Logit with multiple alternatives

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Outline

Random utility

For all $i \in \mathcal{C}_n$

$$U_{in} = V_{in} + \varepsilon_{in}$$

- What is $\mathcal{C}_n$?
- What is ε_{in}?
- What is V_{in}?

Outline

Choice set

Universal choice set

- All potential alternatives for the population
- Restricted to relevant alternatives

Mode choice

- driving alone
- sharing a ride
- taxi
- motorcycle
- bicycle
- walking
- transit bus
- rail rapid transit

Choice set

Individual's choice set

- No driver license
- No auto available
- Awareness of transit services
- Transit services unreachable
- Walking not an option for long distance

Mode choice

- ~~driving alone~~
- sharing a ride
- taxi
- motorcycle
- bicycle
- ~~walking~~
- ~~transit bus~~
- rail rapid transit

Choice set

Choice set generation is tricky

- How to model "awareness"?
- What does "long distance" exactly mean?
- What does "unreachable" exactly mean?

We assume here deterministic rules

- Car is available if n has a driver license and a car is available in the household
- Walking is available if trip length is shorter than 4km.

Outline

Error terms

Main assumption

ε_{in} are

- extreme value EV(0,μ),
- independent and
- identically distributed.

Comments

- Independence: across i and n.
- Identical distribution: same scale parameter μ across i and n.
- Scale must be normalized: $\mu = 1$.

Derivation of the logit model

Assumptions

- $\mathcal{C}_n = \{1, \ldots, J_n\}$
- $U_{in} = V_{in} + \varepsilon_{in}$
- $\varepsilon_{in} \sim \text{EV}(0, \mu)$
- ε_{in} i.i.d.

Choice model

$$P(i|\mathcal{C}_n) = \Pr(V_{in} + \varepsilon_{in} \geq \max_{j=1,\ldots,J_n} V_{jn} + \varepsilon_{jn})$$

Assume without loss of generality (wlog) that $i = 1$

$$P(1|\mathcal{C}_n) = P(V_{1n} + \varepsilon_{1n} \geq \max_{j=2,\ldots,J_n} V_{jn} + \varepsilon_{jn})$$

Derivation of the logit model

Composite alternative

- Define a composite alternative: "anything but alternative one"
- Associated utility:

$$U^* = \max_{j=2,\ldots,J_n} (V_{jn} + \varepsilon_{jn})$$

- From a property of the EV distribution

$$U^* \sim \text{EV}\left(\frac{1}{\mu} \ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}, \mu\right)$$

Derivation of the logit model

Composite alternative

From another property of the EV distribution

$$U^* = V^* + \varepsilon^*$$

where

$$V^* = \frac{1}{\mu} \ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}$$

and

$$\varepsilon^* \sim \text{EV}(0, \mu)$$

Derivation of the logit model

Binary choice

$$\begin{aligned} P(1|\mathcal{C}_n) &= P(V_{1n} + \varepsilon_{1n} \geq \max_{j=2,\ldots,J_n} V_{jn} + \varepsilon_{jn}) \\ &= P(V_{1n} + \varepsilon_{1n} \geq V^* + \varepsilon^*) \end{aligned}$$

ε_{1n} and ε^* are both EV(0,μ).

Binary logit

$$P(1|\mathcal{C}_n) = \frac{e^{\mu V_{1n}}}{e^{\mu V_{1n}} + e^{\mu V^*}}$$

where

$$V^* = \frac{1}{\mu} \ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}$$

Derivation of the logit model

We have

$$e^{\mu V^*} = e^{\ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}} = \sum_{j=2}^{J_n} e^{\mu V_{jn}}$$

and

$$\begin{aligned} P(1|\mathcal{C}_n) &= \frac{e^{\mu V_{1n}}}{e^{\mu V_{1n}} + e^{\mu V^*}} \\ &= \frac{e^{\mu V_{1n}}}{e^{\mu V_{1n}} + \sum_{j=2}^{J_n} e^{\mu V_{jn}}} \\ &= \frac{e^{\mu V_{1n}}}{\sum_{j=1}^{J_n} e^{\mu V_{jn}}} \end{aligned}$$

Scale parameter

- The scale parameter μ is not identifiable: $\mu = 1$.
- Warning: not identifiable $\neq$ not existing

$\mu \to 0$, that is variance goes to infinity

$$\lim_{\mu \to 0} P(i|C_n) = \frac{1}{J_n} \quad \forall i \in \mathcal{C}_n$$

Scale parameter

$\mu \to +\infty$, that is variance goes to zero

$$\begin{aligned}\lim_{\mu\to\infty} P(i|C_n) &= \lim_{\mu\to\infty} \frac{1}{1+\sum_{j\neq i} e^{\mu(V_{jn}-V_{in})}} \\ &= \begin{cases} 1 & \text{if } V_{in} > \max_{j\neq i} V_{jn} \\ 0 & \text{if } V_{in} < \max_{j\neq i} V_{jn} \end{cases}\end{aligned}$$

What if there are ties?

$V_{in} = \max_{j\in\mathcal{C}_n} V_{jn},\ i = 1,\ldots,J_n^*$

$$P(i|\mathcal{C}_n) = \frac{1}{J_n^*} \quad i = 1,\ldots,J_n^* \ \text{ and } P(i|\mathcal{C}_n) = 0 \quad i = J_n^*+1,\ldots,J_n$$

Outline

Systematic part of the utility function

$$V_{in} = V(z_{in}, S_n)$$

- z_{in} is a vector of attributes of alternative i for individual n
- S_n is a vector of socio-economic characteristics of n

Functional form: linear utility

Notation

$$x_{in} = (z_{in}, S_n)$$

Linear-in-parameters utility functions

$$V_{in} = V(z_{in}, S_n) = V(x_{in}) = \sum_k \beta_k (x_{in})_k$$

Not as restrictive as it may seem

Outline

Explanatory variables: alternatives attributes

Numerical and continuous

- $(z_{in})_k \in \mathbb{R}, \forall i, n, k$
- Associated with a specific unit

Examples

- Auto in-vehicle time (in min.)
- Transit in-vehicle time (in min.)
- Auto out-of-pocket cost (in cents)
- Transit fare (in cents)
- Walking time to the bus stop (in min.)

Straightforward modeling

Explanatory variables: alternatives attributes

- V_{in} is unitless
- Therefore, β depends on the unit of the associated attribute
- Example: consider two specifications

$$\begin{aligned} V_{in} &= \beta_1 \mathrm{TT}_{in} + \cdots \\ V_{in} &= \beta_1' \mathrm{TT}_{in}' + \cdots \end{aligned}$$

- If TT_{in} is a number of minutes, the unit of β_1 is 1/min
- If TT_{in}' is a number of hours, the unit of β_1' is 1/hour
- Both models are equivalent, but the estimated value of the coefficient will be different

$$\beta_1 \mathrm{TT}_{in} = \beta_1' \mathrm{TT}_{in}' \Longrightarrow \frac{\mathrm{TT}_{in}}{\mathrm{TT}_{in}'} = \frac{\beta_1'}{\beta_1} = 60$$

Explanatory variables: alternatives attributes

Generic and alternative specific parameters

$$\begin{aligned} V_{\text{auto}} &= \beta_1 \text{TT}_{\text{auto}} + \cdots \\ V_{\text{bus}} &= \beta_1 \text{TT}_{\text{bus}} + \cdots \end{aligned}$$

or

$$\begin{aligned} V_{\text{auto}} &= \beta_1 \text{TT}_{\text{auto}} + \cdots \\ V_{\text{bus}} &= \beta_2 \text{TT}_{\text{bus}} + \cdots \end{aligned}$$

Modeling assumption: a minute has/has not the same marginal utility whether it is incurred on the auto or bus mode

Explanatory variables: socio-eco. characteristics

Numerical and continuous

- $(S_n)_k \in \mathbb{R}, \forall n, k$
- Associated with a specific unit

Examples

- Annual income (in KCHF)
- Age (in years)

Warning: S_n do not depend on i

Explanatory variables: socio-eco. characteristics

They cannot appear in all utility functions

$$\left.\begin{array}{rcl} V_1 &=& \beta_1 x_{11} + \beta_2 \text{income} \\ V_2 &=& \beta_1 x_{21} + \beta_2 \text{income} \\ V_3 &=& \beta_1 x_{31} + \beta_2 \text{income} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} V_1' &=& \beta_1 x_{11} \\ V_2' &=& \beta_1 x_{21} \\ V_3' &=& \beta_1 x_{31} \end{array}\right.$$

In general: alternative specific characteristics

$$\begin{array}{rcll} V_1 &=& \beta_1 x_{11} + \beta_2 \text{income} & +\beta_4 \text{age} \\ V_2 &=& \beta_1 x_{21} + \beta_3 \text{income} & +\beta_5 \text{age} \\ V_3 &=& \beta_1 x_{31} & \end{array}$$

Outline

Discrete variables

Mainly used to capture qualitative attributes

- Level of comfort for the train
- Reliability of the bus
- Color
- Shape
- etc...

or characteristics

- Sex
- Education
- Professional status
- etc.

Discrete variables

Procedure for model specification

- Identify all possible levels of the attribute:
  - Very comfortable,
  - Comfortable,
  - Rather comfortable,
  - Not comfortable.
- Select a base case: very comfortable
- Define numerical attributes
- Adopt a coding convention

Discrete variables

Introduce a 0/1 attribute for all levels except the base case

- z_c for comfortable
- z_{rc} for rather comfortable
- z_{nc} for not comfortable

	z_c	z_{rc}	z_{nc}
very comfortable	0	0	0
comfortable	1	0	0
rather comfortable	0	1	0
not comfortable	0	0	1

If a qualitative attribute has n levels, we introduce $n-1$ variables (0/1) in the model

Comparing two ways of coding

Base: very comfortable

$$V_{in} = \cdots + 0 z_{ivc} + \beta_{\text{c}} z_{ic} + \beta_{\text{rc}} z_{irc} + \beta_{\text{nc}} z_{inc}$$

- β_{c}: difference of utility between comfortable and very comfortable (supposedly negative)
- β_{rc}: difference of utility between rather comfortable and very comfortable (supposedly more negative)
- β_{nc}: difference of utility between not comfortable and very comfortable (supposedly even more negative)

Comparing two ways of coding

Base: comfortable

$$V'_{in} = \cdots + \beta'_{\mathrm{vc}} z_{i\mathrm{vc}} + 0 z_{ic} + \beta'_{\mathrm{rc}} z_{i\mathrm{rc}} + \beta'_{\mathrm{nc}} z_{i\mathrm{nc}}$$

- β'_{vc}: difference of utility between very comfortable and comfortable (supposedly positive)
- β'_{rc}: difference of utility between rather comfortable and comfortable (supposedly negative)
- β'_{nc}: difference of utility between not comfortable and comfortable (supposedly more negative)

Discrete variables

Example of estimation with Biogeme

	Model 1	Model 2
ASC	0.574	0.574
BETA_VC	0.000	0.918
BETA_C	-0.919	0.000
BETA_RC	-1.015	-0.096
BETA_NC	-2.128	-1.210

Outline

Nonlinear transformations of the variables

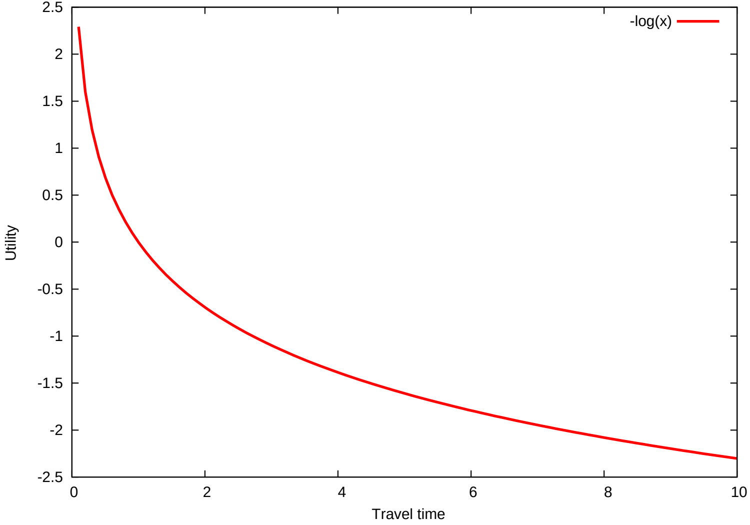
Example with travel time

- Compare a trip of 5 min with a trip of 10 min
- Compare a trip of 120 min with a trip of 125 min
- Utility difference: $\beta_T \times$ 5 min, in both cases.

Behavioral assumption

One more minute of travel is not perceived the same way for short trips as for long trips

Nonlinear transformations of the variables



Nonlinear transformations of the variables

Assumption 1: the marginal impact of travel time is constant

$$V_i = \beta_T \text{time}_i + \cdots$$

Assumption 2: the marginal impact of travel time decreases with travel time

$$V_i = \beta_T \ln(\text{time}_i) + \cdots$$

Remarks

- It is still a linear-in-parameters form
- The unit, the value, and the interpretation of β_T is different

Nonlinear transformations of the variables

Data can be preprocessed to account for nonlinearities

$$V_{in} = V(h(z_{in}, S_n)) = \sum_k \beta_k (h(z_{in}, S_n))_k$$

It is linear-in-parameter, even with h nonlinear.

Categories

Same assumption: sensitivity to travel time varies with travel time

- Log transform is not the only specification
- Another possibility: categories of trips
  - Short trips: 0–90 min.
  - Medium strips: 90–180 min.
  - Long trips: 180–270 min.
  - Very long trips: 270 min. and more

Specifications

- Categories with constants (inferior solution)
- Piecewise linear specification (spline)

Categories with constants

Same specification as for discrete variables

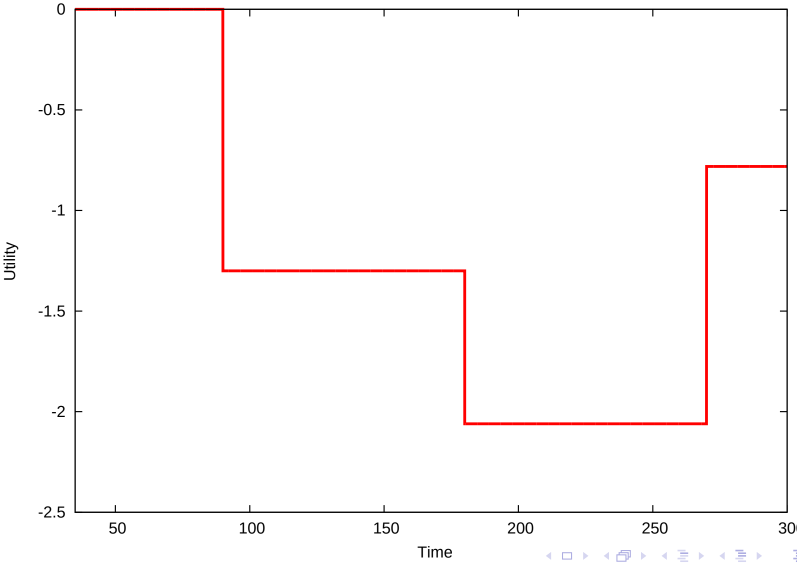
$$V_i = \beta_{T1} x_{T1} + \beta_{T2} x_{T2} + \beta_{T3} x_{T3} + \beta_{T4} x_{T4} + \ldots$$

with

- $x_{T1} = 1$ if $TT_i \in [0–90[$, 0 otherwise
- $x_{T2} = 1$ if $TT_i \in [90–180[$, 0 otherwise
- $x_{T3} = 1$ if $TT_i \in [180–270[$, 0 otherwise
- $x_{T4} = 1$ if $TT_i \in [270–[$, 0 otherwise

One β must be normalized to 0.

Categories with constants



Categories with constants

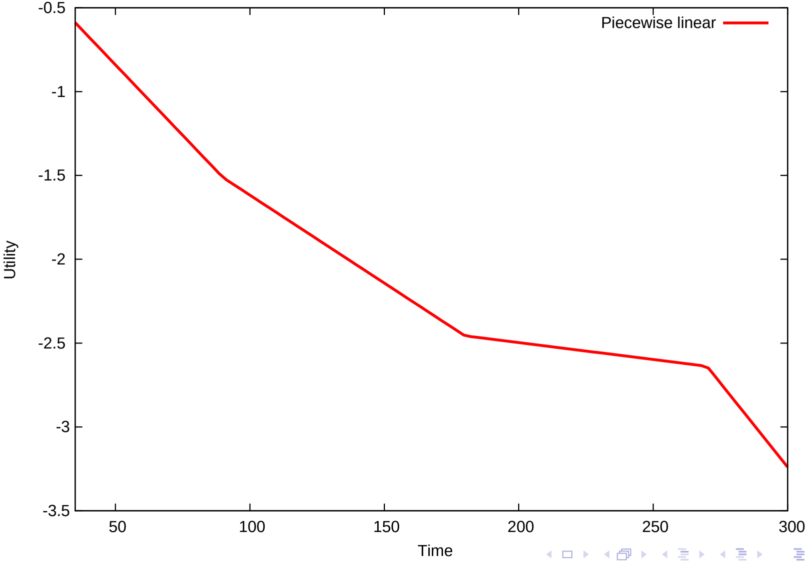
Drawbacks

- No sensitivity to travel time within the intervals
- Discontinuous utility function (jumps)
- Need for many small intervals
- Results may vary significantly with the definition of the intervals

Appropriate when

- Categories have been used in the survey (income, age)
- Definition of categories is natural (weekday)

Piecewise linear specification



Piecewise linear specification

Features

- Capture the sensitivity within the intervals
- Enforce continuity of the utility function

$$V_i = \beta_{T1} x_{T1} + \beta_{T2} x_{T2} + \beta_{T3} x_{T3} + \beta_{T4} x_{T4} + \ldots$$

where

$$x_{T1} = \begin{cases} t & \text{if } t < 90 \\ 90 & \text{otherwise} \end{cases} \qquad x_{T2} = \begin{cases} 0 & \text{if } t < 90 \\ t - 90 & \text{if } 90 \leq t < 180 \\ 90 & \text{otherwise} \end{cases}$$

$$x_{T3} = \begin{cases} 0 & \text{if } t < 180 \\ t - 180 & \text{if } 180 \leq t < 270 \\ 90 & \text{otherwise} \end{cases} \qquad x_{T4} = \begin{cases} 0 & \text{if } t < 270 \\ t - 270 & \text{otherwise} \end{cases}$$

Piecewise linear specification

Note: coding in Biogeme for interval [a:a+b[

$$x_{Ti} = \begin{cases} 0 & \text{if } t < a \\ t - a & \text{if } a \leq t < a + b \\ b & \text{otherwise} \end{cases} \qquad x_{Ti} = \max(0, \min(t - a, b))$$

$$\begin{aligned} x_{T1} &= \min(t, 90) \\ x_{T2} &= \max(0, \min(t - 90, 90)) \\ x_{T3} &= \max(0, \min(t - 180, 90)) \\ x_{T4} &= \max(0, t - 270) \end{aligned}$$

```
TRAIN_TT1  =  min( TRAIN_TT , 90)
TRAIN_TT2  =  max(0,min( TRAIN_TT - 90, 90))
TRAIN_TT3  =  max(0,min( TRAIN_TT - 180 , 90))
TRAIN_TT4  =  max(0,TRAIN_TT - 270)
```

Piecewise linear specification

Examples:

t	TT1	TT2	TT3	TT4
40	40	0	0	0
100	90	10	0	0
200	90	90	20	0
300	90	90	90	30

Box-Cox transforms

Box and Cox, J. of the Royal Statistical Society (1964)

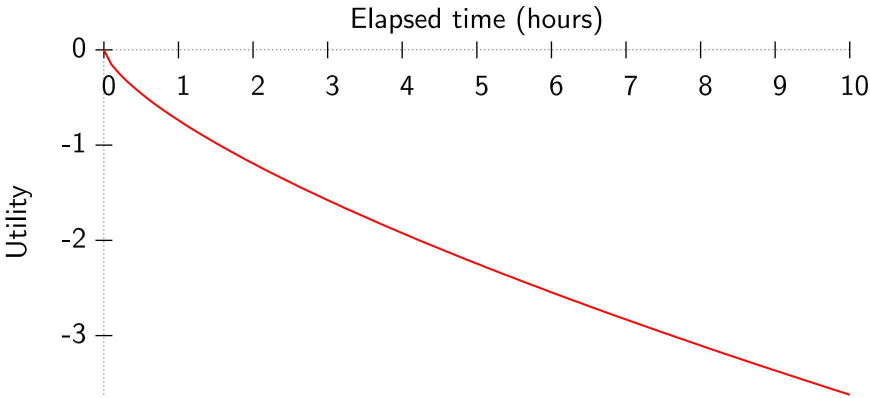
$$V_i = \beta x_i(\lambda) + \cdots$$

where

$$x_i(\lambda) = \begin{cases} \dfrac{x_i^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln x_i & \text{if } \lambda = 0. \end{cases}$$

and $x_i > 0$.

Box-Cox transforms



Box-Cox transforms

Box-Tukey

If $x_i \leq 0$, let α such that $x_i + \alpha > 0$ and

$$x_i(\lambda, \alpha) = \begin{cases} \dfrac{(x_i + \alpha)^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln(x_i + \alpha) & \text{if } \lambda = 0. \end{cases}$$

Box-Cox transforms

Other power transforms are possible:

Manly, Biometrics (1971)

$$x_i(\lambda) = \begin{cases} \frac{e^{x_i\lambda}-1}{\lambda} & \text{if } \lambda \neq 0 \\ x_i & \text{if } \lambda = 0. \end{cases}$$

John and Draper, Applied Statistics (1980)

$$x_i(\lambda) = \begin{cases} \text{sign}(x_i)\frac{(|x_i|+1)^\lambda-1}{\lambda} & \text{if } \lambda \neq 0 \\ \text{sign}(x_i)\ln(|x_i|+1) & \text{if } \lambda = 0. \end{cases}$$

Box-Cox transforms

Other power transforms are possible:

Yeo and Johnson, Biometrika (2000)

$$
x_i(\lambda) = \begin{cases}
\dfrac{(x_i+1)^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0, x_i \geq 0; \\[2ex]
\ln(x_i+1) & \text{if } \lambda = 0, x_i \geq 0; \\[2ex]
\dfrac{(1-x_i)^{2-\lambda} - 1}{\lambda - 2} & \text{if } \lambda \neq 2, x_i < 0; \\[2ex]
-\ln(1-x_i) & \text{if } \lambda = 2, x_i < 0.
\end{cases}
$$

Power series

Taylor expansion

$$V_i = \beta_1 T + \beta_2 T^2 + \beta_3 T^3 + \dots$$

- In practice, these terms can be very correlated
- Difficult to interpret
- Risk of over fitting

Outline

Interactions

Motivation

- All individuals in a population are not alike
- Socio-economic characteristics define segments in the population
- How to capture heterogeneity?

Interactions of attributes and characteristics

Remember...

$$V_{in} = V(h(z_{in}, S_n)) = \sum_k \beta_k (h(z_{in}, S_n))_k$$

Examples of h for interactions

- cost / income
- distance / out-of-vehicle time (= speed)

Segmentation

The population is divided into a finite number of segments

- Each individual belongs to exactly one segment
- Example: gender (M,F) and house location (metro, suburb, perimeter areas)
- 6 segments: (M, m), (M, s), (M, p), (F, m), (F, s), (F, p).

Segmentation

Specification

$$\beta_{M,m} TT_{M,m} + \beta_{M,s} TT_{M,s} + \beta_{M,p} TT_{M,p} +$$
$$\beta_{F,m} TT_{F,m} + \beta_{F,s} TT_{F,s} + \beta_{F,p} TT_{F,p} +$$

$TT_i = TT$ if indiv. belongs to segment i, and 0 otherwise

Remarks

- For a given individual, exactly one of these terms is non zero.
- The number of segments grows exponentially with the number of variables.

Variable parameters

Taste parameter varies with a continuous socio-economic characteristics

Example: the cost parameter varies with income

$$\beta_{\text{cost}} = \hat{\beta}_{\text{cost}} \left(\frac{\text{inc}}{\text{inc}_{\text{ref}}} \right)^{\lambda} \text{ with } \lambda = \frac{\partial \beta_{\text{cost}}}{\partial \text{inc}} \frac{\text{inc}}{\beta_{\text{cost}}}$$

Remarks

- λ must be estimated
- Utility is not linear-in-parameters anymore
- Reference value is arbitrary
- Several characteristics can be combined:

$$\beta_{\text{cost}} = \hat{\beta}_{\text{cost}} \left(\frac{\text{inc}}{\text{inc}_{\text{ref}}} \right)^{\lambda_1} \left(\frac{\text{age}}{\text{age}_{\text{ref}}} \right)^{\lambda_2}$$

Outline

Heteroscedasticity

Assumption: variance of error terms is different across individuals

Assume there are two different groups such that

$$\begin{aligned} U_{in_1} &= V_{in_1} + \varepsilon_{in_1} \\ U_{in_2} &= V_{in_2} + \varepsilon_{in_2} \end{aligned}$$

and $\mathrm{Var}(\varepsilon_{in_2}) = \alpha^2 \,\mathrm{Var}(\varepsilon_{in_1})$

Logit is homoscedastic

- ε_{in} i.i.d. across both i and n.
- How can we specify the model in order to use logit?

Motivation

- People have different level of knowledge (e.g. taxi drivers)
- Different sources of data

Heteroscedasticity

Solution: include scale parameters

$$\begin{array}{rclclclcl} \alpha U_{in_1} & = & \alpha V_{in_1} & + & \alpha\varepsilon_{in_1} & = & \alpha V_{in_1} & + & \varepsilon'_{in_1} \\ U_{in_2} & = & V_{in_2} & + & \varepsilon_{in_2} & = & V_{in_2} & + & \varepsilon'_{in_2} \end{array}$$

where ε'_{in_1} and ε'_{in_2} are i.i.d.

Remarks

- Even if $V_{in_1} = \sum_j \beta_j x_{jin_1}$ is linear-in-parameters, $\alpha V_{in_1} = \sum_j \alpha\beta_j x_{jin_1}$ is not.
- Normalization: a different scale parameter can be estimated for each segment of the population, except one that must be normalized.

Outline

A case study

Choice of residential telephone services

- Household survey conducted in Pennsylvania, USA, 1984
- Revealed preferences
- 434 observations

A case study

Telephone services and availability

	metro, suburban & some perimeter areas	other perimeter areas	non-metro areas
Budget Measured	yes	yes	yes
Standard Measured	yes	yes	yes
Local Flat	yes	yes	yes
Extended Area Flat	no	yes	no
Metro Area Flat	yes	yes	no

A case study

Universal choice set

$$\mathcal{C} = \{\mathrm{BM}, \mathrm{SM}, \mathrm{LF}, \mathrm{EF}, \mathrm{MF}\}$$

Specific choice sets

- Metro, suburban & some perimeter areas: {BM,SM,LF,MF}
- Other perimeter areas: $\mathcal{C}$
- Non-metro areas: {BM,SM,LF}

A case study

Specification table

	β_1	β_2	β_3	β_4	β_5
BM	0	0	0	0	$\ln(\text{cost}(\text{BM}))$
SM	1	0	0	0	$\ln(\text{cost}(\text{SM}))$
LF	0	1	0	0	$\ln(\text{cost}(\text{LF}))$
EF	0	0	1	0	$\ln(\text{cost}(\text{EF}))$
MF	0	0	0	1	$\ln(\text{cost}(\text{MF}))$

A case study

Utility functions

$$
\begin{aligned}
V_{\text{BM}} &= && \beta_5 \ln(\text{cost}_{\text{BM}}) \\
V_{\text{SM}} &= \beta_1 &+& \beta_5 \ln(\text{cost}_{\text{SM}}) \\
V_{\text{LF}} &= \beta_2 &+& \beta_5 \ln(\text{cost}_{\text{LF}}) \\
V_{\text{EF}} &= \beta_3 &+& \beta_5 \ln(\text{cost}_{\text{EF}}) \\
V_{\text{MF}} &= \beta_4 &+& \beta_5 \ln(\text{cost}_{\text{MF}})
\end{aligned}
$$

A case study

Specification table II

	β_1	β_2	β_3	β_4	β_5	β_6	β_7
BM	0	0	0	0	ln(cost(BM))	users	0
SM	1	0	0	0	ln(cost(SM))	users	0
LF	0	1	0	0	ln(cost(LF))	0	1 if metro/suburb
EF	0	0	1	0	ln(cost(EF))	0	0
MF	0	0	0	1	ln(cost(MF))	0	0

A case study

Utility functions

$$
\begin{array}{rcccccccc}
V_{\text{BM}} & = & & & \beta_5 \ln(\text{cost}_{\text{BM}}) & + & \beta_6 \text{users} & & \\
V_{\text{SM}} & = & \beta_1 & + & \beta_5 \ln(\text{cost}_{\text{SM}}) & + & \beta_6 \text{users} & & \\
V_{\text{LF}} & = & \beta_2 & + & \beta_5 \ln(\text{cost}_{\text{LF}}) & & & + & \beta_7 \text{MS} \\
V_{\text{EF}} & = & \beta_3 & + & \beta_5 \ln(\text{cost}_{\text{EF}}) & & & & \\
V_{\text{MF}} & = & \beta_4 & + & \beta_5 \ln(\text{cost}_{\text{MF}}) & & & &
\end{array}
$$

Maximum likelihood estimation

Logit Model

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}}}{\sum_{j\in\mathcal{C}_n} e^{V_{jn}}}$$

Log-likelihood of a sample

$$\mathcal{L}(\beta_1,\ldots,\beta_K) = \sum_{n=1}^{N}\left(\sum_{j=1}^{J} y_{jn} \ln P_n(j|\mathcal{C}_n)\right)$$

where $y_{jn} = 1$ if ind. n has chosen alt. j, 0 otherwise

Maximum likelihood estimation

Logit model

$$\begin{aligned} \ln P_n(i|\mathcal{C}_n) &= \ln \frac{e^{V_{in}}}{\sum_{j\in\mathcal{C}_n} e^{V_{jn}}} \\ &= V_{in} - \ln(\textstyle\sum_{j\in\mathcal{C}_n} e^{V_{jn}}) \end{aligned}$$

Log-likelihood of a sample for logit

$$\mathcal{L}(\beta_1,\ldots,\beta_K) = \sum_{n=1}^{N}\sum_{i=1}^{J} y_{in}\left(V_{in} - \ln\sum_{j\in\mathcal{C}_n} e^{V_{jn}}\right)$$

Maximum likelihood estimation

The maximum likelihood estimation problem

$$\max_{\beta \in \mathbb{R}^K} \mathcal{L}(\beta)$$

- Nonlinear optimization
- If the V's are linear-in-parameters, the function is concave

Maximum likelihood estimation

Numerical issue

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}}}{\sum_{j\in\mathcal{C}_n} e^{V_{jn}}}$$

Largest value that can be stored in a computer $\approx 10^{308}$, that is

$$e^{709.783}$$

It is equivalent to compute

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}-V_{in}}}{\sum_{j\in\mathcal{C}_n} e^{V_{jn}-V_{in}}} = \frac{1}{\sum_{j\in\mathcal{C}_n} e^{V_{jn}-V_{in}}}$$

Outline

Simple models

Null model

$$U_i = \varepsilon_i \quad \forall i$$

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}}}{\sum_{j\in\mathcal{C}_n} e^{V_{jn}}} = \frac{e^0}{\sum_{j\in\mathcal{C}_n} e^0} = \frac{1}{\#\mathcal{C}_n}$$

$$\mathcal{L} = \sum_n \ln \frac{1}{\#\mathcal{C}_n} = -\sum_n \ln(\#\mathcal{C}_n)$$

Simple models

Constants only [Assume $\mathcal{C}_n = \mathcal{C}$, $\forall n$]

$$U_i = c_i + \varepsilon_i \quad \forall i$$

In the sample of size n, there are n_i persons choosing alt. i.

$$\ln P(i) = c_i - \ln(\sum_j e^{c_j})$$

If $\mathcal{C}_n$ is the same for all people choosing i, the log-likelihood for this part of the sample is

$$\mathcal{L}_i = n_i c_i - n_i \ln(\sum_j e^{c_j})$$

Simple models

Constants only (ctd)

The total log-likelihood is

$$\mathcal{L} = \sum_j n_j c_j - n \ln(\sum_j e^{c_j})$$

At the maximum, the derivatives must be zero

$$\frac{\partial \mathcal{L}}{\partial c_1} = n_1 - n \frac{e^{c_1}}{\sum_j e^{c_j}} = n_1 - nP(1) = 0.$$

Simple models

Constants only (ctd.)

Therefore,

$$P(1) = \frac{n_1}{n}$$

Conclusion

If all alternatives are always available, a model with only Alternative Specific Constants reproduces exactly the market shares in the sample

Back to the case study

Alt.	n_i	n_i/n	c_i	e^{c_i}	P(i)
BM	73	0.168	0.247	1.281	0.168
SM	123	0.283	0.769	2.158	0.283
LF	178	0.410	1.139	3.123	0.410
EF	3	0.007	-2.944	0.053	0.007
MF	57	0.131	0.000	1.000	0.131
	434	1.000			

Null-model: $\mathcal{L}$ = -434 ln(5) = -698.496

Warning: these results have been obtained assuming that all alternatives are always available