Theoretical foundations – 2.4 Random utility theory

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Mathematical derivation of the choice model

We derive here the general random utility model. Although the derivation is quite straightforward, it is also technical. It may be skipped without loss of continuity in the course.

Consider the choice model with J_n alternatives

$$P(i|\mathcal{C}_n) = \Pr(U_{in} \geq U_{jn}, \forall j = 1, \ldots, J_n), \tag{1}$$

where

$$U_{in} = V_{in} + \varepsilon_{in}. \tag{2}$$

Denote by

$$\varepsilon_n = (\varepsilon_{1n}, \ldots, \varepsilon_{J_n n})$$

the vector of J_n error terms. If ε_n is a multivariate random variable with CDF $F_{\varepsilon_n}(\varepsilon_1, \ldots, \varepsilon_{J_n})$ and pdf

$$f_{\varepsilon_n}(\varepsilon_1, \ldots, \varepsilon_{J_n}) = \frac{\partial^{J_n} F}{\partial \varepsilon_1 \cdots \partial \varepsilon_{J_n}}(\varepsilon_1, \ldots, \varepsilon_{J_n}), \tag{3}$$

then

$$\begin{aligned} P_n(i|\mathcal{C}_n) = \int_{\varepsilon_i=-\infty}^{+\infty} \int_{\varepsilon_1=-\infty}^{V_{in}-V_{1n}+\varepsilon_i} \cdots \\ \int_{\varepsilon_{i-1}=-\infty}^{V_{in}-V_{i-1n}+\varepsilon_i} \int_{\varepsilon_{i+1}=-\infty}^{V_{in}-V_{i+1n}+\varepsilon_i} \cdots \\ \int_{\varepsilon_{J_n}=-\infty}^{V_{1n}-V_{J_n n}+\varepsilon_1} f_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{J_n}) d\varepsilon. \end{aligned} \tag{4}$$

and

$$P_n(i|\mathcal{C}_n) = \int_{\varepsilon=-\infty}^{+\infty} \frac{\partial F_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}}{\partial \varepsilon_i}(\ldots, V_{in}-V_{(i-1)n}+\varepsilon, \varepsilon, V_{in}-V_{(i+1)n}+\varepsilon, \ldots)d\varepsilon. \tag{5}$$

Therefore, if the CDF is available in closed form, the choice model is obtained by solving a uni-dimensional integral, which can be done analytically for simple models, and numerically for more complex ones.

Proof. We prove the result for alternative 1 without loss of generality, in order to simplify the notations.

Using (2) into (1), we obtain

$$P(1|\mathcal{C}_n) = \Pr(V_{2n} + \varepsilon_{2n} \leq V_{1n} + \varepsilon_{1n}, \ldots, V_{J_n n} + \varepsilon_{J_n n} \leq V_{1n} + \varepsilon_{1n}), \tag{6}$$

or, gathering the random terms on one side, and the deterministic ones on the other side,

$$P_n(1|\mathcal{C}_n) = \Pr(\varepsilon_{2n} - \varepsilon_{1n} \leq V_{1n} - V_{2n}, \ldots, \varepsilon_{J_n n} - \varepsilon_{1n} \leq V_{1n} - V_{J_n n}). \tag{7}$$

We consider the following change of variables:

$$\xi_{1n} = \varepsilon_{1n},\ \xi_{jn} = \varepsilon_{jn} - \varepsilon_{1n},\ j = 2, \ldots, J_n, \tag{8}$$

that is, in matrix notations,

$$\xi_n = \begin{pmatrix} \xi_{1n} \\ \xi_{2n} \\ \vdots \\ \xi_{(J_n-1)n} \\ \xi_{J_n n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ & & \vdots & & \\ -1 & 0 & \cdots & 1 & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \\ \vdots \\ \varepsilon_{(J_n-1)n} \\ \varepsilon_{J_n n} \end{pmatrix} = M\varepsilon_n.$$

Note that the determinant of the change of variables matrix M is 1, so that ε_n and ξ_n have the same pdf. The model in the new variables becomes

$$P_n(1|\mathcal{C}_n) = \Pr(\xi_{2n} \leq V_{1n} - V_{2n}, \ldots, \xi_{J_n n} \leq V_{1n} - V_{J_n n}).$$

Therefore,

$$P_n(1|\mathcal{C}_n) = F_{\xi_{1n},\xi_{2n},\ldots,\xi_{J_n}}(+\infty, V_{1n} - V_{2n}, \ldots, V_{1n} - V_{J_n n})$$

from the definition of a cumulative distribution function. As the CDF is obtained by integrating the pdf, we have

$$P_n(1|\mathcal{C}_n) = \int_{\xi_1=-\infty}^{+\infty} \int_{\xi_2=-\infty}^{V_{1n}-V_{2n}} \cdots \int_{\xi_{J_n}=-\infty}^{V_{1n}-V_{J_n n}} f_{\xi_{1n},\xi_{2n},\ldots,\xi_{J_n}}(\xi_1, \xi_2, \ldots, \xi_{J_n}) d\xi.$$

Now we come back to the original variables, exploiting the fact that the pdf of ξ_n and ε_n are identical:

$$P_n(1|\mathcal{C}_n) = \int_{\varepsilon_1=-\infty}^{+\infty} \int_{\varepsilon_2=-\infty}^{V_{1n}-V_{2n}+\varepsilon_1} \cdots \int_{\varepsilon_{J_n}=-\infty}^{V_{1n}-V_{J_n n}+\varepsilon_1} f_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{J_n}) d\varepsilon.$$

By integrating over all dimensions except the first one, we obtain:

$$P_n(1|\mathcal{C}_n) = \int_{\varepsilon_1=-\infty}^{+\infty} \frac{\partial F_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}}{\partial \varepsilon_1}(\varepsilon_1, V_{1n} - V_{2n} + \varepsilon_1, \ldots, V_{1n} - V_{J_n n} + \varepsilon_1) d\varepsilon_1.$$

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