
Sampling

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Introduction

- Does the sample perfectly reflect the population?
- Is it desirable to perform random sampling?
- How will other sampling strategies affect the model estimates?
- What are the specific implications for discrete choice?

Types of variables

- Exogenous/independent variables (denoted by x)
 - age, gender, income, prices
 - Not modeled, treated as given in the population
 - May be subject to *what if* policy manipulations
- Endogenous/dependent variables (denoted by i)
- choice
- Causal model $P(i|x; \theta)$

Types of variables

- The nature of a variable depends on the application
- Example: residential location
 - Endogenous in a house choice study
 - Exogenous in a study about transport mode choice to work
- A model $P(i|x; \theta)$ may fit the data and describe correlation between i and x without being a causal model. Example: $P(\text{crime}|\text{temp})$ and $P(\text{temp}|\text{crime})$.
- Critical to identify the causal relationship and, therefore, exogenous and endogenous variables.

Sampling strategies

- Simple Random Sample (SRS)
 - Probability of being drawn: R
 - R is identical for each individual
 - Convenient for model estimation and forecasting
 - Very difficult to conduct in practice
- Exogenously Stratified Sample (XSS)
 - Probability of being drawn: $R(x)$
 - $R(x)$ varies with variables other than i
 - May also vary with variables outside the model
 - Examples:
 - oversampling of workers for mode choice
 - oversampling of women for baby food choice
 - undersampling of old people for choice of a retirement plan

Sampling strategies

- Endogenously Stratified Sample (ESS)
 - Probability of being drawn: $R(i, x)$
 - $R(i, x)$ varies with dependent variables
 - Examples:
 - oversampling of bus riders
 - products with small market shares: if SRS, likely that no observation of i in the sample (ex: Ferrari)
 - oversampling of current customers

Sampling strategies

- Special case: pure choice-based sampling
 - Probability of being drawn: $R(i)$
 - $R(i)$ varies only with dependent variables

Sampling strategies

In practice, groups are defined, and individuals are sampled randomly within each group.

Let's consider each sampling scheme on the following example:

- Exogenous variable: travel time by car
- Endogenous variable: transportation mode

Sampling strategies

Simple Random Sampling (SRS): one group = population

		Drive alone	Carpooling	Transit
Travel time by car	≤ 15			
	$> 15, \leq 30$			
	> 30			

Sampling strategies

Exogenously Stratified Sample (XSS):

		Drive alone	Carpooling	Transit
Travel time by car	≤ 15			
	$> 15, \leq 30$			
	> 30			

Sampling strategies

Pure choice-based sampling:

		Drive alone	Carpooling	Transit
Travel time by car	≤ 15			
	$> 15, \leq 30$			
	> 30			

Sampling strategies

Endogenously Stratified Sample (ESS):

		Drive alone	Carpooling	Transit
Travel time by car	≤ 15			
	$> 15, \leq 30$			
	> 30			

Sampling strategies

If (i, x) belongs to group g , we can write

$$R(i, x) = \frac{H_g N_s}{W_g N}$$

where

- H_g is the fraction of the group corresponding to (i, x) in the sample
- W_g is the fraction of the group corresponding to (i, x) in the population
- N_s is the sample size
- N is the population size

Sampling strategies

- H_g and N_s are decided by the analyst
- W_g can be expressed as

$$W_g = \int_x \left(\sum_{i \in \mathcal{C}_g} P(i|x, \theta) \right) p(x) dx$$

which is a function of θ

Sampling strategies

- If group g contains all alternatives, then

$$\sum_{i \in \mathcal{C}_g} P(i|x, \theta) = 1$$

and $W_g = \int_{x \in g} p(x) dx$ does not depend on θ

- This can happen only if groups are not defined based on the alternatives.

Sampling strategies

Population	i=0	i=1		
x=0	300000	100000	400000	40%
x=1	510000	90000	600000	60%
	810000	190000	1000000	
	81%	19%		

SRS

x=0	1/1000	1/1000		
x=1	1/1000	1/1000		
x=0	300	100	400	40%
x=1	510	90	600	60%
	810	190	1000	
	81%	19%		

Sampling strategies

Population	i=0	i=1		
x=0	300000	100000	400000	40%
x=1	510000	90000	600000	60%
	810000	190000	1000000	
	81%	19%		

XSS

x=0	1/1600	1/1600		
x=1	1/800	1/800		
x=0	187.5	62.5	250	25%
x=1	637.5	112.5	750	75%
	825	175	1000	
	83%	18%		

Sampling strategies

Population	i=0	i=1		
x=0	300000	100000	400000	40%
x=1	510000	90000	600000	60%
	810000	190000	1000000	
	81%	19%		

ESS

x=0	1/1190	1/595
x=1	1/1190	1/595

x=0	252.1	168.1	420.2	42%
x=1	428.6	151.3	579.9	58%
	680.7	319.3	1000	
	68%	32%		

Estimation

- Define s_n has the event of individual n being in the sample
- Maximum Likelihood:

$$\max_{\theta} \mathcal{L}(\theta) = \sum_{n=1}^N \ln \Pr(i_n, x_n | s_n; \theta)$$

- The joint probability for an individual to
 - be in the sample (s_n)
 - be exposed to exogenous variables x_n
 - choose the observed alternative (i_n)

is denoted

$$\Pr(i_n, x_n, s_n; \theta)$$

- Two ways to derive it

Estimation

$$\begin{aligned}\Pr(i_n, x_n, s_n; \theta) &= \Pr(i_n, x_n | s_n; \theta) \Pr(s_n; \theta) \\ &= \Pr(s_n | i_n, x_n; \theta) \Pr(i_n | x_n; \theta) p(x_n).\end{aligned}$$

$$\Pr(i_n, x_n | s_n; \theta) \Pr(s_n; \theta) = \Pr(s_n | i_n, x_n; \theta) \Pr(i_n | x_n; \theta) p(x_n)$$

- $\Pr(i_n, x_n | s_n; \theta)$: term for the ML
- $\Pr(s_n; \theta) = \sum_z \sum_{j \in \mathcal{C}} \Pr(s_n | j, z; \theta) \Pr(j | z; \theta) \Pr(z)$
- $\Pr(s_n | i_n, x_n; \theta)$: probability to be sampled, that is $R(i_n, x_n; \theta)$
- $\Pr(i_n | x_n; \theta)$: choice model $P(i_n | x_n; \theta)$

$$\Pr(i_n, x_n | s_n; \theta) = \frac{R(i_n, x_n; \theta) P(i_n | x_n; \theta) p(x_n)}{\sum_z \sum_{j \in \mathcal{C}} R(j, z; \theta) P(j | z; \theta) p(z)}$$

Estimation

$$\Pr(i_n, x_n | s_n; \theta) = \frac{R(i_n, x_n; \theta) P(i_n | x_n; \theta) p(x_n)}{\sum_z \sum_{j \in \mathcal{C}} R(j, z; \theta) P(j | z; \theta) p(z)}$$

- In general, impossible to handle
- Namely, $p(z)$ is usually not available
- Two possibilities:
 1. It does simplify when the sampling is exogenous
 2. If not, we use Conditional Maximum Likelihood instead.
 - (a) Case of logit
 - (b) Case of MEV
 - (c) Other models

Exogenous Sample Maximum Likelihood

- If the sample is simple or exogenous :

$$R(i, x; \theta) = R(x) \quad \forall i, \theta$$

$$\begin{aligned} \Pr(i_n, x_n | s_n; \theta) &= \frac{R(i_n, x_n; \theta) P(i_n | x_n; \theta) p(x_n)}{\sum_z \sum_{j \in \mathcal{C}} R(j, z; \theta) P(j | z; \theta) p(z)} \\ &= \frac{R(x_n) P(i_n | x_n; \theta) p(x_n)}{\sum_z \sum_{j \in \mathcal{C}} R(z) P(j | z; \theta) p(z)} \\ &= \frac{R(x_n) P(i_n | x_n; \theta) p(x_n)}{\sum_z R(z) p(z) \sum_{j \in \mathcal{C}} P(j | z; \theta)} \\ &= \frac{R(x_n) P(i_n | x_n; \theta) p(x_n)}{\sum_z R(z) p(z)} \end{aligned}$$

Exogenous Sample Maximum Likelihood

$$\Pr(i_n, x_n | s_n; \theta) = \frac{R(x_n)P(i_n | x_n; \theta)p(x_n)}{\sum_z R(z)p(z)}$$

- Taking the log for the maximum likelihood

$$\ln \Pr(i_n, x_n | s_n; \theta) = \ln P(i_n | x_n; \theta) + \ln R(x_n) + \ln p(x_n) - \ln \sum_z R(z)p(z)$$

- For the maximization, terms not depending on θ are irrelevant

$$\operatorname{argmax}_{\theta} \sum_n \ln \Pr(i_n, x_n | s_n; \theta) = \operatorname{argmax}_{\theta} \sum_n \ln P(i_n | x_n; \theta)$$

- Same procedure as for SRS

Conditional Maximum Likelihood

- Instead of solving

$$\max_{\theta} \sum_n \ln \Pr(i_n, x_n | s_n; \theta)$$

- we solve

$$\max_{\theta} \sum_n \ln \Pr(i_n | x_n, s_n; \theta)$$

- CML is consistent but not efficient
- Let's again derive $\Pr(i_n, x_n, s_n; \theta)$ in two ways

Conditional Maximum Likelihood

$$\begin{aligned}\Pr(i_n, x_n, s_n; \theta) &= \Pr(i_n | x_n, s_n; \theta) \Pr(s_n | x_n; \theta) p(x_n) \\ &= \Pr(s_n | i_n, x_n; \theta) \Pr(i_n | x_n; \theta) p(x_n).\end{aligned}$$

$$\Pr(i_n | x_n, s_n; \theta) \Pr(s_n | x_n; \theta) = \Pr(s_n | i_n, x_n; \theta) \Pr(i_n | x_n; \theta).$$

- $\Pr(i_n | x_n, s_n; \theta)$: term for the CML
- $\Pr(s_n | x_n; \theta) = \sum_{j \in \mathcal{C}} \Pr(s_n | j, x_n; \theta) \Pr(j | x_n; \theta)$
- $\Pr(s_n | i_n, x_n; \theta)$: probability to be sampled, that is $R(i_n, x_n; \theta)$
- $\Pr(i_n | x_n; \theta)$: choice model $P(i_n | x_n; \theta)$

$$\Pr(i_n | x_n, s_n; \theta) = \frac{R(i_n, x_n; \theta) P(i_n | x_n; \theta)}{\sum_{j \in \mathcal{C}} R(j, x_n; \theta) P(j | x_n; \theta)}$$

CML with logit and ESS

Assume now logit and $R(i_n, x_n; \theta) = R(i_n; \theta)$

$$P(i_n | x_n; \theta = \beta) = \frac{e^{V_{i_n}(x_n, \beta)}}{\sum_k e^{V_k(x_n, \beta)}} = \frac{e^{V_{i_n}(x_n, \beta)}}{D}$$

where $D = \sum_j e^{V_j(x_n, \beta)}$ Then

$$\begin{aligned} \Pr(i_n | x_n, s_n; \theta) &= \frac{R(i_n; \theta) P(i_n | x_n; \theta)}{\sum_{j \in \mathcal{C}} R(j; \theta) P(j | x_n; \theta)} \\ &= \frac{DR(i_n; \theta) e^{V_{i_n}(x_n, \beta)}}{D \sum_{j \in \mathcal{C}} R(j; \theta) e^{V_j(x_n, \beta)}} \\ &= \frac{e^{V_{i_n}(x_n, \beta) + \ln R(i_n; \theta)}}{\sum_{j \in \mathcal{C}} e^{V_j(x_n, \beta) + \ln R(j; \theta)}} \end{aligned}$$

CML with logit and ESS

- Let's define J additional unknown parameters

$$\omega_j = \ln R(j; \theta)$$

- Assume that each utility has an ASC, so that

$$V_{i_n}(x_n, \beta) = \tilde{V}_{i_n}(x_n, \beta) + \gamma_i$$

- The CML involves

$$\Pr(i_n | x_n, s_n; \theta) = \frac{e^{\tilde{V}_{i_n}(x_n, \beta) + \gamma_i + \omega_i}}{\sum_{j \in \mathcal{C}} e^{\tilde{V}_j(x_n, \beta) + \gamma_j + \omega_j}}$$

- It is exactly ESML, except that γ_i is replaced by $\gamma_i + \omega_i$

CML with logit and ESS

If the logit model has a full set of constants, ESML yields consistent estimates of all parameters except the constants with Endogenous Sampling Strategy

Example

- $i = 0$ stay on defined benefit pension plan
- $i = 1$ switch to defined contribution plan
- $x = 1$ switching penalty
- $x = 0$ no switching penalty

Population

	$i=0$	$i=1$		
$x=0$	300000	100000	400000	0.4
$x=1$	510000	90000	600000	0.6
	810000	190000	1000000	
	0.81	0.19		

Example

Simple model:

$$\begin{aligned}V_0 &= 0 \\V_1 &= \alpha + \beta x\end{aligned}$$

$$P(0|x) = \frac{1}{1 + e^{\alpha + \beta x}}, \quad P(1|x) = \frac{e^{\alpha + \beta x}}{1 + e^{\alpha + \beta x}} = \frac{1}{1 + e^{-\alpha - \beta x}}$$

Easy to estimate:

$$P(1|0) = \frac{1}{1 + e^{-\alpha}}, \quad P(0|0) = 1 - P(1|0) = \frac{e^{-\alpha}}{1 + e^{-\alpha}}$$

Therefore

$$e^{\alpha} = \frac{P(1|0)}{P(0|0)}, \quad \text{and} \quad \alpha = \ln \frac{P(1|0)}{P(0|0)}$$

Example

$$P(1|1) = \frac{1}{1 + e^{-\alpha-\beta}}, \quad P(0|1) = 1 - P(1|1) = \frac{e^{-\alpha-\beta}}{1 + e^{-\alpha-\beta}}$$

Therefore

$$e^{\alpha+\beta} = \frac{P(1|1)}{P(0|1)}, \quad e^{\beta} = e^{-\alpha} \frac{P(1|1)}{P(0|1)}$$

and

$$e^{\beta} = \frac{P(0|0) P(1|1)}{P(1|0) P(0|1)} \quad \text{and} \quad \beta = \ln \left(\frac{P(0|0) P(1|1)}{P(1|0) P(0|1)} \right)$$

Example

	$i=0$	$i=1$		
$x=0$	300000	100000	400000	40%
$x=1$	510000	90000	600000	60%
	810000	190000	1000000	
	81%	19%		

$$P(1|0) = 0.25 \quad \alpha = -1.09861$$

$$P(0|0) = 0.75 \quad \beta = -0.63599$$

$$P(1|1) = 0.15$$

$$P(0|1) = 0.85$$

Example

SRS: $R = 1/1000$

	$i = 0$	$i = 1$		
$x = 0$	300	100	400	40%
$x = 1$	510	90	600	60%
	810	190	1000	
	81%	19%		

$$P(1|0) = 0.25 \quad \alpha = -1.09861$$

$$P(0|0) = 0.75 \quad \beta = -0.63599$$

$$P(1|1) = 0.15$$

$$P(0|1) = 0.85$$

Retrieve the true parameters

Example

XSS: $R(x = 0) = 1/1600$, $R(x = 1) = 1/800$

	$i = 0$	$i = 1$		
$x = 0$	187.5	62.5	250	25%
$x = 1$	637.5	112.5	750	75%
	825	175	1000	
	82.5%	17.5%		

$$P(1|0) = 0.25 \quad \alpha = -1.09861$$

$$P(0|0) = 0.75 \quad \beta = -0.63599$$

$$P(1|1) = 0.15$$

$$P(0|1) = 0.85$$

Retrieve the true parameters

Example

Important note:

- Although the sampling strategy is exogenous, the market shares in the sample do not reflect the true market shares.
- Omitting an explanatory variable may therefore bias the results
- In this example, a model with only the constant will reproduce the market shares of the sample.

Example

ERS: $R(i = 0) = 1/1190$, $R(i = 1) = 1/595$

	$i = 0$	$i = 1$		
$x = 0$	252	168	420	42%
$x = 1$	429	151	580	58%
	681	319	1000	
	68.1%	31.9%		

$$P(1|0) = 0.4 \quad \alpha = -0.40547$$

$$P(0|0) = 0.6 \quad \beta = -0.63599$$

$$P(1|1) = 0.26087$$

$$P(0|1) = 0.73913$$

Retrieve the true value of β

Example

True α	-1.09861	$\ln R(i = 0)$	-7.08171
Estim. α	-0.40547	$\ln R(i = 1)$	-6.38856
Diff	0.693147	Diff	0.693147

- We have estimated

$$V_0 = 0 + \ln R(i = 0) = -7.08171$$

$$\begin{aligned} V_1 &= \beta x + \alpha + \ln R(i = 1) = \beta x - 1.09861 - 6.38856 \\ &= \beta x - 7.487173 \end{aligned}$$

- Shift both constants by 7.08171

$$V_0 = 0$$

$$V_1 = \beta x - 0.40547$$

CML with MEV and ESS

- What about MEV model?
- Same derivation as for logit
- Recent result by Bierlaire, Bolduc & McFadden (2008)

CML with MEV and ESS

Assume now MEV and $R(i_n, x_n; \theta) = R(i_n; \theta)$

$$P(i_n | x_n; \theta = \beta) = \frac{e^{V_{i_n}(x_n, \beta) + \ln G_{i_n}(\cdot)}}{\sum_k e^{V_k(x_n, \beta) + \ln G_k(\cdot)}} = \frac{e^{V_{i_n}(x_n, \beta) + \ln G_{i_n}(\cdot)}}{D}$$

where $G_k(\cdot) = G_k(e^{V_1}, \dots, e^{V_J})$. Then

$$\begin{aligned} \Pr(i_n | x_n, s_n; \theta) &= \frac{R(i_n; \theta) P(i_n | x_n; \theta)}{\sum_{j \in \mathcal{C}} R(j; \theta) P(j | x_n; \theta)} \\ &= \frac{DR(i_n; \theta) e^{V_{i_n}(x_n, \beta) + \ln G_{i_n}(\cdot)}}{D \sum_{j \in \mathcal{C}} R(j; \theta) e^{V_j(x_n, \beta) + \ln G_j(\cdot)}} \\ &= \frac{e^{V_{i_n}(x_n, \beta) + \ln G_{i_n}(\cdot) + \ln R(i_n; \theta)}}{\sum_{j \in \mathcal{C}} e^{V_j(x_n, \beta) + \ln G_j(\cdot) + \ln R(j; \theta)}} \end{aligned}$$

CML with MEV and ESS

- Let's define J additional unknown parameters

$$\omega_j = \ln R(j; \theta)$$

- The CML involves

$$\Pr(i_n | x_n, s_n; \theta) = \frac{e^{V_{i_n}(x_n, \beta) + \ln G_{i_n}(\cdot) + \omega_{i_n}}}{\sum_{j \in \mathcal{C}} e^{V_j(x_n, \beta) + \ln G_j(\cdot) + \omega_j}}$$

- Here, because there are constants **inside** $G_j(\cdot)$, the parameters ω cannot be “absorbed” by the constants.
- ESML cannot be used
- But CML is not difficult in this case.

MEV and sampling

Claims in the literature (both erroneous):

- Koppelman, Garrow and Nelson (2005)
 - ESML estimator can also be used for nested logit
 - Consistent est. for all parameters but the constants
 - Consistent est. of the constants obtained by subtracting $\ln R(i, z) / \mu_{m_i}$
- Bierlaire, Bolduc and McFadden (2003)
 - ESML estimator can be used for any MEV model
 - It provides consistent est. for all parameters except the constants.
 - Consistent est. of the constants obtained by subtracting $\ln R(i, z)$

MEV and sampling

- Koppelman et al.:
 - Flaw in the proof
 - Results not confirmed numerically.
- Bierlaire et al.:
 - Flaw in the proof
 - Need for another estimator

Illustration

- Pseudo-synthetic data
- Data base: SP mode choice for future highspeed train in Switzerland (Swissmetro)
- Alternatives:
 1. Regular train (TRAIN),
 2. Swissmetro (SM), the future high speed train,
 3. Driving a car (CAR).
- Generation of a synthetic population of 507600 individuals

Illustration

- Attributes are random perturbations of actual attributes
- Assumed true choice model: NL

Param.	Value	Alternatives		
		TRAIN	SM	CAR
ASC_CAR	-0.1880	0	0	1
ASC_SM	0.1470	0	1	0
B_TRAIN_TIME	-0.0107	travel time	0	0
B_SM_TIME	-0.0081	0	travel time	0
B_CAR_TIME	-0.0071	0	0	travel time
B_COST	-0.0083	travel cost	travel cost	travel cost

Illustration

- Nesting structure:

	μ_m	TRAIN	SM	CAR
NESTA	2.27	1	0	1
NESTB	1.0	0	1	0

Illustration

- 100 samples drawn from the population

Strata	$W_g N_P$	W_g	H_g	$H_g N_s$	R_g
TRAIN	67938	13.4%	60%	3000	4.42E-02
SM	306279	60.3%	20%	1000	3.26E-03
CAR	133383	26.3%	20%	1000	7.50E-03
Total	507600	1	1	5000	

- Estimation of 100 models
- Empirical mean and std dev of the estimates

Illustration

	True	ESML			New estimator		
		Mean	<i>t</i> -test	Std. dev.	Mean	<i>t</i> -test	Std. dev.
ASC_SM	0.1470	-2.2479	-25.4771	0.0940	-2.4900	-23.9809	0.1100
ASC_CAR	-0.1880	-0.8328	-7.3876	0.0873	-0.1676	0.1581	0.1292
BCOST	-0.0083	-0.0066	2.6470	0.0007	-0.0083	0.0638	0.0008
BTIME_TRAIN	-0.0107	-0.0094	1.4290	0.0009	-0.0109	-0.1774	0.0009
BTIME_SM	-0.0081	-0.0042	3.1046	0.0013	-0.0080	0.0446	0.0014
BTIME_CAR	-0.0071	-0.0065	0.9895	0.0007	-0.0074	-0.3255	0.0007
NestParam	2.2700	2.7432	1.7665	0.2679	2.2576	-0.0609	0.2043
S_SM_Shifted	-2.6045						
S_CAR_Shifted	-1.7732				-1.7877	-0.0546	0.2651
ASC_SM+S_SM	-2.4575				-2.4900	-0.2958	0.1100

CML for MEV with ESS

- Except in very specific cases, ESML provides biased estimated for non-logit MEV models
- Due to the logit-like form of the MEV model, a new simple estimator has been proposed
- It allows to estimate selection bias from the data

Weighted Exogenous Sample Maximum Likelihood

- Manski and Lerman (1977)
- Assumes that $R(i, x)$ is known
- Equivalently, assume that H_g and W_g are known for each group as

$$R(i, x) = \frac{H_g N_s}{W_g N}$$

- Solution of

$$\max_{\theta} \mathcal{L}(\theta) = \sum_{n=1}^N \frac{1}{R(i_n, x_n)} \ln P(i_n | x_n; \theta)$$

- This is a weighted version of the ESML
- In Biogeme, simply define weights

Summary

- With SRS and XSS: use ESML
 - $\max_{\theta} \sum_n \ln P(i_n | x_n; \theta)$
 - Classical procedure, available in most packages
- With ESS and logit: use ESML and correct the constants
- With ESS and MEV: estimate the bias from data
 - Require a specific procedure
 - Available in Biogeme
- General case: use WESML