Tests

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Outline

1. Introduction
2. Case study
3. Informal tests
4. t-test
5. Wald test
   - Linear restrictions
   - Nonlinear restrictions
6. Likelihood ratio test
   - Test of generic attributes
   - Test of taste variations
7. Tests of Nonlinear Specifications
   - Piecewise linear
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   - Cox test
   - Davidson and McKinnon J-test
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10. Market segments
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Introduction

Modeling

- Impossible to determine the most appropriate model specification
- A good fit does not mean a good model
- Formal testing is necessary, but not sufficient
- No clear-cut rules can be given
- Subjective judgments of the analyst
- Good modeling = good judgment + good analysis

Wilkinson (1999) "The grammar of graphics". Springer

... some researchers who use statistical methods pay more attention to goodness of fit than to the meaning of the model... Statisticians must think about what the models mean, regardless of fit, or they will promulgate nonsense.

Introduction

Hypothesis testing

Two propositions

- H_0 null hypothesis
- H_1 alternative hypothesis

Analogy with a court trial

- H_0: the defendant
- "Presumed innocent until proved guilty"
- H_0 is accepted, unless the data argue strongly to the contrary
- Benefit of the doubt

Introduction

Errors are always possible

	Accept H_0	Reject H_0
H_0 is true		Type I error (prob. α)
H_0 is false	Type II error (prob. β)	

In the court...

- Type I error: send an innocent to jail
- Type II error: free a culprit

Introduction

Errors

- For a given sample size N, there is a trade-off between α and β.
- The only way to reduce both Type I and Type II error probabilities is to increase N.
- $\pi = 1 - \beta$ is the power of the test, that is the probability of rejecting H_0 when H_0 is false.
- H_1 is usually a composite hypothesis. π can only be determined for a simple hypothesis.
- In general, α is fixed by the analyst, and the power is maximized by the test.

Introduction

Type I error: incorrectly reject

$$\begin{aligned} P\,(\text{Type I error}) &= P(H_0 \text{ rejected}, H_0 \text{ true}) \\ &= P(H_0 \text{ rejected} | H_0 \text{ true}) P(H_0 \text{ true}) \\ &= \alpha\lambda \end{aligned}$$

Type II error: incorrectly accept

$$\begin{aligned} P\,(\text{Type II error}) &= P(H_0 \text{ accepted}, H_0 \text{ false}) \\ &= P(H_0 \text{ accepted} | H_0 \text{ false}) P(H_0 \text{ false}) \\ &= \beta(1-\lambda) = (1-\pi)(1-\lambda) \end{aligned}$$

Introduction

Impact of an error

$$\sum_{\text{possible errors}} P\left(\text{error}\right) \text{Cost}\left(\text{error}\right) = \alpha\lambda C_I + (1-\pi)(1-\lambda)C_{II}$$

Classical hypothesis testing

- We believe in H_0: $\lambda \approx 1$.
- $C_I \approx C_{II}$
- Choose small α
- Assume H_0 true unless proven otherwise

Introduction

Impact of an error

$$\sum_{\text{possible errors}} P\,(\text{error})\,\text{Cost}\,(\text{error}) = \alpha\lambda C_I + (1-\pi)(1-\lambda)C_{II}$$

Specification testing

- No strong believe in H_0: $\lambda \approx 0.5$.
- Type I: include irrelevant variables, loss of efficiency
- Type II: omit relevant variables, specification error
- C_{II} larger than C_I
- Choose larger α

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Case study: choice of airline itinerary

Survey

- Conducted by the Boeing Company (fall 2004)
- Sample of the customers of an Internet airline booking service

Booking

The Internet service

- takes a specific user request for travel in a city pair
- interrogates the web sites of airlines that provide service in that market
- returns to the user a compiled list of available itineraries

Case study: choice of airline itinerary

Questionnaire

- Random selection of customers for the survey
- Three alternatives based on the origin-destination market request that the respondent entered into the itinerary search engine:
  1. a non stop flight
  2. a flight with 1 stop on the same airline
  3. a flight with 1 stop and a change of airline

Demographic data

- age
- gender
- income
- occupation
- education

Context data

- desired departure time
- trip purpose
- who is paying for the trip
- the number of passengers traveling together

Case study: choice of airline itinerary

Pick Your Preferred Flight

Three flight options are described for your trip from Chicago to San Diego . These are options that might be available on this route or might be new options actively being considered for this route as well as replacing some options that are offered now. The options differ from each other in one or more of the features described on the left.

Please evaluate these options, assuming that eveything about the options is the same except these particular features. Indicate your choices at the bottom of the appropriate column and press the Continue button.

FEATURES	Non-Stop (Option 1)	1 Stop (Option 2)	1 Stop (Option 3)
Departure time (local)	6:00 PM	4:30 PM	6:00 PM
Arrival time (local)	8:14 PM	8:44 PM	9:44 PM
Total time in air	4 hr 14 min	4 hr 44 min	4 hr 44 min
Total trip time	4 hr 14 min	6 hr 14 min	5 hr 44 min
Legroom	typical legroom	2-in more of legroom	4-in more of legroom
Airline [Airplane]	Depart Chicago Continental Airlines [B737] to San Diego	Depart Chicago Southwest Airlines [A320], connecting with Southwest Airlines [MD80] to San Diego	Depart Chicago Northwest Airlines [MD80], connecting with American Airlines [DC9] to San Diego
Fare	$565	$485	$620

1. Which is MOST attractive? ○ Option 1 ○ Option 2 ○ Option 3

2. Which is LEAST attractive? ○ Option 1 ○ Option 2 ○ Option 3

3. If these were the ONLY three options available, I would NOT make this trip by air. ○ Yes ○ No

Case study: choice of airline itinerary

Sample

- origin-destination city pairs in the USA
- 3609 respondents
- 1 choice each
- we consider only data corresponding to leisure trips

Logit model

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.879	0.219	-4.02	0.00
2	One stop–multiple airlines dummy	-1.27	0.227	-5.60	0.00
3	Round trip fare ($100)	-1.81	0.151	-11.99	0.00
4	Elapsed time (hours)	-0.303	0.0778	-3.90	0.00
5	Leg room (inches), if male (non stop)	0.100	0.0330	3.04	0.00
6	Leg room (inches), if female (non stop)	0.182	0.0318	5.71	0.00
7	Leg room (inches), if male (one stop)	0.113	0.0297	3.80	0.00
8	Leg room (inches), if female (one stop)	0.0931	0.0273	3.41	0.00
9	Being early (hours)	-0.151	0.0189	-7.99	0.00
10	Being late (hours)	-0.0975	0.0167	-5.83	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.300	0.141	-2.12	0.03
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0847	0.157	-0.54	0.59
13	Male dummy (one stop–same airline)	0.100	0.133	0.75	0.45
14	Male dummy (one stop–multiple airlines)	0.189	0.144	1.31	0.19
15	Round trip fare / income ($100/$1000)	-23.8	8.09	-2.94	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\hat{\beta}) = -1640.525$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 2308.689$$
$$\rho^2 = 0.413$$
$$\bar{\rho}^2 = 0.408$$

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Informal tests

Sign of the coefficients

All coefficients have the correct sign

Trade-offs

- quantity that one variable should vary in order to compensate a small variation of another variable
- consider x_k, x_ℓ and alternative i

$$\frac{\partial U_{in}/\partial x_k}{\partial U_{in}/\partial x_\ell}$$

- Unit: unit of x_ℓ divided by the unit of x_k

Airline itinerary choice

Utility function

$$U_{in} = \cdots + \beta_3 \text{round trip fare} + \beta_4 \text{elapsed time} + \cdots + \beta_{15} \frac{\text{round trip fare}}{\text{income}} + \varepsilon_{in},$$

Trade-off between time and cost

$$\frac{\partial U_{in}/\partial \text{elapsed time}}{\partial U_{in}/\partial \text{round trip fare}} = \frac{\widehat{\beta}_4}{\widehat{\beta}_3 + \frac{\widehat{\beta}_{15}}{\text{income}}} \frac{\$100}{\text{hour}}, \forall i \in \mathcal{C}_n.$$

Trade-off for the sample average income: 107 (kUSD/year)

$$\frac{-0.303}{-1.81 + (\frac{-23.8}{107})} = 0.149 \; \$100/\text{hour} = \$14.9/\text{hour}, \forall i \in \mathcal{C}_n.$$

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t-test

Question

Is the parameter θ significantly different from a given value θ^*?

- $H_0 : \theta = \theta^*$
- $H_1 : \theta \neq \theta^*$

Statistic

Under H_0, if $\hat{\theta}$ is normally distributed with known variance σ^2

$$\frac{\hat{\theta} - \theta^*}{\sigma} \sim N(0, 1).$$

Therefore

$$P(-1.96 \leq \frac{\hat{\theta} - \theta^*}{\sigma} \leq 1.96) = 0.95 = 1 - 0.05$$

t-test

Test

$$P(-1.96 \leq \frac{\hat{\theta} - \theta^*}{\sigma} \leq 1.96) = 0.95 = 1 - 0.05$$

H_0 can be rejected at the 5% level ($\alpha = 0.05$) if

$$\left| \frac{\hat{\theta} - \theta^*}{\sigma} \right| \geq 1.96.$$

Comments

- If $\hat{\theta}$ asymptotically normal
- If variance unknown
- A t test should be used with n degrees of freedom.
- When $n \geq 30$, the Student t distribution is well approximated by a $N(0, 1)$

Estimator of the asymptotic variance for ML

Cramer-Rao Bound with the estimated parameters

$$\hat{V}_{CR} = -\nabla^2 \ln L(\hat{\theta})^{-1}$$

Berndt, Hall, Hall & Haussman (BHHH) estimator

$$\hat{V}_{BHHH} = \left(\sum_{i=1}^{n} \hat{g}_i \hat{g}_i^T \right)^{-1}$$

where

$$\hat{g}_i = \frac{\partial \ln P_n(i; \mathcal{C}, \theta)}{\partial \theta}$$

Estimator of the asymptotic variance for ML

Robust estimator

$$\hat{V}_{CR}\hat{V}_{BHHH}^{-1}\hat{V}_{CR}$$

- The three are asymptotically equivalent
- This one is more robust when the model is misspecified
- Biogeme uses Cramer-Rao and the robust estimators

t-test

p value

- probability to get a t statistic at least as large (in absolute value) as the one reported, under the null hypothesis
- the null hypothesis is rejected when the p-value is lower than the significance level (typically 0.05)

Case study

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.879	0.219	-4.02	0.00
2	One stop–multiple airlines dummy	-1.27	0.227	-5.60	0.00
3	Round trip fare ($100)	-1.81	0.151	-11.99	0.00
4	Elapsed time (hours)	-0.303	0.0778	-3.90	0.00
5	Leg room (inches), if male (non stop)	0.100	0.0330	3.04	0.00
6	Leg room (inches), if female (non stop)	0.182	0.0318	5.71	0.00
7	Leg room (inches), if male (one stop)	0.113	0.0297	3.80	0.00
8	Leg room (inches), if female (one stop)	0.0931	0.0273	3.41	0.00
9	**Being early (hours)**	-0.151	0.0189	-7.99	0.00
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12	More than 2 air trips per year (one stop–multiple airlines)	-0.0847	0.157	-0.54	0.59
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H_0: "the fact of being early does not play a role in the choice"

t-test = -7.99. Rejected at the 5% level.

t-test

Comparing two coefficients

$$H_0 : \beta_1 = \beta_2.$$

The t statistic is given by

$$\frac{\widehat{\beta}_1 - \widehat{\beta}_2}{\sqrt{\mathrm{Var}(\widehat{\beta}_1 - \widehat{\beta}_2)}}$$

$$\mathrm{Var}(\widehat{\beta}_1 - \widehat{\beta}_2) = \mathrm{Var}(\widehat{\beta}_1) + \mathrm{Var}(\widehat{\beta}_2) - 2\,\mathrm{Cov}(\widehat{\beta}_1, \widehat{\beta}_2)$$

Case study

Specifications

We compare two specifications:

- the elapsed time coefficient is generic.
- the elapsed time coefficient is alternative specific.

Specification with generic elapsed time coefficients

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	*t*-stat	*p*-value
1	One stop–same airline dummy	-0.964	0.216	-4.47	0.00
2	One stop–multiple airlines dummy	-1.36	0.224	-6.09	0.00
3	**Elapsed time (hours)**	-0.301	0.0778	-3.87	0.00
4	Round trip fare ($100)	-1.80	0.150	-11.97	0.00
5	Leg room (inches), if female	0.132	0.0220	6.00	0.00
6	Leg room (inches), if male	0.107	0.0232	4.62	0.00
7	Being early (hours)	-0.151	0.0188	-8.04	0.00
8	Being late (hours)	-0.0958	0.0167	-5.74	0.00
9	More than 2 air trips per year (one stop–same airline)	-0.309	0.141	-2.20	0.03
10	More than 2 air trips per year (one stop–multiple airlines)	-0.0931	0.157	-0.59	0.55
11	Male dummy (one stop–same airline)	0.201	0.125	1.60	0.11
12	Male dummy (one stop–multiple airlines)	0.294	0.132	2.23	0.03
13	Round trip fare / income ($100/$1000)	-24.1	8.07	-2.98	0.00

Summary statistics

Number of observations = 2544

$$
\begin{aligned}
\mathcal{L}(0) &= -2794.870 \\
\mathcal{L}(c) &= -2203.160 \\
\mathcal{L}(\widehat{\beta}) &= -1642.796 \\
-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] &= 2304.148 \\
\rho^2 &= 0.412 \\
\bar{\rho}^2 &= 0.408
\end{aligned}
$$

Specification with alternative specific elapsed time coefficients

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	*t*-stat	*p*-value
1	One stop–same airline dummy	-1.17	0.278	-4.19	0.00
2	One stop–multiple airlines dummy	-1.45	0.292	-4.98	0.00
3	**Elapsed time (hours) (non stop)**	-0.341	0.0854	-3.99	0.00
4	**Elapsed time (hours) (one stop–same airline)**	-0.291	0.0822	-3.54	0.00
5	**Elapsed time (hours) (one stop–multiple airlines)**	-0.310	0.0802	-3.87	0.00
6	Round trip fare ($100)	-1.78	0.151	-11.84	0.00
7	Leg room (inches), if male	0.108	0.0232	4.65	0.00
8	Leg room (inches), if female	0.132	0.0221	5.99	0.00
9	Being early (hours)	-0.151	0.0188	-8.02	0.00
10	Being late (hours)	-0.0960	0.0167	-5.73	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.307	0.141	-2.18	0.03
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0910	0.157	-0.58	0.56
13	Male dummy (one stop–same airline)	0.199	0.126	1.59	0.11
14	Male dummy (one stop–multiple airlines)	0.293	0.132	2.21	0.03
15	Round trip fare / income ($100/$1000)	-24.0	8.09	-2.97	0.00

Summary statistics

Number of observations = 2544

$$
\begin{aligned}
\mathcal{L}(0) &= -2794.870 \\
\mathcal{L}(c) &= -2203.160 \\
\mathcal{L}(\widehat{\beta}) &= -1641.932 \\
-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] &= 2305.875 \\
\rho^2 &= 0.413 \\
\bar{\rho}^2 &= 0.407
\end{aligned}
$$

Tests

Asymptotic covariance matrix

	β_3	β_4	β_5
β_3	0.00729	0.00627	0.006
β_4	0.00627	0.00676	0.00553
β_5	0.006	0.00553	0.00643

$H_0 : \beta_3 = \beta_4$

$$\begin{aligned}\text{Var}(\widehat{\beta}_3 - \widehat{\beta}_4) &= \text{Var}(\widehat{\beta}_3) + \text{Var}(\widehat{\beta}_4) - 2\,\text{Cov}(\widehat{\beta}_3, \widehat{\beta}_4) \\ &= 0.00729 + 0.00676 - 2 \times 0.00627 = 0.00151\end{aligned}$$

$$\frac{\widehat{\beta}_3 - \widehat{\beta}_4}{\sqrt{\text{Var}(\widehat{\beta}_3 - \widehat{\beta}_4)}} = \frac{-0.341 - (-0.291)}{\sqrt{0.00151}} = -1.287$$

Cannot reject H_0

Tests

$H_0 : \beta_3 = \beta_4$

$$\frac{-0.341 - (-0.291)}{\sqrt{0.00729 + 0.00676 - 2 \times 0.00627}} = -1.287$$

$H_0 : \beta_4 = \beta_5$

$$\frac{-0.291 - (-0.310)}{\sqrt{0.00676 + 0.00643 - 2 \times 0.00553}} = 0.412.$$

$H_0 : \beta_3 = \beta_5$

$$\frac{-0.341 - (-0.310)}{\sqrt{0.00729 + 0.00643 - 2 \times 0.006}} = -0.747.$$

None can be rejected.

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Wald test

Linear restrictions

- β is the $P \times 1$ vector of parameters. R is a $Q \times P$ matrix of linear restrictions
- $\mathrm{H}_0 : \mathrm{R}\beta = r \qquad \mathrm{H}_1 : \mathrm{R}\beta \neq r$
- $\hat{\beta} - \beta \xrightarrow[n \to +\infty]{d} \mathcal{N}\left(0, \hat{V}_\beta\right)$

Statistic

Under H_0,

$$W = \left(\mathrm{R}\hat{\beta} - r\right)^T \left(\mathrm{R}\hat{V}_\beta \mathrm{R}^T\right)^{-1} \left(\mathrm{R}\hat{\beta} - r\right) \sim \chi^2_Q$$

- can be used for testing single parameter equal to one specific value: W is equal to the square of the t-stat and follows asymptotically a χ^2 distribution with 1 degree of freedom

Specification with alternative specific elapsed time coefficients

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1	One stop–same airline dummy	-1.17	0.278	-4.19	0.00
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4	**Elapsed time (hours) (one stop–same airline)**	-0.291	0.0822	-3.54	0.00
5	**Elapsed time (hours) (one stop–multiple airlines)**	-0.310	0.0802	-3.87	0.00
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\mathcal{L}(\widehat{\beta}) &= -1641.932 \\
-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] &= 2305.875 \\
\rho^2 &= 0.413 \\
\bar{\rho}^2 &= 0.407
\end{aligned}
$$

Wald test

- H_0: the elapsed time coefficient is generic.
- $H_0 : \beta_3 = \beta_4 = \beta_5 \Leftrightarrow H_0 : \beta_3 - \beta_4 = 0, \beta_3 - \beta_5 = 0.$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & -1 & \cdots & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \vdots \\ \beta_{15} \end{bmatrix}}_{\beta} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{r}$$

Wald test

Note that

$$R\hat{V}_\beta R^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.00729 & 0.00627 & 0.006 \\ 0.00627 & 0.00676 & 0.00553 \\ 0.006 & 0.00553 & 0.00643 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$R\hat{V}_\beta R^T = \begin{bmatrix} 0.00151 & 0.00055 \\ 0.00055 & 0.00172 \end{bmatrix}$$

Wald test

$$R\hat{\beta} = \begin{bmatrix} -0.050 \\ -0.031 \end{bmatrix}, \left(R\hat{V}_\beta R^T\right)^{-1} = \begin{bmatrix} 749.5533 & -239.6827 \\ -239.6827 & 658.0381 \end{bmatrix}$$

$$W = \begin{bmatrix} -0.050 & -0.031 \end{bmatrix} \begin{bmatrix} 749.5533 & -239.6827 \\ -239.6827 & 658.0381 \end{bmatrix} \begin{bmatrix} -0.050 \\ -0.031 \end{bmatrix} = 1.763$$

$W \sim \chi^2_2$ (2 linear restrictions). We reject H_0 at level of risk α if $W > C_{1-\alpha}$.

$$\Pr\left(W > C_{1-\alpha}\right) = \alpha \Leftarrow C = 5.9915 \text{ for } \alpha = 0.05$$

H_0 is not rejected

Wald test

Nonlinear restrictions

- $\text{H}_0 : c(\beta) = r \qquad \text{H}_1 : c(\beta) \neq r$
- $c()$ is a $\mathcal{C}^1$-class monotonic function
- $\hat{\beta} - \beta \xrightarrow[n\to+\infty]{\text{d}} \mathcal{N}\left(0, \hat{V}_\beta\right)$

Statistic

Under H_0,

$$W = \left(c\left(\hat{\beta}\right) - r\right)^T \left(\frac{\partial c\left(\hat{\beta}\right)}{\partial \beta} \hat{V}_\beta \frac{\partial c\left(\hat{\beta}\right)}{\partial \theta^T}\right)^{-1} \left(c\left(\hat{\beta}\right) - r\right) \sim \chi^2_Q$$

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Likelihood ratio test

Comparing two models

- Used for "nested" hypotheses
- One model is a special case of the other obtained from a set of linear restrictions on the parameters
- H_0: the restricted model is the true model.

Statistic

Under H_0,

$$-2(\mathcal{L}(\hat{\beta}_R) - \mathcal{L}(\hat{\beta}_U)) \sim \chi^2_{(K_U - K_R)}$$

- $\mathcal{L}(\hat{\beta}_R)$ is the log likelihood of the restricted model
- $\mathcal{L}(\hat{\beta}_U)$ is the log likelihood of the unrestricted model
- K_R is the number of parameters in the restricted model
- K_U is the number of parameters in the unrestricted model

Restricted models

Equal probability

$$P(i|\mathcal{C}_n) = \frac{1}{J_n}$$

- Restrictions: $\beta_1 = \beta_2 = \ldots = \beta_K = 0$
- Statistic: $-2(\mathcal{L}(0) - \mathcal{L}(\widehat{\beta}))$, where $\mathcal{L}(0) = -\sum_{n=1}^{N} \log(J_n)$.
- Distributed as χ^2_K.
- In practice, H_0 is often rejected.

Summary statistics

Number of observations = 2544

$\mathcal{L}(0)$	=	-2794.870
$\mathcal{L}(c)$	=	-2203.160
$\mathcal{L}(\hat{\beta})$	=	-1640.525
$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})]$	=	2308.689
ρ^2	=	0.413
$\bar{\rho}^2$	=	0.408

Restricted models

Constants only

- Restrictions: all parameters except the ASCs are zero.
- Statistic: $-2(\mathcal{L}(c) - \mathcal{L}(\widehat{\beta}))$.
- If all alternatives are always available:

$$\mathcal{L}(c) = \sum_{i=1}^{J} N_i \ln(\frac{N_i}{N})$$

where N_i is the number of obs. selecting alternative i

- Base model: $-2(-2203.160 - (-1640.525)) = 1125.27$
- 15 parameters, 2 constants: χ^2_{13} (90%: 19.81, 95%: 22.36).
- Restrictions rejected.

Unrestricted model

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.879	0.219	-4.02	0.00
2	One stop–multiple airlines dummy	-1.27	0.227	-5.60	0.00
3	Round trip fare ($100)	-1.81	0.151	-11.99	0.00
4	Elapsed time (hours)	-0.303	0.0778	-3.90	0.00
5	**Leg room (inches), if male (non stop)**	0.100	0.0330	3.04	0.00
6	**Leg room (inches), if female (non stop)**	0.182	0.0318	5.71	0.00
7	**Leg room (inches), if male (one stop)**	0.113	0.0297	3.80	0.00
8	**Leg room (inches), if female (one stop)**	0.0931	0.0273	3.41	0.00
9	Being early (hours)	-0.151	0.0189	-7.99	0.00
10	Being late (hours)	-0.0975	0.0167	-5.83	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.300	0.141	-2.12	0.03
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0847	0.157	-0.54	0.59
13	Male dummy (one stop–same airline)	0.100	0.133	0.75	0.45
14	Male dummy (one stop–multiple airlines)	0.189	0.144	1.31	0.19
15	**Round trip fare / income ($100/$1000)**	-23.8	8.09	-2.94	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\hat{\beta}) = -1640.525$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 2308.689$$
$$\rho^2 = 0.413$$
$$\bar{\rho}^2 = 0.408$$

Restricted model: leg room coefficient generic, no interaction round trip fare / income

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.922	0.215	-4.28	0.00
2	One stop–multiple airlines dummy	-1.31	0.222	-5.89	0.00
3	Round trip fare ($100)	-2.16	0.103	-20.92	0.00
4	Elapsed time (hours)	-0.302	0.0778	-3.88	0.00
5	**Leg room (inches), if male**	0.108	0.0233	4.66	0.00
6	**Leg room (inches), if female**	0.131	0.0219	5.99	0.00
7	Being early (hours)	-0.150	0.0188	-7.97	0.00
8	Being late (hours)	-0.0946	0.0166	-5.70	0.00
9	More than two air trips per year (one stop–same airline)	-0.349	0.138	-2.52	0.01
10	More than two air trips per year (one stop–multiple airlines)	-0.153	0.153	-1.00	0.32
11	Male dummy (one stop–same airline)	0.188	0.125	1.51	0.13
12	Male dummy (one stop–multiple airlines)	0.288	0.132	2.18	0.03

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\hat{\beta}) = -1652.573$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 2284.594$$
$$\rho^2 = 0.409$$
$$\bar{\rho}^2 = 0.404$$

Testing restrictions

Linear restrictions

- $\beta_5 = \beta_7$,
- $\beta_6 = \beta_8$,
- $\beta_{15} = 0$.

Test

- Unrestricted model: $\mathcal{L}(\hat{\beta}) = -1640.525$, 15 parameters
- Restricted model: $\mathcal{L}(\hat{\beta}) = -1652.573$, 12 parameters
- Test: $-2(-1652.573 + 1640.525) = 24.096$
- Threshold: $\chi^2_{3,0.05} = 7.81$
- H_0 is rejected at 5% level

Test of generic attributes

- Generic specification = restrictions that coefficients are equal across alternatives.
- Likelihood ratio test is appropriate

$$-2(\mathcal{L}(\widehat{\beta}_G) - \mathcal{L}(\widehat{\beta}_{AS})) \sim \chi^2_{K_{AS}-K_G}$$

Alternative specific elapsed time coefficients

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-1.17	0.278	-4.19	0.00
2	One stop–multiple airlines dummy	-1.45	0.292	-4.98	0.00
3	**Elapsed time (hours) (non stop)**	-0.341	0.0854	-3.99	0.00
4	**Elapsed time (hours) (one stop–same airline)**	-0.291	0.0822	-3.54	0.00
5	**Elapsed time (hours) (one stop–multiple airlines)**	-0.310	0.0802	-3.87	0.00
6	Round trip fare ($100)	-1.78	0.151	-11.84	0.00
7	Leg room (inches), if male	0.108	0.0232	4.65	0.00
8	Leg room (inches), if female	0.132	0.0221	5.99	0.00
9	Being early (hours)	-0.151	0.0188	-8.02	0.00
10	Being late (hours)	-0.0960	0.0167	-5.73	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.307	0.141	-2.18	0.03
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0910	0.157	-0.58	0.56
13	Male dummy (one stop–same airline)	0.199	0.126	1.59	0.11
14	Male dummy (one stop–multiple airlines)	0.293	0.132	2.21	0.03
15	Round trip fare / income ($100/$1000)	-24.0	8.09	-2.97	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\widehat{\beta}) = -1641.932$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] = 2305.875$$
$$\rho^2 = 0.413$$
$$\bar{\rho}^2 = 0.407$$

Generic elapsed time coefficients

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.964	0.216	-4.47	0.00
2	One stop–multiple airlines dummy	-1.36	0.224	-6.09	0.00
3	**Elapsed time (hours)**	-0.301	0.0778	-3.87	0.00
4	Round trip fare ($100)	-1.80	0.150	-11.97	0.00
5	Leg room (inches), if female	0.132	0.0220	6.00	0.00
6	Leg room (inches), if male	0.107	0.0232	4.62	0.00
7	Being early (hours)	-0.151	0.0188	-8.04	0.00
8	Being late (hours)	-0.0958	0.0167	-5.74	0.00
9	More than 2 air trips per year (one stop–same airline)	-0.309	0.141	-2.20	0.03
10	More than 2 air trips per year (one stop–multiple airlines)	-0.0931	0.157	-0.59	0.55
11	Male dummy (one stop–same airline)	0.201	0.125	1.60	0.11
12	Male dummy (one stop–multiple airlines)	0.294	0.132	2.23	0.03
13	Round trip fare / income ($100/$1000)	-24.1	8.07	-2.98	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\widehat{\beta}) = -1642.796$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] = 2304.148$$
$$\rho^2 = 0.412$$
$$\bar{\rho}^2 = 0.408$$

Test of generic attributes

- Alternative specific model: $\mathcal{L}(\hat{\beta}) = -1641.932$, 15 parameters
- Generic model: $\mathcal{L}(\hat{\beta}) = -1642.796$, 13 parameters
- Test: $-2(-1642.796 + 1641.932) = 1.728$
- Threshold: $\chi^2_{2,0.05} = 5.99$
- H_0 cannot be rejected at 5% level.

Notes

- Same conclusion as using the t-test.
- It is not always necessarily the case.

Test of taste variations

Segmentation

- Classify the data into G groups. Size of group g: N_g.
- The same specification is considered for each group.
- A different set of parameters is estimated for each group.
- Restrictions:

$$\beta^1 = \beta^2 = \ldots = \beta^G$$

  where β^g is the vector of coefficients of market segment g.
- Statistic:

$$-2\left[\mathcal{L}_N(\widehat{\beta}) - \sum_{g=1}^{G} \mathcal{L}_{N_g}(\widehat{\beta}^g)\right]$$

- χ^2 with $\sum_{g=1}^{G} K_g - K$ degrees of freedom.
- In general, $\sum_{g=1}^{G} K_g - K = (G-1)K$.

Example: segment by trip purpose

Sample

- Full data set: 3609 observations
- Leisure trips: 2544 observations
- Non leisure trips: 1065 observations

Hypothesis

H_0: the true parameters are the same for leisure and non leisure trips.

Base specification with full data set

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.942	0.190	-4.95	0.00
2	One stop–multiple airlines dummy	-1.29	0.198	-6.53	0.00
3	Round trip fare ($100)	-1.60	0.124	-12.83	0.00
4	Elapsed time (hours)	-0.299	0.0672	-4.45	0.00
5	Leg room (inches), if male (non stop)	0.108	0.0268	4.03	0.00
6	Leg room (inches), if female (non stop)	0.141	0.0272	5.18	0.00
7	Leg room (inches), if male (one stop)	0.125	0.0250	4.99	0.00
8	Leg room (inches), if female (one stop)	0.0850	0.0233	3.64	0.00
9	Being early (hours)	-0.140	0.0162	-8.64	0.00
10	Being late (hours)	-0.105	0.0138	-7.61	0.00
11	More than 2 air trips per year (one stop–same airline)	0.0263	0.114	0.23	0.82
12	More than 2 air trips per year (one stop–multiple airlines)	0.0144	0.123	0.12	0.91
13	Male dummy (one stop–same airline)	0.100	0.133	0.75	0.45
14	Male dummy (one stop–multiple airlines)	0.189	0.144	1.31	0.19
15	Round trip fare / income ($100/$1000)	-24.8	7.57	-3.27	0.00

Summary statistics

Number of observations = 3609

$$\mathcal{L}(0) = -3964.892$$
$$\mathcal{L}(\widehat{\beta}) = -2300.453$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] = 3328.878$$
$$\rho^2 = 0.420$$
$$\bar{\rho}^2 = 0.416$$

Estimation results by trip purpose

Parameter number	Description	Coefficient estimate (Rob. asympt. std error) Leisure	Non Leisure
1	One stop–same airline dummy	-0.879	-1.37
		(-4.02)	(-3.36)
2	One stop–multiple airlines dummy	-1.27	-1.58
		(-5.60)	(-3.62)
3	Round trip fare ($100)	-1.81	-1.29
		(-11.99)	(-6.32)
4	Elapsed time (hours)	-0.303	-0.300
		(-3.90)	(-2.24)
5	Leg room (inches), if male (non stop)	0.100	0.110
		(3.04)	(2.38)
6	Leg room (inches), if female (non stop)	0.182	0.0212
		(5.71)	(0.39)
7	Leg room (inches), if male (one stop)	0.113	0.166
		(3.80)	(3.58)
8	Leg room (inches), if female (one stop)	0.0931	0.0661
		(3.41)	(1.37)
9	Being early (hours)	-0.151	-0.118
		(-7.99)	(-3.43)
10	Being late (hours)	-0.0975	-0.126
		(-5.83)	(-4.86)
11	More than 2 air trips per year (one stop–same airline)	-0.300	0.0308
		(-2.12)	(0.11)

Estimation results by trip purpose (ctd.)

Parameter number	Description	Coefficient estimate (Rob. asympt. std error) Leisure	Non Leisure
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0847	0.0611
		(-0.54)	(0.19)
13	Male dummy (one stop–same airline)	0.100	-0.0446
		(0.75)	(-0.19)
14	Male dummy (one stop–multiple airlines)	0.189	-0.349
		(1.31)	(-1.39)
15	Round trip fare / income ($100/$1000)	-23.8	-17.6
		(-2.94)	(-1.24)
Summary statistics			
Number of observations by market segment (total: 3609)		2544	1065
$\mathcal{L}_{N_g}(\widehat{\beta})$		-1640.525	-629.08
$\mathcal{L}(0)$ = -3964.892			
$\sum_g \mathcal{L}_{N_g}(\widehat{\beta})$= -2269.605			
$-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})]$= 3390.574			
ρ^2 = 0.428			
$\bar{\rho}^2$ = 0.420			

H_0: there is no taste variation across trip purpose

Estimation results

Model	$\mathcal{L}(\widehat{\beta})$	Sample size	K
Restricted	-2300.453	3609	15
Leisure	-1640.525	2544	15
Non leisure	-629.080	1065	15
Unrestricted	-2269.605	3609	30

Likelihood ratio test

$$-2\left[\mathcal{L}_N(\widehat{\beta}) - \sum_{g=1}^{G} \mathcal{L}_{N_g}(\widehat{\beta}^g)\right] = -2(-2300.453 + 2269.605) = 61.696.$$

$$\chi^2_{15,0.05} = 25.00.$$

The hypothesis is rejected.

Outline

1 Introduction
2 Case study
3 Informal tests
4 t-test
5 Wald test
- Linear restrictions
- Nonlinear restrictions

6 Likelihood ratio test
- Test of generic attributes
- Test of taste variations

7 Tests of Nonlinear Specifications
- Piecewise linear
- Power series
- Box-Cox

8 Non nested hypotheses
- Cox test
- Davidson and McKinnon J-test
- Adjusted likelihood ratio index

9 Outlier analysis
10 Market segments
11 Conclusions
12 Appendix

Tests of Nonlinear Specifications

- Consider a variable x of the model (elapsed time, say)
- Unrestricted model: V is a nonlinear function of x
- Restricted model: V is a linear function of x
- We consider the following nonlinear specifications:
  - Piecewise linear
  - Power series
  - Box-Cox transforms
- For each of them, the linear specification is obtained using simple restrictions on the nonlinear specification

Piecewise linear specification

Model

- Partition the range of values of x into M intervals $[a_m, a_{m+1}]$, $m = 1, \ldots, M$
- For example, the partition [0–2], [2–4], [4–8], [8–] corresponds to

$$M = 4, a_1 = 0, a_2 = 2, a_3 = 4, a_4 = 8, a_5 = +\infty$$

- The slope of the utility function may vary across intervals
- Therefore, there will be M parameters instead of 1
- The function must be continuous

Piecewise linear specification

Specifications

- Linear specification:

$$V_i = \beta x_i + \cdots$$

- Piecewise linear specification

$$V_i = \sum_{m=1}^{M} \beta_m x_{im} + \cdots$$

where

$$x_{im} = \max(0, \min(x - a_m, a_{m+1} - a_m))$$

that is

$$x_{im} = \begin{cases} 0 & \text{if } x < a_m \\ x - a_m & \text{if } a_m \leq x < a_{m+1} \\ a_{m+1} - a_m & \text{if } a_{m+1} \leq x \end{cases}$$

Piecewise linear specification

Example: $M = 4, a_1 = 0, a_2 = 2, a_3 = 4, a_4 = 8, a_5 = +\infty$

x	x_1	x_2	x_3	x_4
1	1	0	0	0
3	2	1	0	0
7	2	2	3	0
11	2	2	4	3

Estimation results: piecewise linear specification

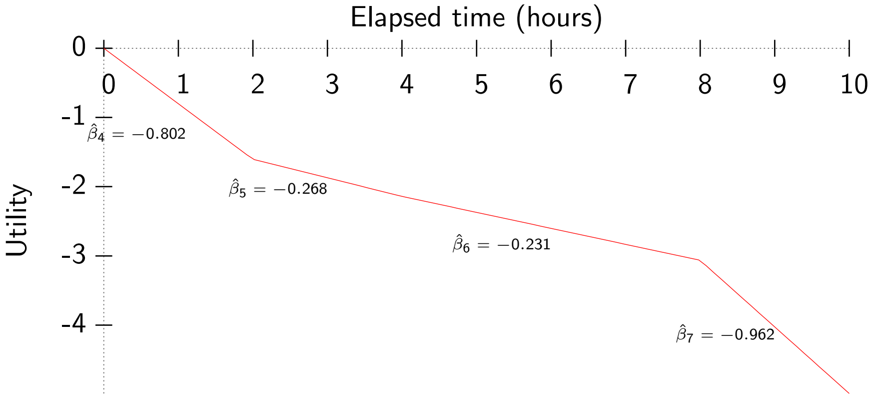
Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.933	0.225	-4.14	0.00
2	One stop–multiple airlines dummy	-1.32	0.232	-5.71	0.00
3	Round trip fare ($100)	-1.80	0.153	-11.82	0.00
4	**Elapsed time (0 - 2 hours)**	-0.802	0.241	-3.32	0.00
5	**Elapsed time (2 - 4 hours)**	-0.268	0.100	-2.67	0.01
6	**Elapsed time (4 - 8 hours)**	-0.231	0.0834	-2.77	0.01
7	**Elapsed time ($>$ 8 hours)**	-0.962	0.319	-3.02	0.00
8	Leg room (inches), if male (non stop)	0.104	0.0331	3.13	0.00
9	Leg room (inches), if female (non stop)	0.185	0.0320	5.79	0.00
10	Leg room (inches), if male (one stop)	0.118	0.0297	3.98	0.00
11	Leg room (inches), if female (one stop)	0.0939	0.0274	3.42	0.00
12	Being early (hours)	-0.150	0.0190	-7.87	0.00
13	Being late (hours)	-0.0988	0.0167	-5.90	0.00
14	More than 2 air trips per year (one stop–same airline)	-0.283	0.141	-2.00	0.05
15	More than 2 air trips per year (one stop–multiple airlines)	-0.0791	0.158	-0.50	0.62
16	Male dummy (one stop–same airline)	0.0838	0.134	0.63	0.53
17	Male dummy (one stop–multiple airlines)	0.181	0.144	1.26	0.21
18	Round trip fare / income ($100/$1000)	-23.1	8.17	-2.82	0.00

Summary statistics

Number of observations = 2544

$$\begin{aligned}
\mathcal{L}(0) &= -2794.870 \\
\mathcal{L}(c) &= -2203.160 \\
\mathcal{L}(\widehat{\beta}) &= -1634.131 \\
-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] &= 2321.478 \\
\rho^2 &= 0.415 \\
\bar{\rho}^2 &= 0.409
\end{aligned}$$

Piecewise linear specification



Estimation results: linear specification

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.879	0.219	-4.02	0.00
2	One stop–multiple airlines dummy	-1.27	0.227	-5.60	0.00
3	Round trip fare ($100)	-1.81	0.151	-11.99	0.00
4	**Elapsed time (hours)**	-0.303	0.0778	-3.90	0.00
5	Leg room (inches), if male (non stop)	0.100	0.0330	3.04	0.00
6	Leg room (inches), if female (non stop)	0.182	0.0318	5.71	0.00
7	Leg room (inches), if male (one stop)	0.113	0.0297	3.80	0.00
8	Leg room (inches), if female (one stop)	0.0931	0.0273	3.41	0.00
9	Being early (hours)	-0.151	0.0189	-7.99	0.00
10	Being late (hours)	-0.0975	0.0167	-5.83	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.300	0.141	-2.12	0.03
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0847	0.157	-0.54	0.59
13	Male dummy (one stop–same airline)	0.100	0.133	0.75	0.45
14	Male dummy (one stop–multiple airlines)	0.189	0.144	1.31	0.19
15	Round trip fare / income ($100/$1000)	-23.8	8.09	-2.94	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\hat{\beta}) = -1640.525$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 2308.689$$
$$\rho^2 = 0.413$$
$$\bar{\rho}^2 = 0.408$$

H_0: the linear specification is the correct model

Tested restrictions

$$\beta_4 = \beta_5 = \beta_6 = \beta_7$$

Statistic

$$-2(-1640.525 - (-1634.131)) = 12.788$$

Threshold

$$\chi^2_{3,0.05} = 7.81$$

The linear specification is rejected

Power series

- Idea: if the utility function is nonlinear in x, it can be approximated by a polynomial of degree M
- Linear specification:

$$V_i = \beta x_i + \cdots$$

- Power series

$$V_i = \sum_{m=1}^{M} \beta_m x_i^m + \cdots$$

Estimation results: power series specification

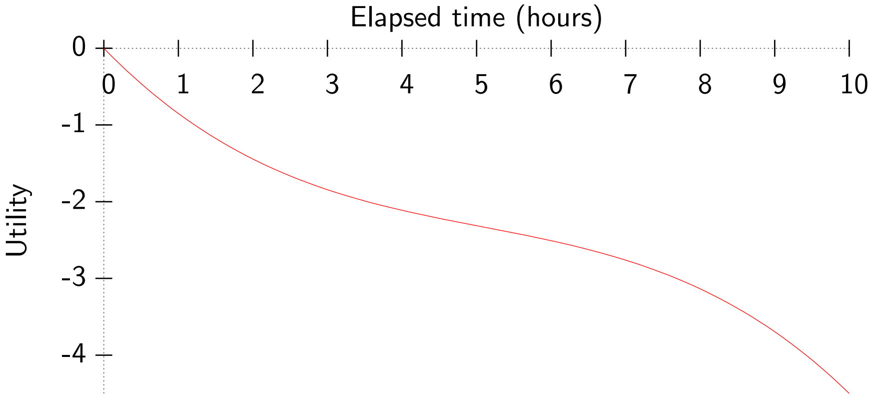
Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.912	0.224	-4.08	0.00
2	One stop–multiple airlines dummy	-1.30	0.230	-5.64	0.00
3	Round trip fare ($100)	-1.80	0.153	-11.80	0.00
4	**Elapsed time (hours)**	-1.00	0.235	-4.27	0.00
5	**Elapsed time^2 (hours^2)**	0.160	0.0507	3.14	0.00
6	**Elapsed time^3 (hours^3)**	-0.0105	0.00347	-3.03	0.00
7	Leg room (inches), if male (non stop)	0.104	0.0332	3.14	0.00
8	Leg room (inches), if female (non stop)	0.185	0.0320	5.78	0.00
9	Leg room (inches), if male (one stop)	0.118	0.0298	3.94	0.00
10	Leg room (inches), if female (one stop)	0.0932	0.0274	3.40	0.00
11	Being early (hours)	-0.150	0.0191	-7.88	0.00
12	Being late (hours)	-0.0986	0.0167	-5.90	0.00
13	More than 2 air trips per year (one stop–same airline)	-0.279	0.142	-1.97	0.05
14	More than 2 air trips per year (one stop–multiple airlines)	-0.0727	0.157	-0.46	0.64
15	Male dummy (one stop–same airline)	0.0879	0.134	0.66	0.51
16	Male dummy (one stop–multiple airlines)	0.184	0.144	1.27	0.20
17	Round trip fare / income ($100/$1000)	-23.2	8.22	-2.82	0.00

Summary statistics

Number of observations = 2544

$$\begin{aligned}
\mathcal{L}(0) &= -2794.870 \\
\mathcal{L}(c) &= -2203.160 \\
\mathcal{L}(\widehat{\beta}) &= -1635.347 \\
-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] &= 2319.046 \\
\rho^2 &= 0.415 \\
\bar{\rho}^2 &= 0.409
\end{aligned}$$

Power series: $M=3$



Estimation results: linear specification

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.879	0.219	-4.02	0.00
2	One stop–multiple airlines dummy	-1.27	0.227	-5.60	0.00
3	Round trip fare ($100)	-1.81	0.151	-11.99	0.00
4	**Elapsed time (hours)**	-0.303	0.0778	-3.90	0.00
5	Leg room (inches), if male (non stop)	0.100	0.0330	3.04	0.00
6	Leg room (inches), if female (non stop)	0.182	0.0318	5.71	0.00
7	Leg room (inches), if male (one stop)	0.113	0.0297	3.80	0.00
8	Leg room (inches), if female (one stop)	0.0931	0.0273	3.41	0.00
9	Being early (hours)	-0.151	0.0189	-7.99	0.00
10	Being late (hours)	-0.0975	0.0167	-5.83	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.300	0.141	-2.12	0.03
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0847	0.157	-0.54	0.59
13	Male dummy (one stop–same airline)	0.100	0.133	0.75	0.45
14	Male dummy (one stop–multiple airlines)	0.189	0.144	1.31	0.19
15	Round trip fare / income ($100/$1000)	-23.8	8.09	-2.94	0.00

Summary statistics

Number of observations = 2544

$$\begin{aligned}
\mathcal{L}(0) &= -2794.870 \\
\mathcal{L}(c) &= -2203.160 \\
\mathcal{L}(\hat{\beta}) &= -1640.525 \\
-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] &= 2308.689 \\
\rho^2 &= 0.413 \\
\bar{\rho}^2 &= 0.408
\end{aligned}$$

H_0: the linear specification is the correct model

Tested restrictions

$$\beta_5 = \beta_6 = 0$$

Statistic

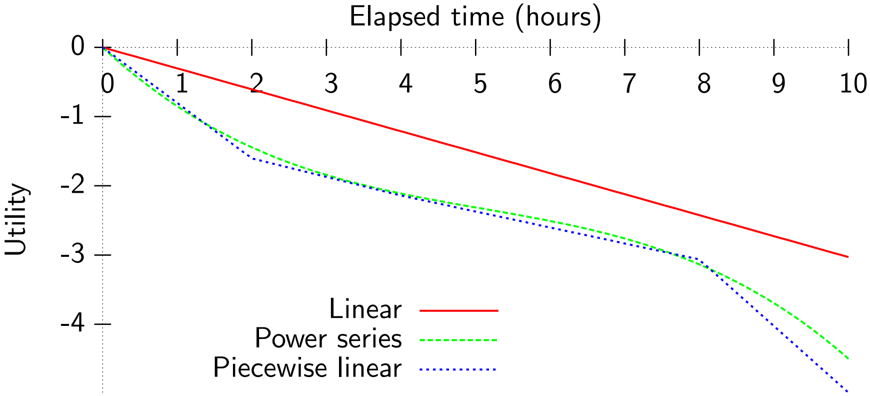
$$-2(-1640.525 - (-1635.347)) = 10.356$$

Threshold

$$\chi^2_{2,0.05} = 5.99$$

The linear specification is rejected

Comparing the specifications



Box-Cox transform

Definition

- Let $x > 0$ be a positive variable
- Its Box-Cox transform is defined as

$$B(x, \lambda) = \begin{cases} \dfrac{x^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln x & \text{if } \lambda = 0. \end{cases}$$

where $\lambda \in \mathbb{R}$ is a parameter.

Continuity

$$\lim_{\lambda \to 0} \frac{x^\lambda - 1}{\lambda} = \ln x.$$

Box-Cox transform

Linear specification

$$V_i = \beta x_i + \cdots$$

Box-Cox specification

$$V_i = \beta B(x, \lambda) + \cdots$$

Properties

- Convex if $\lambda > 1$
- Linear if $\lambda = 1$
- Concave if $\lambda < 1$

Box-Cox transform

Estimation

- λ is estimated from data
- Utility function not linear-in-parameters

Testing the linear specification

- Restriction: $\lambda = 1$.
- Likelihood ratio test
- *t*-test can also be used

Estimation results: Box-Cox specification

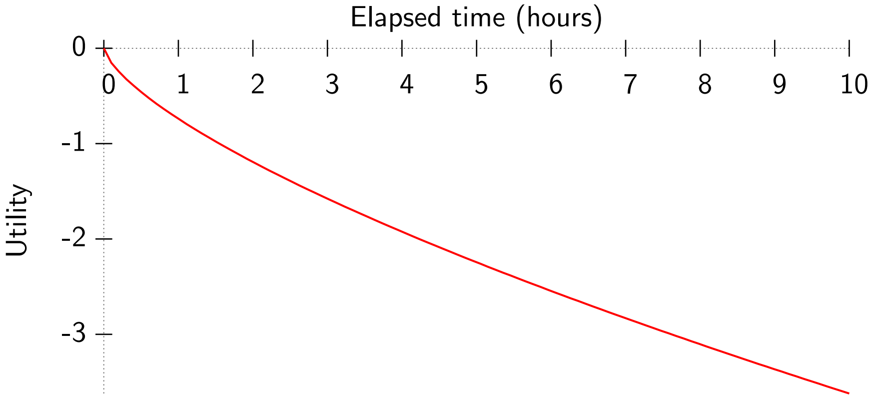
Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.832	0.224	-3.72	0.00
2	One stop–multiple airlines dummy	-1.23	0.231	-5.31	0.00
3	Round trip fare ($100)	-1.79	0.151	-11.79	0.00
4	Elapsed time (hours)	-0.510	0.174	-2.93	0.00
5	Leg room (inches), if male (non stop)	0.101	0.0331	3.06	0.00
6	Leg room (inches), if female (non stop)	0.181	0.0319	5.69	0.00
7	Leg room (inches), if male (one stop)	0.114	0.0297	3.84	0.00
8	Leg room (inches), if female (one stop)	0.0948	0.0275	3.45	0.00
9	Being early (hours)	-0.151	0.0190	-7.95	0.00
10	Being late (hours)	-0.0977	0.0168	-5.82	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.295	0.141	-2.09	0.04
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0790	0.157	-0.50	0.62
13	Male dummy (one stop–same airline)	0.0993	0.133	0.74	0.46
14	Male dummy (one stop–multiple airlines)	0.188	0.144	1.31	0.19
15	Round trip fare / income ($100/$1000)	-23.7	8.10	-2.92	0.00
16	**Box-Cox Elapsed time (hours): λ**	0.690	0.213	3.24	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\widehat{\beta}) = -1639.317$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] = 2311.106$$
$$\rho^2 = 0.413$$
$$\bar{\rho}^2 = 0.408$$

Box-Cox transform



H_0: the linear specification is the correct model

t-test

- $\lambda = 0.690$
- Robust asymptotic standard error $= 0.213$
- $H_0 : \lambda = 1$
- Test:

$$\frac{0.690 - 1}{0.213} = -1.46$$

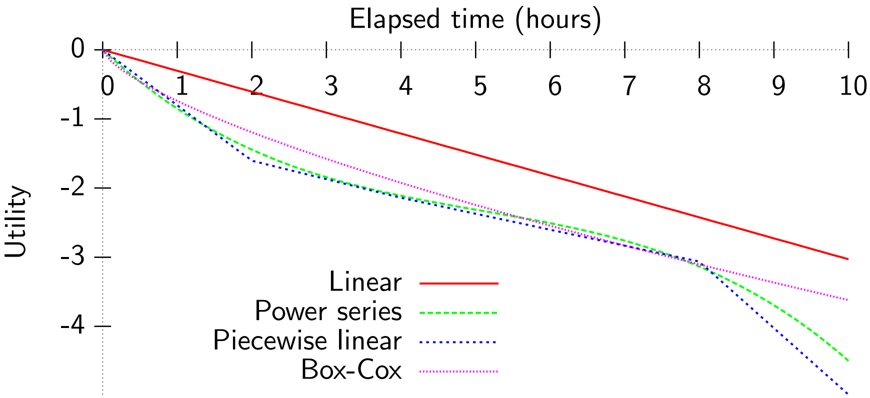
- The hypothesis cannot be rejected at the 5% level.

H_0: the linear specification is the correct model

Likelihood ratio test

- Unrestricted model: -1639.317
- Restricted model: -1640.525
- Test: $-2(-1640.525 + 1639.317) = 2.416$
- Threshold: $\chi^2_{1,0.05} = 3.84$
- The hypothesis cannot be rejected at the 5% level.

Comparing the specifications



Outline

1. Introduction
2. Case study
3. Informal tests
4. t-test
5. Wald test
   - Linear restrictions
   - Nonlinear restrictions
6. Likelihood ratio test
   - Test of generic attributes
   - Test of taste variations
7. Tests of Nonlinear Specifications
   - Piecewise linear
   - Power series
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8. Non nested hypotheses
   - Cox test
   - Davidson and McKinnon J-test
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Non nested hypotheses

Nested hypotheses

- Restricted and unrestricted models
- Linear restrictions
- H_0: restricted model is correct
- Test: likelihood ratio test

Non nested hypotheses

- Need to compare two models
- None of them is a restriction of the other
- Likelihood ratio test cannot be used

Model 1

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.879	0.219	-4.02	0.00
2	One stop–multiple airlines dummy	-1.27	0.227	-5.60	0.00
3	Round trip fare ($100)	-1.81	0.151	-11.99	0.00
4	Elapsed time (hours)	-0.303	0.0778	-3.90	0.00
5	Leg room (inches), if male (non stop)	0.100	0.0330	3.04	0.00
6	Leg room (inches), if female (non stop)	0.182	0.0318	5.71	0.00
7	Leg room (inches), if male (one stop)	0.113	0.0297	3.80	0.00
8	Leg room (inches), if female (one stop)	0.0931	0.0273	3.41	0.00
9	**Being early (hours)**	-0.151	0.0189	-7.99	0.00
10	**Being late (hours)**	-0.0975	0.0167	-5.83	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.300	0.141	-2.12	0.03
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0847	0.157	-0.54	0.59
13	Male dummy (one stop–same airline)	0.100	0.133	0.75	0.45
14	Male dummy (one stop–multiple airlines)	0.189	0.144	1.31	0.19
15	Round trip fare / income ($100/$1000)	-23.8	8.09	-2.94	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\hat{\beta}) = -1640.525$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 2308.689$$
$$\rho^2 = 0.413$$
$$\bar{\rho}^2 = 0.408$$

Model 2

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.857	0.219	-3.91	0.00
2	One stop–multiple airlines dummy	-1.26	0.228	-5.52	0.00
3	Round trip fare ($100)	-1.79	0.150	-11.97	0.00
4	Elapsed time (hours)	-0.309	0.0780	-3.96	0.00
5	Leg room (inches), if male (non stop)	0.0967	0.0328	2.95	0.00
6	Leg room (inches), if female (non stop)	0.181	0.0315	5.74	0.00
7	Leg room (inches), if male (one stop)	0.113	0.0297	3.82	0.00
8	Leg room (inches), if male (one stop)	0.0918	0.0272	3.37	0.00
9	**Being early^2 (hours^2)**	-0.0111	0.00169	-6.58	0.00
10	**Being late^2 (hours^2)**	-0.00731	0.00166	-4.39	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.300	0.141	-2.12	0.03
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0809	0.157	-0.52	0.61
13	Male dummy (one stop–same airline)	0.114	0.133	0.86	0.39
14	Male dummy (one stop–multiple airlines)	0.194	0.143	1.36	0.18
15	Round trip fare / income ($100/$1000)	-23.8	8.12	-2.93	0.00

Summary statistics

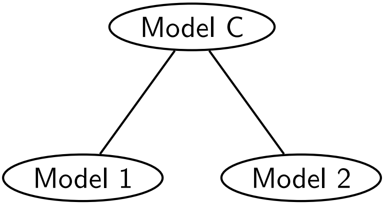
Number of observations = 2544

$$\begin{aligned}
\mathcal{L}(0) &= -2794.870 \\
\mathcal{L}(c) &= -2203.160 \\
\mathcal{L}(\widehat{\beta}) &= -1649.407 \\
-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] &= 2290.925 \\
\rho^2 &= 0.410 \\
\bar{\rho}^2 &= 0.404
\end{aligned}$$

Cox test

Back to nested hypotheses

- We want to test model 1 against model 2
- We generate a composite model C such that both models 1 and 2 are restricted cases of model C.



Cox test

Testing

- We test 1 against C using the likelihood ratio test
- We test 2 against C using the likelihood ratio test
- Possible outcomes:
  - Only one of the two models is rejected. Keep the other.
  - Both models are rejected. Better models should be developed.
  - Both models are accepted. Use another test.

Cox test

Models

- $M_1 : U_{in} = \cdots + \beta x_{in} + \cdots + \varepsilon_{in}^{(1)}$
- $M_2 : U_{in} = \cdots + \theta x_{in}^2 + \cdots + \varepsilon_{in}^{(2)}$
- $M_C : U_{in} = \cdots + \beta x_{in} + \theta x_{in}^2 + \cdots + \varepsilon_{in}.$

Testing M_1 against M_C

Restrictions: $\theta = 0$

Testing M_2 against M_C

Restrictions: $\beta = 0$

Cox test: illustration

Estimation results

	Model	$\mathcal{L}(\widehat{\beta})$	K
M_1	Linear specification	-1640.525	15
M_2	Quadratic specification	-1649.407	15
M_C	Composite	-1640.487	17

Tests

	Statistic	Threshold	Outcome
M_1 vs M_C	0.076	5.99	Cannot reject M_1
M_2 vs M_C	17.84	5.99	Reject M_2

Davidson and McKinnon J-test

- Model 1: $U_n^{(1)} = V_n^{(1)}(x_n^{(1)};\beta) + \varepsilon_n^{(1)}$
- Model 2: $U_n^{(2)} = V_n^{(2)}(x_n^{(2)};\gamma) + \varepsilon_n^{(2)}$
- Hypothesis H_0: model 1 is correct.
- Procedure:
  1. Estimate model 2 and obtain $\widehat{\gamma}$.
  2. Consider the composite model

$$U_n^{(1)} = (1-\alpha)V_n^{(1)}(x_n^{(1)};\beta) + \alpha V_n^{(2)}(x_n^{(2)};\widehat{\gamma}) + \varepsilon_n.$$

  3. Estimate β and α.
  4. Under H_0, we have $\alpha = 0$.
  5. It can be tested with a t-test.

Linear specification

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.879	0.219	-4.02	0.00
2	One stop–multiple airlines dummy	-1.27	0.227	-5.60	0.00
3	Round trip fare ($100)	-1.81	0.151	-11.99	0.00
4	Elapsed time (hours)	-0.303	0.0778	-3.90	0.00
5	Leg room (inches), if male (non stop)	0.100	0.0330	3.04	0.00
6	Leg room (inches), if female (non stop)	0.182	0.0318	5.71	0.00
7	Leg room (inches), if male (one stop)	0.113	0.0297	3.80	0.00
8	Leg room (inches), if female (one stop)	0.0931	0.0273	3.41	0.00
9	**Being early (hours)**	-0.151	0.0189	-7.99	0.00
10	**Being late (hours)**	-0.0975	0.0167	-5.83	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.300	0.141	-2.12	0.03
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0847	0.157	-0.54	0.59
13	Male dummy (one stop–same airline)	0.100	0.133	0.75	0.45
14	Male dummy (one stop–multiple airlines)	0.189	0.144	1.31	0.19
15	Round trip fare / income ($100/$1000)	-23.8	8.09	-2.94	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\hat{\beta}) = -1640.525$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\hat{\beta})] = 2308.689$$
$$\rho^2 = 0.413$$
$$\bar{\rho}^2 = 0.408$$

Quadratic specification

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	*t*-stat	*p*-value
1	One stop–same airline dummy	-0.857	0.219	-3.91	0.00
2	One stop–multiple airlines dummy	-1.26	0.228	-5.52	0.00
3	Round trip fare ($100)	-1.79	0.150	-11.97	0.00
4	Elapsed time (hours)	-0.309	0.0780	-3.96	0.00
5	Leg room (inches), if male (non stop)	0.0967	0.0328	2.95	0.00
6	Leg room (inches), if female (non stop)	0.181	0.0315	5.74	0.00
7	Leg room (inches), if male (one stop)	0.113	0.0297	3.82	0.00
8	Leg room (inches), if male (one stop)	0.0918	0.0272	3.37	0.00
9	**Being early^2 (hours^2)**	-0.0111	0.00169	-6.58	0.00
10	**Being late^2 (hours^2)**	-0.00731	0.00166	-4.39	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.300	0.141	-2.12	0.03
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0809	0.157	-0.52	0.61
13	Male dummy (one stop–same airline)	0.114	0.133	0.86	0.39
14	Male dummy (one stop–multiple airlines)	0.194	0.143	1.36	0.18
15	Round trip fare / income ($100/$1000)	-23.8	8.12	-2.93	0.00

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\widehat{\beta}) = -1649.407$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] = 2290.925$$
$$\rho^2 = 0.410$$
$$\bar{\rho}^2 = 0.404$$

Testing the linear specification

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.878	0.205	-4.29	0.00
2	One stop–multiple airlines dummy	-1.27	0.213	-5.98	0.00
3	Round trip fare ($100)	-1.81	0.141	-12.82	0.00
4	Elapsed time (hours)	-0.304	0.0728	-4.17	0.00
5	Leg room (inches), if male (non stop)	0.100	0.0308	3.25	0.00
6	Leg room (inches), if female (non stop)	0.182	0.0298	6.10	0.00
7	Leg room (inches), if male (one stop)	0.113	0.0278	4.07	0.00
8	Leg room (inches), if female (one stop)	0.0930	0.0256	3.64	0.00
9	**Being early (hours)**	-0.149	0.0189	-7.88	0.00
10	**Being late (hours)**	-0.0964	0.0163	-5.93	0.00
11	More than 2 air trips per year (one stop–same airline)	-0.300	0.132	-2.27	0.02
12	More than 2 air trips per year (one stop–multiple airlines)	-0.0849	0.147	-0.58	0.56
13	Male dummy (one stop–same airline)	0.100	0.125	0.81	0.42
14	Male dummy (one stop–multiple airlines)	0.190	0.135	1.41	0.16
15	Round trip fare / income ($100/$1000)	-23.8	7.57	-3.14	0.00
16	α	-0.0698	0.301	-0.23	0.82

Summary statistics

Number of observations = 2544

$$\mathcal{L}(0) = -2794.870$$
$$\mathcal{L}(c) = -2203.160$$
$$\mathcal{L}(\widehat{\beta}) = -1640.493$$
$$-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] = 2308.754$$
$$\rho^2 = 0.413$$
$$\bar{\rho}^2 = 0.407$$

H_0: the linear specification is correct

Test

- Under H_0, $\alpha = 0$.
- t-test: -0.23
- Linear model cannot be rejected.

Testing the quadratic specification

Parameter number	Description	Coeff. estimate	Robust Asympt. std. error	t-stat	p-value
1	One stop–same airline dummy	-0.868	3.77	-0.23	0.82
2	One stop–multiple airlines dummy	-1.27	3.92	-0.32	0.75
3	Round trip fare ($100)	-1.80	2.60	-0.69	0.49
4	Elapsed time (hours)	-0.301	1.34	-0.22	0.82
5	Leg room (inches), if male (non stop)	0.0972	0.569	0.17	0.86
6	Leg room (inches), if female (non stop)	0.184	0.550	0.34	0.74
7	Leg room (inches), if male (one stop)	0.115	0.513	0.22	0.82
8	Leg room (inches), if female (one stop)	0.0919	0.471	0.20	0.85
9	**Being early^2 (hours^2)**	-0.0126	0.0283	-0.44	0.66
10	**Being late^2 (hours^2)**	-0.00982	0.0294	-0.33	0.74
11	More than 2 air trips per year (one stop–same airline)	-0.303	2.43	-0.12	0.90
12	Male dummy (one stop–multiple airlines)	-0.0759	2.70	-0.03	0.98
13	Male dummy (one stop–same airline)	0.113	2.30	0.05	0.96
14	Male dummy (one stop–multiple airlines)	0.189	2.48	0.08	0.94
15	Round trip fare / income ($100/$1000)	-23.8	140.	-0.17	0.86
16	α	1.06	0.272	3.89	0.00

Summary statistics

Number of observations = 2544

$$
\begin{aligned}
\mathcal{L}(0) &= -2794.870 \\
\mathcal{L}(c) &= -2203.160 \\
\mathcal{L}(\widehat{\beta}) &= -1640.492 \\
-2[\mathcal{L}(0) - \mathcal{L}(\widehat{\beta})] &= 2308.756 \\
\rho^2 &= 0.413 \\
\bar{\rho}^2 &= 0.407
\end{aligned}
$$

H_0: the quadratic specification is correct

Test

- Under H_0, $\alpha = 0$.
- t-test: 3.89
- Quadratic model can be rejected.

Adjusted likelihood ratio index

Likelihood ratio index

$$\rho^2 = 1 - \frac{\mathcal{L}(\hat{\beta})}{\mathcal{L}(0)}$$

- $\rho^2 = 0$: trivial model, equal probabilities
- $\rho^2 = 1$: perfect fit.

Adjusted likelihood ratio index

- ρ^2 is increasing with the number of parameters.
- A higher fit (that is a higher ρ^2) does not mean a better model.
- An adjustment is needed.

$$\bar{\rho}^2 = 1 - \frac{\mathcal{L}(\hat{\beta}) - K}{\mathcal{L}(0)}$$

Test

Compare model M_1 and model M_2

- Null hypothesis: model M_1 is correct
- We expect that the best model corresponds to the largest $\bar{\rho}^2$.
- We will be wrong if M_1 is the true model and M_2 produces a better fit.
- What is the probability that this happens?

$$\Pr(\bar{\rho}_2^2 - \bar{\rho}_1^2 > z) \leq \Phi\{-\sqrt{-2z\mathcal{L}(0) + (K_1 - K_2)}\}, \qquad z > 0,$$

  where
  - $\bar{\rho}_\ell{}^2$ is the adjusted likelihood ratio index of model $\ell = 1, 2$
  - K_ℓ is the number of parameters of model ℓ
  - Φ is the standard normal CDF.
- If this probability is low, M_1 can be rejected.

Adjusted likelihood ratio index

Back to the example

	$\bar{\rho}^2$	# parameters
Model 1 (linear)	0.408	15
Model 2 (quadratic)	0.404	15

$$\begin{aligned}\Phi\{-\sqrt{2zN \ln J + (K_1 - K_2)}\} &= \Phi\{-\sqrt{2 \times 0.004 \times 2544 \times \ln 3}\} \\ &= \Phi(-4.73) \\ &= 0.00000113,\end{aligned}$$

Therefore, the linear specification is preferred.

Adjusted likelihood ratio index

In practice

- if the sample is large enough (i.e. more than 250 observations),
- if the models have the same number of parameters,
- if the values of the $\bar{\rho}^2$ differ by 0.01 or more,
- the model with the lower $\bar{\rho}^2$ is almost certainly incorrect.

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Outlier analysis

Procedure

- Apply the model on the sample
- Examine observations where the predicted probability is the smallest for the observed choice
- Test model sensitivity to outliers, as a small probability has a significant impact on the log likelihood
- Potential causes of low probability:
  - Coding or measurement error in the data
  - Model misspecification
  - Unexplainable variation in choice behavior

Outlier analysis

Coding or measurement error in the data

- Look for signs of data errors
- Correct or remove the observation

Model misspecification

- Seek clues of missing variables from the observation
- Keep the observation and improve the model

Unexplainable variation in choice behavior

- Keep the observation
- Avoid over fitting of the model to the data

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5. Wald test
   - Linear restrictions
   - Nonlinear restrictions
6. Likelihood ratio test
   - Test of generic attributes
   - Test of taste variations
7. Tests of Nonlinear Specifications
   - Piecewise linear
   - Power series
   - Box-Cox
8. Non nested hypotheses
   - Cox test
   - Davidson and McKinnon J-test
   - Adjusted likelihood ratio index
9. Outlier analysis
10. **Market segments**
11. Conclusions
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Market segments

Procedure

- Compare predicted vs. observed shares per segment
- Let N_g be the set of sampled individuals in segment g
- Observed share for alt. i and segment g

$$S_g(i) = \sum_{n \in N_g} y_{in} / N_g$$

- Predicted share for alt. i and segment g

$$\hat{S}_g(i) = \sum_{n \in N_g} P_n(i) / N_g$$

Market segments

Note

- With a full set of constants for segment g:

$$\sum_{n \in N_g} y_{in} = \sum_{n \in N_g} P_n(i)$$

- Do not saturate the model with constants

Outline

1. Introduction
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   - Test of generic attributes
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7. Tests of Nonlinear Specifications
   - Piecewise linear
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Conclusions

- Tests are designed to check meaningful hypotheses
- Do not test hypotheses that do not make sense
- Do not apply the tests blindly
- Always use your judgment.

90%, 95% and 99% of the χ^2 distribution with K degrees of freedom

K	90%	95%	99%	K	90%	95%	99%
1	2.706	3.841	6.635	21	29.615	32.671	38.932
2	4.605	5.991	9.210	22	30.813	33.924	40.289
3	6.251	7.815	11.345	23	32.007	35.172	41.638
4	7.779	9.488	13.277	24	33.196	36.415	42.980
5	9.236	11.070	15.086	25	34.382	37.652	44.314
6	10.645	12.592	16.812	26	35.563	38.885	45.642
7	12.017	14.067	18.475	27	36.741	40.113	46.963
8	13.362	15.507	20.090	28	37.916	41.337	48.278
9	14.684	16.919	21.666	29	39.087	42.557	49.588
10	15.987	18.307	23.209	30	40.256	43.773	50.892
11	17.275	19.675	24.725	31	41.422	44.985	52.191
12	18.549	21.026	26.217	32	42.585	46.194	53.486
13	19.812	22.362	27.688	33	43.745	47.400	54.776
14	21.064	23.685	29.141	34	44.903	48.602	56.061
15	22.307	24.996	30.578	35	46.059	49.802	57.342
16	23.542	26.296	32.000	36	47.212	50.998	58.619
17	24.769	27.587	33.409	37	48.363	52.192	59.893
18	25.989	28.869	34.805	38	49.513	53.384	61.162
19	27.204	30.144	36.191	39	50.660	54.572	62.428
20	28.412	31.410	37.566	40	51.805	55.758	63.691