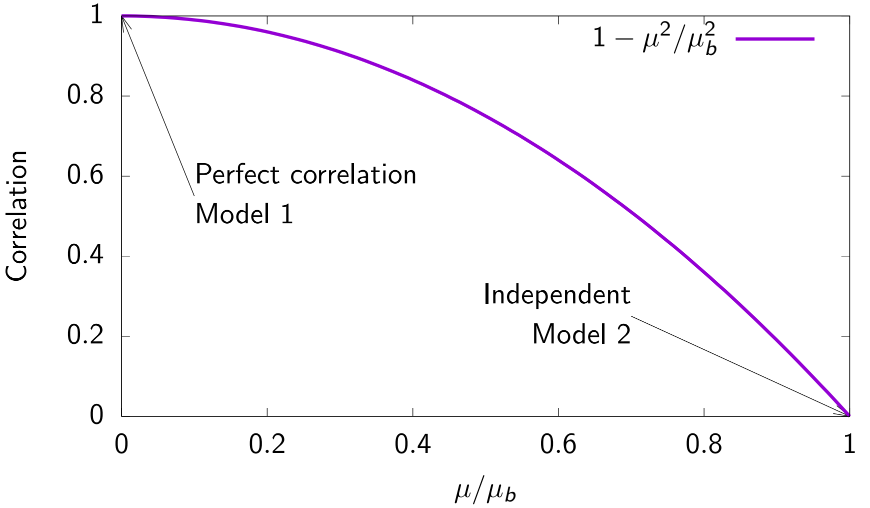
Utility of bus when $\mu_b \to \infty$

$$\lim_{\mu_b \to \infty} \beta T + \frac{1}{\mu_b} \ln 2 = \beta T$$

Nested Logit Model



Nested Logit Model



Solving the paradox

If $\frac{\mu}{\mu_b} \to 0$, we have

$$
\begin{aligned}
P(\text{car}) &= & & 1/2 \\
P(\text{bus}) &= & & 1/2 \\
P(\text{red bus}|\text{bus}) &= & & 1/2 \\
P(\text{blue bus}|\text{bus}) &= & & 1/2 \\
P(\text{red bus}) &= P(\text{red bus}|\text{bus})P(\text{bus}) &=& 1/4 \\
P(\text{blue bus}) &= P(\text{blue bus}|\text{bus})P(\text{bus}) &=& 1/4
\end{aligned}
$$

Nested logit model

Comments

- A group of similar alternatives is called a nest
- Each alternative belongs to exactly one nest.
- The model is named Nested Logit
- The ratio μ/μ_b must be estimated from the data
- $0 < \mu/\mu_b \leq 1$ (between models 1 and 2)
- Going down the tree, μ's must increase, variance must decrease

Outline

Airline itinerary case study: Logit model

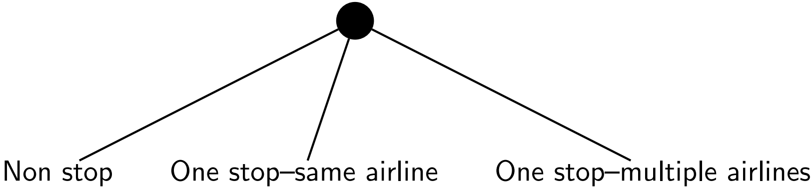
Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.898	0.218	-4.13	0.00
2	One stop–multiple airlines dummy	-1.24	0.223	-5.58	0.00
3	Round trip fare ($100)	-1.81	0.152	-11.90	0.00
4	Elapsed time (0 - 2 hours)	-0.856	0.226	-3.79	0.00
5	Elapsed time (2 - 8 hours)	-0.241	0.0820	-2.93	0.00
6	Elapsed time ($>$ 8 hours)	-0.936	0.314	-2.99	0.00
7	Leg room (inches), if male (non stop)	0.0972	0.0330	2.94	0.00
8	Leg room (inches), if female (non stop)	0.193	0.0315	6.15	0.00
9	Leg room (inches), if male (one stop)	0.128	0.0290	4.42	0.00
10	Leg room (inches), if female (one stop)	0.0845	0.0259	3.26	0.00
11	Being early (hours)	-0.150	0.0190	-7.89	0.00
12	Being late (hours)	-0.0993	0.0167	-5.94	0.00
13	More than 2 air trips per year (one stop–same airline)	-0.279	0.141	-1.98	0.05
14	More than 2 air trips per year (one stop–multiple airlines)	-0.0670	0.157	-0.43	0.67
15	Round trip fare / income ($100/$1000)	-23.0	8.11	-2.83	0.00

Summary statistics

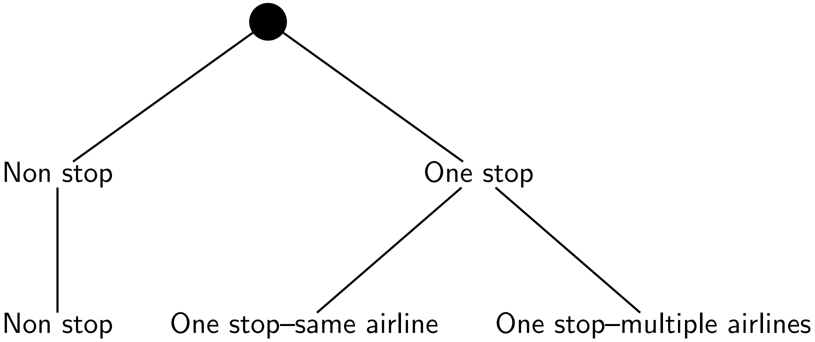
Number of observations = 2544

$$
\begin{aligned}
\mathcal{L}(0) &= -2794.870 \\
\mathcal{L}(c) &= -2203.160 \\
\mathcal{L}(\widehat{\beta}) &= -1635.068 \\
-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] &= 2319.603 \\
\rho^2 &= 0.415 \\
\bar{\rho}^2 &= 0.410
\end{aligned}
$$

Logit model



Nested logit model



Nested logit

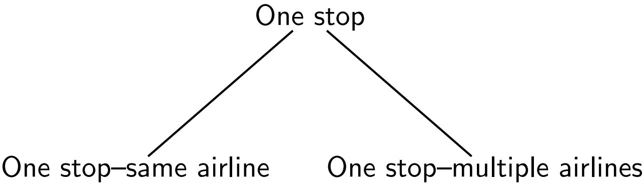
Marginal and conditional probabilities

$$\begin{aligned}
\Pr(\text{NS}) &= \Pr(\text{NS}|\text{Non stop}) \Pr(\text{Non stop}|\{\text{Non stop}, \text{One stop}\}), \\
\Pr(\text{SAME}) &= \Pr(\text{SAME}|\text{One stop}) \Pr(\text{One stop}|\{\text{Non stop}, \text{One stop}\}), \\
\Pr(\text{MULT}) &= \Pr(\text{MULT}|\text{One stop}) \Pr(\text{One stop}|\{\text{Non stop}, \text{One stop}\}).
\end{aligned}$$

Note

$$\Pr(\text{NS}|\text{Non stop}) = 1$$

Nest “one stop”



Nest "one stop"

Parameter number	Description	
1	One stop–same airline dummy	
2	One stop–multiple airlines dummy	normalized to 0
3	Round trip fare ($100)	
4	Elapsed time (0 - 2 hours)	no data to estimate
5	Elapsed time (2 - 8 hours)	
6	Elapsed time ($>$ 8 hours)	
7	Leg room (inches), if male (non stop)	alt. not in the model
8	Leg room (inches), if female (non stop)	alt. not in the model
9	Leg room (inches), if male (one stop)	
10	Leg room (inches), if female (one stop)	
11	Being early (hours)	
12	Being late (hours)	
13	More than 2 air trips per year (one stop–same airline)	
14	More than 2 air trips per year (one stop–multiple airlines)	normalized to 0
15	Round trip fare / income ($100/$1000)	

Nest “one stop”

Binary choice

- SAME: “One stop–same airline”
- MULT: “One stop–multiple airlines”

Specification of the utility functions

- Same as logit model.
- Up to normalization.
  - MULT constant normalized to zero
  - “More than two air trips per year (MULT)” normalized to 0
  - “Elapsed time (0–2 hours)” cannot be identified due to absence of data.

Nest “one stop”

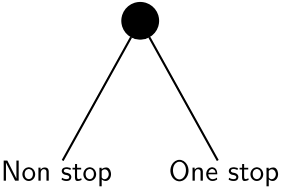
Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop, same airline dummy	0.469	0.188	2.50	0.01
3	Round trip fare ($100)	-2.87	0.624	-4.61	0.00
5	Elapsed time (2–8 hours)	-0.387	0.136	-2.84	0.00
6	Elapsed time ($>$ 8 hours)	-2.33	0.759	-3.07	0.00
9	Leg room (inches), if male (one stop)	0.170	0.0464	3.65	0.00
10	Leg room (inches), if female (one stop)	0.104	0.0421	2.46	0.01
11	Being early (hours)	-0.250	0.0422	-5.91	0.00
12	Being late (hours)	-0.0942	0.0286	-3.29	0.00
13	More than two air trips per year (one stop, same airline)	-0.220	0.218	-1.01	0.31
15	Round trip fare / income ($100/$1000)	-37.8	40.8	-0.93	0.35

Summary statistics

Number of observations = 846

$$\mathcal{L}(0) = -586.403$$
$$\mathcal{L}(c) = -585.258$$
$$\mathcal{L}(\hat{\beta}) = -318.994$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 534.816$$
$$\rho^2 = 0.456$$
$$\bar{\rho}^2 = 0.439$$

Upper level



Upper level

Parameter number	Description	
0	Non stop	replaces parameter 2
1	One stop–same airline dummy	already estimated
2	One stop–multiple airlines dummy	not an alternative
3	Round trip fare ($100)	already estimated
4	Elapsed time (0 - 2 hours)	
5	Elapsed time (2 - 8 hours)	already estimated
6	Elapsed time ($>$ 8 hours)	already estimated
7	Leg room (inches), if male (non stop)	
8	Leg room (inches), if female (non stop)	
9	Leg room (inches), if male (one stop)	already estimated
10	Leg room (inches), if female (one stop)	already estimated
11	Being early (hours)	already estimated
12	Being late (hours)	already estimated
12'	More than 2 air trips per year (non stop)	replaces parameter 14
13	More than 2 air trips per year (one stop–same airline)	already estimated
14	More than 2 air trips per year (one stop–multiple airlines)	not an alternative
15	Round trip fare / income ($100/$1000)	already estimated

Upper level

Binary choice

- Non stop
- One stop

Non stop: specification of the utility functions

- Same as logit.
- Up to normalization.
  - Alternative specific constant
  - "More than 2 air trips per year"
  - Coefficients 0 and 12' are replacing 2 and 14 in logit.
- Coefficients already estimated at the lower level are not re-estimated.
- 5 coefficients must be estimated.

Upper level

One stop: specification

$$\tilde{V}_{\text{One stop}} = E[\max(U_{\text{SAME}}, U_{\text{MULT}})]$$
$$= \frac{1}{\mu_{\text{One stop}}} \log(e^{\mu_{\text{One stop}} V_{\text{SAME}}} + e^{\mu_{\text{One stop}} V_{\text{MULT}}}).$$

Normalization

$$\mu_{\text{One stop}} = 1$$
$$\tilde{V}_{\text{One stop}} = \log(e^{V_{\text{SAME}}} + e^{V_{\text{MULT}}}).$$

Upper level

Logit

$$\Pr(\text{One stop}|\{\text{Non stop}, \text{One stop}\}) = \frac{e^{\mu \tilde{V}_{\text{One stop}}}}{e^{\mu V_{\text{NS}}} + e^{\mu \tilde{V}_{\text{One stop}}}}.$$

Comment

As $\mu_{\text{One stop}}$ has been normalized, μ is identified.

Upper level

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
0	Non stop dummy	4.90	0.817	6.00	0.00
4	Elapsed time (0–2 hours)	-1.60	0.405	-3.95	0.00
7	Leg room (inches), if male (non stop)	0.170	0.0584	2.91	0.00
8	Leg room (inches), if female (non stop)	0.338	0.0565	5.98	0.00
12'	More than 2 air trips per year (non stop)	0.219	0.215	1.02	0.31
16	μ	0.526	0.0307	-15.42[1]	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -1763.366$$
$$\mathcal{L}(c) = -1617.902$$
$$\mathcal{L}(\hat{\beta}) = -1298.135$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 930.463$$
$$\rho^2 = 0.264$$
$$\bar{\rho}^2 = 0.260$$

[1] t-test against 1

Full model (sequential estimation)

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
0	Non stop dummy	4.90	0.817	6.00	0.00
1	One stop, same airline dummy	0.469	0.188	2.50	0.01
3	Round trip fare ($100)	-2.87	0.624	-4.61	0.00
4	Elapsed time (0–2 hours)	-1.60	0.405	-3.95	0.00
5	Elapsed time (2–8 hours)	-0.387	0.136	-2.84	0.00
6	Elapsed time ($>$ 8 hours)	-2.33	0.759	-3.07	0.00
7	Leg room (inches), if male (non stop)	0.170	0.0584	2.91	0.00
8	Leg room (inches), if female (non stop)	0.338	0.0565	5.98	0.00
9	Leg room (inches), if male (one stop)	0.170	0.0464	3.65	0.00
10	Leg room (inches), if female (one stop)	0.104	0.0421	2.46	0.01
11	Being early (hours)	-0.250	0.0422	-5.91	0.00
12	Being late (hours)	-0.0942	0.0286	-3.29	0.00
12'	More than 2 air trips per year (non stop)	0.219	0.215	1.02	0.31
13	More than two air trips per year (one stop, same airline)	-0.220	0.218	-1.01	0.31
15	Round trip fare / income ($100/$1000)	-37.8	40.8	-0.93	0.35
16	μ	0.526	0.0307	-15.42[1]	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2349.769$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\hat{\beta}) = -1617.129 (= -1298.135 - 318.994)$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 1465.28$$
$$\rho^2 = 0.312$$
$$\bar{\rho}^2 = 0.305$$

Logit vs. Nested logit

Scale

- What is being estimated: $\mu\beta$
- Logit: $\mu = 1$ (normalized)
- Nested logit: $\mu = 0.526$
- Make sure to compare $\mu\beta$ across models.

Examples

Parameter	Logit	Nested logit	Scaled
Round trip fare ($100)	-1.81	-2.87	-1.51
Elapsed time (0–2 hours)	-0.856	-1.60	-0.84
Leg room (inches), if female (non stop)	0.193	0.338	0.178

Logit vs. Nested logit

Normalization of constants

Parameter number	Description	Logit	Nested logit	Nested logit (scaled)	Nested logit (scaled & shifted)
0	Non stop	0.0	4.90	2.58	0.0
1	One stop–same airline dummy	-0.898	0.469	0.247	-2.33
2	One stop–multiple airlines dummy	-1.24	0.0	0.0	-2.58
12'	$>$ 2 trips/y (non stop)	0.0	0.219	0.115	0.0
13	$>$ 2 trips/y (one stop–same airline)	-0.279	-0.220	-0.115	-0.231
14	$>$ 2 trips/y (one stop–multiple airlines)	-0.0670	0.0	0.0	-0.115

Logit vs Nested logit

t-test

Logit: $\mu = 1$

$$\frac{0.526 - 1}{0.0307} = -15.42$$

Reject $H_0 : \mu = 1$

Likelihood ratio test

- Logit: -1635.068
- Nested logit: -1617.129
- Likelihood ratio test: -2(-1635.07+1617.13)=35.378
- Threshold: $\chi^2_{1,0.95} = 3.84$
- Logit is rejected.

Estimation

Sequential estimation

- Estimate first the lower levels.
- Transfer the estimated utility function to estimate the upper level.
- Consistent estimator.
- Not efficient.

Full information maximum likelihood

- All parameters estimated together.
- Consistent.
- Efficient.

Full model (full information estimation)

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
0	Non stop dummy	1.74	0.337	5.16	0.00
1	One stop, same airline dummy	0.437	0.183	2.39	0.02
3	Round trip fare ($100)	-2.81	0.315	-8.91	0.00
4	Elapsed time (0–2 hours)	-1.49	0.417	-3.57	0.00
5	Elapsed time (2–8 hours)	-0.348	0.112	-3.10	0.00
6	Elapsed time ($>$ 8 hours)	-1.62	0.506	-3.21	0.00
7	Leg room (inches), if male (non stop)	0.168	0.0587	2.86	0.00
8	Leg room (inches), if female (non stop)	0.330	0.0624	5.28	0.00
9	Leg room (inches), if male (one stop)	0.175	0.0396	4.41	0.00
10	Leg room (inches), if female (one stop)	0.112	0.0344	3.25	0.00
11	Being early (hours)	-0.234	0.0338	-6.92	0.00
12	Being late (hours)	-0.135	0.0241	-5.61	0.00
12'	More than two air trips per year (non stop)	0.199	0.243	0.82	0.41
13	More than two air trips per year (one stop, same airline)	-0.237	0.210	-1.13	0.26
15	Round trip fare / income ($100/$1000)	-36.4	14.3	-2.55	0.01
16	μ	0.546	0.0595	-7.62[1]	0.00

Summary statistics

Number of observations = 2544

$$
\begin{aligned}
\mathcal{L}(0) &= -2794.870 \\
\mathcal{L}(c) &= -2203.160 \\
\mathcal{L}(\hat{\beta}) &= -1613.858 \\
-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] &= 2362.022 \\
\rho^2 &= 0.423 \\
\bar{\rho}^2 &= 0.417
\end{aligned}
$$

Normalize $\mu = 1$, estimate μ_m

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.710	0.169	-4.20	0.00
2	One stop–multiple airlines	-0.949	0.173	-5.47	0.00
3	Round trip fare ($100)	-1.54	0.149	-10.29	0.00
4	Elapsed time (0–2 hours)	-0.815	0.215	-3.80	0.00
5	Elapsed time (2–8 hours)	-0.190	0.0610	-3.12	0.00
6	Elapsed time ($>$ 8 hours)	-0.887	0.267	-3.32	0.00
7	Leg room (inches), if male (non stop)	0.0919	0.0310	2.96	0.00
8	Leg room (inches), if female (non stop)	0.180	0.0296	6.08	0.00
9	Leg room (inches), if male (one stop)	0.0954	0.0219	4.35	0.00
10	Leg room (inches), if female (one stop)	0.0610	0.0193	3.16	0.00
11	Being early (hours)	-0.128	0.0160	-7.97	0.00
12	Being late (hours)	-0.0739	0.0141	-5.23	0.00
13	More than two air trips per year (one stop–same airline)	-0.239	0.124	-1.93	0.05
14	More than two air trips per year (one stop–multiple airlines)	-0.109	0.132	-0.82	0.41
15	Round trip fare / income ($100/$1000)	-19.9	7.47	-2.66	0.01
16	μ_m	1.83	0.199	4.17[1]	0.00

Summary statistics

Number of observations = 2544

$$
\begin{aligned}
\mathcal{L}(0) &= -2794.870 \\
\mathcal{L}(c) &= -2203.160 \\
\mathcal{L}(\hat{\beta}) &= -1613.858 \\
-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] &= 2362.022 \\
\rho^2 &= 0.423 \\
\bar{\rho}^2 &= 0.417
\end{aligned}
$$

Normalization of the nested logit model

Best practice

- Normalize μ to 1.
- Estimate μ_m for each nest.
- As $0 \leq \mu/\mu_m \leq 1$, then $\mu_m \geq 1$.

Large models

- Normalize all μ_m to 1.
- Estimate μ.
- Note that it is not the most general specification.
- Imposing the same scale parameter for each nest is a strong assumption.
- Motivated only by models with a very high number of nests.

Outline

Derivation from random utility

- Let $\mathcal{C}$ be the choice set.
- Let $\mathcal{C}_1, \ldots, C_M$ be a partition of $\mathcal{C}$.
- The model is derived as

$$P(i|\mathcal{C}) = \sum_{m=1}^{M} \Pr(i|m, \mathcal{C}) \Pr(m|\mathcal{C}).$$

- Each i belongs to exactly one nest m.

$$P(i|\mathcal{C}) = \Pr(i|m) \Pr(m|\mathcal{C}).$$

- Utility: error components

$$U_i = V_i + \varepsilon_i = V_i + \varepsilon_m + \varepsilon_{im}.$$

Derivation: $\Pr(i|m)$

$$\begin{aligned}\Pr(i|m) &= \Pr(U_i \geq U_j, j \in \mathcal{C}_m) \\ &= \Pr(V_i + \varepsilon_m + \varepsilon_{im} \geq V_j + \varepsilon_m + \varepsilon_{jm}, j \in \mathcal{C}_m) \\ &= \Pr(V_i + \varepsilon_{im} \geq V_j + \varepsilon_{jm}, j \in \mathcal{C}_m)\end{aligned}$$

Assumption: ε_{im} i.i.d. $\text{EV}(0, \mu_m)$

$$\Pr(i|m) = \frac{e^{\mu_m V_i}}{\sum_{j \in \mathcal{C}_m} e^{\mu_m V_j}}.$$

Derivation: $\Pr(m|\mathcal{C})$

$$\Pr(m|\mathcal{C}) = \Pr\left(\max_{i\in\mathcal{C}_m} U_i \geq \max_{j\in\mathcal{C}_\ell} U_j, \forall \ell \neq m\right)$$

$$= \Pr\left(\varepsilon_m + \max_{i\in\mathcal{C}_m}(V_i + \varepsilon_{im}) \geq \varepsilon_\ell + \max_{j\in\mathcal{C}_\ell}(V_j + \varepsilon_{j\ell}), \forall \ell \neq m\right),$$

As ε_{im} are i.i.d. $\text{EV}(0, \mu_m)$,

$$\max_{i\in\mathcal{C}_m}(V_i + \varepsilon_{im}) \sim \text{EV}(\tilde{V}_m, \mu_m),$$

where

$$\tilde{V}_m = \frac{1}{\mu_m} \ln \sum_{i\in\mathcal{C}_m} e^{\mu_m V_i}.$$

Derivation: $\Pr(m|\mathcal{C})$

Denote

$$\max_{i \in \mathcal{C}_m}(V_i + \varepsilon_{im}) = \tilde{V}_m + \varepsilon'_m,$$

to obtain

$$\Pr(m|\mathcal{C}) = \Pr(\tilde{V}_m + \varepsilon'_m + \varepsilon_m \geq \tilde{V}_\ell + \varepsilon'_\ell + \varepsilon_\ell, \forall \ell \neq m).$$

where

$$\varepsilon'_m \sim \text{EV}(0, \mu_m).$$

Define

$$\tilde{\varepsilon}_m = \varepsilon'_m + \varepsilon_m,$$

to obtain

$$\Pr(m|\mathcal{C}) = \Pr(\tilde{V}_m + \tilde{\varepsilon}_m \geq \tilde{V}_\ell + \tilde{\varepsilon}_\ell, \forall \ell \neq m).$$

Derivation: $\Pr(m|\mathcal{C})$

Assumption: $\tilde{\varepsilon}_m$ i.i.d. EV$(0, \mu)$

$$\Pr(m|\mathcal{C}) = \Pr(\tilde{V}_m + \tilde{\varepsilon}_m \geq \tilde{V}_\ell + \tilde{\varepsilon}_\ell, \forall \ell \neq m)$$

$$= \frac{e^{\mu \tilde{V}_m}}{\sum_{p=1}^{M} e^{\mu \tilde{V}_p}}.$$

We obtain the nested logit model

$$P(i|\mathcal{C}) = \frac{e^{\mu_m V_i}}{\sum_{j \in \mathcal{C}_m} e^{\mu_m V_j}} \frac{e^{\mu \tilde{V}_m}}{\sum_{p=1}^{M} e^{\mu \tilde{V}_p}}$$

$$= \frac{e^{\mu_m V_i}}{\sum_{j \in \mathcal{C}_m} e^{\mu_m V_j}} \frac{\exp\left(\frac{\mu}{\mu_m} \ln \sum_{\ell \in \mathcal{C}_m} e^{\mu_m V_\ell}\right)}{\sum_{p=1}^{M} \exp\left(\frac{\mu}{\mu_p} \ln \sum_{\ell \in \mathcal{C}_p} e^{\mu_p V_{\ell p}}\right)}$$

Nested Logit Model

- If $\frac{\mu}{\mu_m} = 1$, for all m, NL becomes logit.
- Sequential estimation:
  - Estimation of NL decomposed into two estimations of logit
  - Estimator is consistent but not efficient
- Simultaneous estimation:
  - Log-likelihood function is generally non concave
  - No guarantee of global maximum
  - Estimator asymptotically efficient
  - Log likelihood for observation n is

$$\ln P(i_n|\mathcal{C}_n) = \ln P(i_n|\mathcal{C}_{mn}) + \ln P(\mathcal{C}_{mn}|\mathcal{C}_n)$$

where i_n is the chosen alternative.

Correlation

Correlation matrix is block diagonal:

$$\mathrm{Corr}(U_i, U_j) = \begin{cases} 1 & \text{if } i = j, \\ 1 - \dfrac{\mu^2}{\mu_m^2} & \text{if } i \neq j,\ i \text{ and } j \text{ are in the same nest } m, \\ 0 & \text{otherwise.} \end{cases}$$

Variance-covariance matrix is block diagonal:

$$\mathrm{Cov}(U_i, U_j) = \begin{cases} \dfrac{\pi^2}{6\mu^2} & \text{if } i = j, \\ \dfrac{\pi^2}{6\mu^2} - \dfrac{\pi^2}{6\mu_m^2} & \text{if } i \neq j,\ i \text{ and } j \text{ are in the same nest } m, \\ 0 & \text{otherwise.} \end{cases}$$

Summary

- Independence assumption of logit may lead to erroneous forecasts
- Relaxing the assumption: nests
- Closed form model