Theoretical foundations – 2.3 Example

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We illustrate the concept of utility using a simple example of a transportation mode choice, where two alternatives are considered for a commuter trip: car and bus. Each alternative is characterized by two attributes: the travel time and the travel cost, as reported in Table 1

	Attributes	
Alternatives	Travel time (t)	Travel cost (c)
Car (i)	t_i	c_i
Bus (j)	t_j	c_j

Table 1: Attributes of the alternatives

We denote by y_i and y_j the binary variables associated with each alternative:

$$y_i = \begin{cases} 1 & \text{if car is chosen,} \\ 0 & \text{otherwise;} \end{cases}$$

and $y_j = 1 - y_i$, in order to verify the constraint imposing that exactly one alternative is chosen. In terms of the decision problem that the individual is solving, the decision variables and the feasible set are illustrated in Figure 1.

The utility functions associated with each alternative can be written as

$$\begin{aligned} U_i &= -\beta_t t_i - \beta_c c_i, \\ U_j &= -\beta_t t_j - \beta_c c_j, \end{aligned}$$

where $\beta_t > 0$ and $\beta_c > 0$ are parameters.

Note that this specification involves some behavioral assumptions:

- The sign restrictions on the unknown parameters β_t and β_c impose that the value of the utility decreases when one of the variables increases. It is consistent with the behavioral assumption that commuters want to arrive as fast as possible to their destination, at the lowest cost possible.

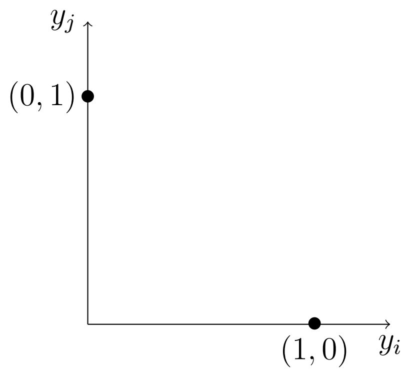

Figure 1: Decision variables and feasible set

- The same coefficients are used for both alternatives. This implies that a modification of the travel time has the same impact on the utility of car and on the utility of bus. The same applies for travel cost. This assumption is debatable. It can be argued that an additional minute spent in the bus, with the possibility to sleep, listen to music, or read, may not be perceived the same way as spending one more minute driving the car. *Quiz: How would you specify a model where the impact of an additional minute in travel time would be different for the two alternatives?*

As a representation of the individual's preferences, the utility is defined up to order preserving transformations. For instance, we can divide each utility by a strictly positive number, without modifying their ranking.

$$
\begin{array}{rcccl}
U'_i & = & -(\beta_t/\beta_c)t_i - c_i & = & -\beta t_i - c_i \\
U'_j & = & -(\beta_t/\beta_c)t_j - c_j & = & -\beta t_j - c_j
\end{array}
$$

where $\beta = \beta_t/\beta_c > 0$ is a parameter. Note that this parameter is converting travel time units into travel cost units, so that they can be combined together in the same utility function.

The behavioral assumption is that alternative i is chosen if $U_i \geq U_j$ or, equivalently, if $U'_i \geq U'_j$. If we ignore ties, we obtain

$$-\beta t_i - c_i < -\beta t_j - c_j,$$

or, equivalently,

$$-\beta(t_i - t_j) < c_i - c_j.$$

Two cases are trivial:

- If $c_j > c_i$ and $t_j > t_i$, the car alternative is both cheaper and faster than the bus alternative. Therefore, $U_i > U_j$ for any $\beta > 0$, and the car is chosen. The car alternative is called a *dominating alternative*.

- Symmetrically, if $c_i > c_j$ and $t_i > t_j$, the car alternative is both more expensive and slower than the bus alternative. Therefore, $U_i < U_j$ for any $\beta > 0$, and the bus is chosen. The car alternative is called a *dominated alternative*.

But what happens when one alternative is cheaper and slower than the other one? In that case, the parameter β captures the trade-off of the decision-maker between the two variables. For instance, assume that the car is cheaper and slower than the bus, that is $c_j > c_i$ and $t_i > t_j$. Alternative j is chosen if

$$-\beta(t_i - t_j) < c_i - c_j,$$

or, as $t_i > t_j$,

$$\beta > \frac{c_j - c_i}{t_i - t_j}.$$

The behavioral question is: Is the traveler willing to pay the extra cost $c_j - c_i$ to save the extra time $t_i - t_j$? The parameter β is capturing this behavioral trade-off. It is called *the value of time*, or the *willingness to pay* to save travel time, and will be discussed in more details later in the course.

This simple example is illustrated in Figure 2. The x-axis corresponds to the difference of travel time $t_i - t_j$. Therefore, negative values correspond to alternative i being faster, and positive values to alternative j being faster. Similarly, the y-axis corresponds to the difference of travel cost $c_i - c_j$. Negative values correspond to alternative i being cheaper, and positive values to alternative j being cheaper.

The north-east quadrant corresponds to situations where alternative j is dominant. Indeed, it is both faster and cheaper. Symmetrically, the south-west quadrant corresponds to situations where alternative i is dominant.

The two other quadrants correspond to situations where there is a trade-off between travel time and travel cost. For a given value of the parameter β, we draw the indifference line, corresponding to situations where the two utilities are equal, that is

$$c_i + \beta t_i = c_j + \beta t_j,$$

or, equivalently,

$$c_i - c_j = -\beta(t_i - t_j).$$

As $\beta > 0$, the slope of this line is negative.

In order to determine the value of β, we collect choice data. We observe a sample of individuals during their commuting trip, and, for each of them, collect:

- the travel time by car t_i,
- the travel time by bus t_j,
- the travel cost by car c_i,
- the travel cost by bus c_j,
- the alternative actually chosen (i or j).

The data is represented in Figure 3, using the following convention:

- each dot corresponds to an individual,
- the x coordinate of the dot corresponds to the associated value of $t_i - t_j$,
- the y coordinate of the dot corresponds to the associated value of $c_i - c_j$,
- the shape of the dot reveals the choice made by the individual.

Therefore, the objective is to find a value of β, that is, to find a slope of the indifference line, such that all dots corresponding to alternative i lie on one side of the line, and all dots corresponding to alternative j lie on the other side. It is clear that the choice of β in Figure 3 does not achieve that. Moreover, it is relatively easy to figure out that it is impossible to find such a slope. There is at least one dot corresponding to alternative i that is surrounded by dots corresponding to alternative j, and that cannot be separated from them using any line.

This inconsistency between the behavioral model and the behavioral observations illustrates the limitations of the utility theory, when confronted to data, and motivates to consider the utility as a random variable. The *random utility theory* is discussed next.

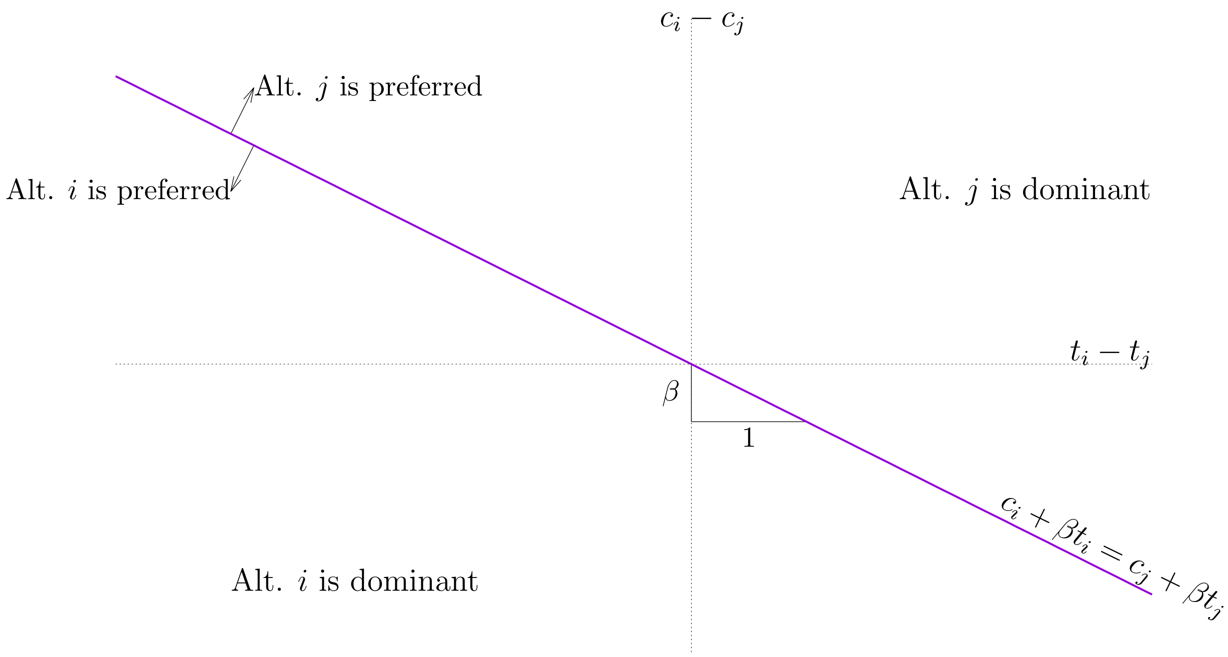

Figure 2: Simple example: two transportation modes

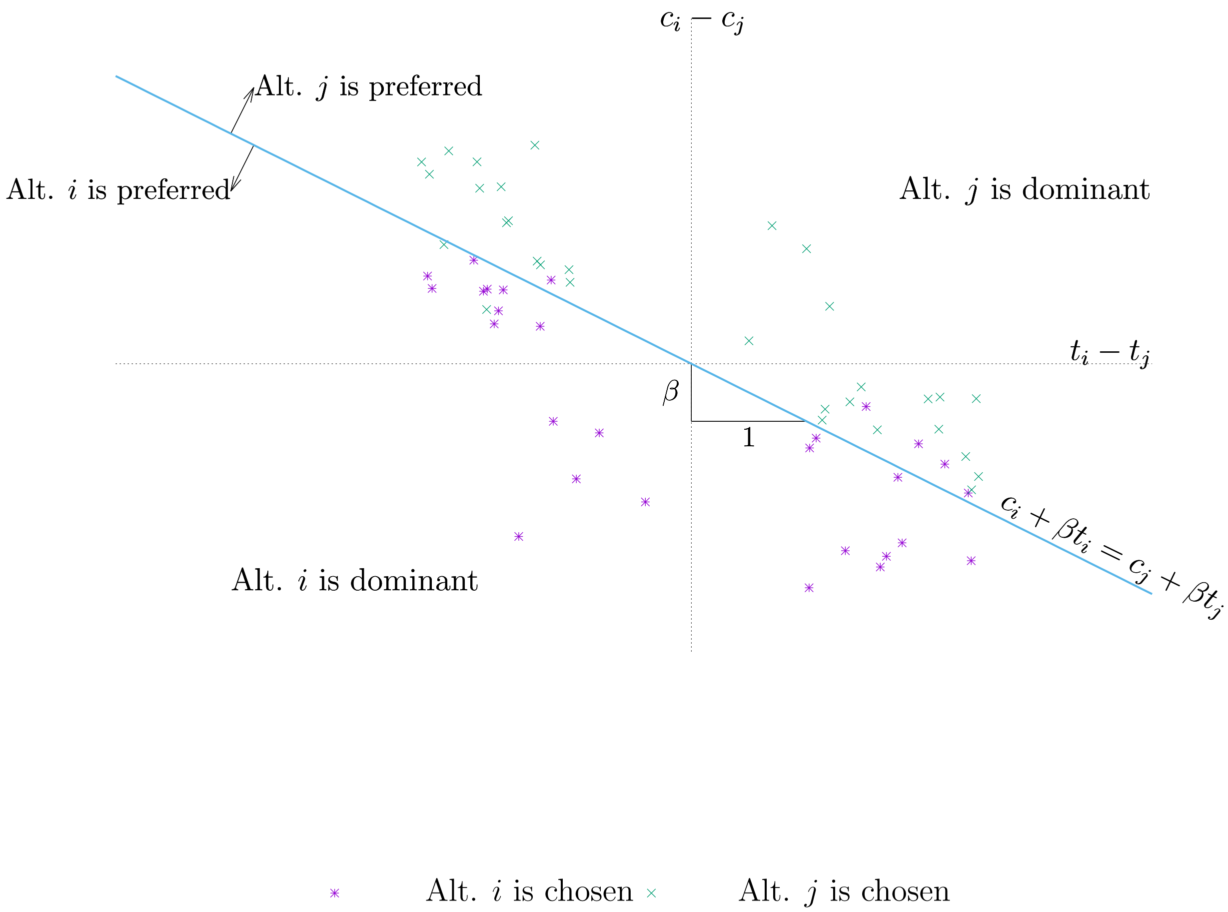

Figure 3: Simple example: two transportation modes with observed choices