Multivariate Extreme Value models

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Transport and Mobility LAboratory

Logit

- Random utility:
$$U_{in} = V_{in} + \varepsilon_{in}$$
- ε_{in} is i.i.d. EV (Extreme Value) distributed
- ε_{in} is the maximum of many r.v. capturing unobservable attributes, measurement and specification errors.
- Key assumption: Independence

Relax the independence assumption

$$\begin{pmatrix} U_{1n} \\ \vdots \\ U_{Jn} \end{pmatrix} = \begin{pmatrix} V_{1n} \\ \vdots \\ V_{Jn} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1n} \\ \vdots \\ \varepsilon_{Jn} \end{pmatrix}$$

that is

$$U_n = V_n + \varepsilon_n$$

and ε_n is a vector of random variables.

Assumption about the random term:
multivariate distribution

Relax the independence assumption

A multivariate random variable ε is represented by a density function

$$f(\varepsilon_1, \ldots, \varepsilon_J)$$

and

$$P(\varepsilon \leq x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_J} f(\varepsilon) d\varepsilon_J \ldots d\varepsilon_1$$

where $x \in \mathbb{R}^J$ is a $J \times 1$ vector of constants.

Probit model

- Multivariate normal variable $N(\mu, \Sigma)$
- $\mu \in \mathbb{R}^J$
- $\Sigma \in \mathbb{R}^{J \times J}$, definite positive
- Density function:

$$f(\varepsilon) = (2\pi)^{-\frac{J}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(\varepsilon-\mu)^T \Sigma^{-1} (\varepsilon-\mu)}$$

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Probit model

Example: trinomial model

$$
\begin{aligned}
U_1 &= V_1 + \varepsilon_1 \\
U_2 &= V_2 + \varepsilon_2 \\
U_3 &= V_3 + \varepsilon_3
\end{aligned}
$$

and $\varepsilon \sim N(0, \Sigma)$. We have $P(2) = P(U_i - U_2 \leq 0 \quad i = 1, 2, 3)$

$$
\begin{aligned}
U_1 - U_2 &= V_1 - V_2 + \varepsilon_1 - \varepsilon_2 \\
U_3 - U_2 &= V_3 - V_2 + \varepsilon_3 - \varepsilon_2
\end{aligned}
$$

Probit model

Matrix notation with

$$\Delta_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\Delta_2 U = \begin{pmatrix} U_1 - U_2 \\ U_3 - U_2 \end{pmatrix} \sim N(\Delta_2 V, \Delta_2 \Sigma \Delta_2^T)$$

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Probit model

In general, we have

$$\Delta_i U \sim N(\Delta_i V, \Delta_i \Sigma \Delta_i^T)$$

and $P(i) =$

$$P(\Delta_i U \leq 0) = \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} f(\Delta_i \varepsilon) d(\Delta_i \varepsilon)_1 \ldots d(\Delta_i \varepsilon)_{J-1}$$

with

$$f(\Delta_i \varepsilon) = (2\pi)^{-\frac{J}{2}} |\Delta_i \Sigma \Delta_i^T|^{-\frac{1}{2}} e^{-\frac{1}{2}(\Delta_i \varepsilon - \Delta_i V)^T (\Delta_i \Sigma \Delta_i^T)^{-1} (\Delta_i \varepsilon - \Delta_i V)}$$

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Probit model

- The integral of the density function has no closed form
- In high dimensions, numerical integration is computationally infeasible
- Therefore, the probit model with more than 5 alternatives is very difficult to use in practice

Relax the independence assumption

Assume that ε_n is a multivariate random variable with

- CDF: $F_{\varepsilon_n}(\xi_1, \ldots, \xi_{J_n})$
- pdf: $f_{\varepsilon_n}(\xi_1, \ldots, \xi_{J_n}) = \frac{\partial^{J_n} F}{\partial \xi_1 \cdots \partial \xi_{J_n}}(\xi_1, \ldots, \xi_{J_n})$.
- The choice probability is

$$P_n(1) = \Pr(V_{2n} + \varepsilon_{2n} \leq V_{1n} + \varepsilon_{1n}, \ldots, V_{Jn} + \varepsilon_{Jn} \leq V_{1n} + \varepsilon_{1n}),$$

or

$$P_n(1) = \Pr(\varepsilon_{2n} - \varepsilon_{1n} \leq V_{1n} - V_{2n}, \ldots, \varepsilon_{Jn} - \varepsilon_{1n} \leq V_{1n} - V_{Jn}).$$

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Relax the independence assumption

Change of variables:

$$\xi_{1n} = \varepsilon_{1n},\ \xi_{in} = \varepsilon_{in} - \varepsilon_{1n},\ i = 2, \ldots, J_n,$$

that is

$$\begin{pmatrix} \xi_{1n} \\ \xi_{2n} \\ \vdots \\ \xi_{(J_n-1)n} \\ \xi_{J_n n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ & & \vdots & & \\ -1 & 0 & \cdots & 1 & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \\ \vdots \\ \varepsilon_{(J_n-1)n} \\ \varepsilon_{J_n n} \end{pmatrix}.$$

and

$$P_n(1) = \Pr(\xi_{2n} \leq V_{1n} - V_{2n}, \ldots, \xi_{J_n n} \leq V_{1n} - V_{J_n n}).$$

Relax the independence assumption

$$P_n(1) = \Pr(\xi_{2n} \leq V_{1n} - V_{2n}, \ldots, \xi_{J_n n} \leq V_{1n} - V_{J_n n}).$$

- Only $J_n - 1$ inequalities.
- ξ_{1n} can take any value.
- Choice probability = CDF of $(\xi_{2n}, \ldots, \xi_{J_n n})$ evaluated at $(V_{1n} - V_{2n}, \ldots, V_{1n} - V_{J_n n})$.

$$\begin{aligned} P_n(1) &= F_{\xi_{1n}, \xi_{2n}, \ldots, \xi_{J_n}}(+\infty, V_{1n} - V_{2n}, \ldots, V_{1n} - V_{J_n n}) \\ &= \int_{\xi_1=-\infty}^{+\infty} \int_{\xi_2=-\infty}^{V_{1n}-V_{2n}} \cdots \int_{\xi_{J_n}=-\infty}^{V_{1n}-V_{J_n n}} f_{\xi_{1n}, \xi_{2n}, \ldots, \xi_{J_n}}(\xi_1, \xi_2, \ldots, \xi_{J_n}) d\xi, \end{aligned}$$

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Relax the independence assumption

$$P_n(1) \int_{\xi_1=-\infty}^{+\infty} \int_{\xi_2=-\infty}^{V_{1n}-V_{2n}} \cdots \int_{\xi_{J_n}=-\infty}^{V_{1n}-V_{J_n n}} f_{\xi_{1n},\xi_{2n},\ldots,\xi_{J_n}}(\xi_1,\xi_2,\ldots,\xi_{J_n})d\xi.$$

- Change of variables: determinant 1.
- pdf of $(\xi_{1n},\ldots,\xi_{J_n n})$ = pdf of $(\varepsilon_{1n},\ldots,\varepsilon_{J_n n})$

$$P_n(1) = \int_{\varepsilon_1=-\infty}^{+\infty} \int_{\varepsilon_2=-\infty}^{V_{1n}-V_{2n}+\varepsilon_1} \cdots \int_{\varepsilon_{J_n}=-\infty}^{V_{1n}-V_{J_n n}+\varepsilon_1} f_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{J_n})d\varepsilon.$$

or

$$P_n(1) = \int_{\varepsilon_1=-\infty}^{+\infty} \frac{\partial F_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}}{\partial \varepsilon_1}(\varepsilon_1, V_{1n}-V_{2n}+\varepsilon_1,\ldots,V_{1n}-V_{J_n n}+\varepsilon_1)d\varepsilon_1.$$

Multivariate Extreme Value model

- $\varepsilon_n = (\varepsilon_{1n}, \ldots, \varepsilon_{Jn})$ follows a multivariate extreme value distribution if it has the CDF:

$$F_{\varepsilon_n}(\varepsilon_{1n}, \ldots, \varepsilon_{Jn}) = e^{-G(e^{-\varepsilon_{1n}}, \ldots, e^{-\varepsilon_{Jn}})},$$

  where $G : \mathbb{R}_+^{J_n} \to \mathbb{R}_+$ is a positive function with positive arguments.
- To be a valid CDF, it must verify the following properties.
- (i) the limit property

$$F_{\varepsilon_n}(\varepsilon_{1n}, \ldots, -\infty, \ldots, \varepsilon_{Jn}) = 0,$$

  or

$$G(y_{1n}, \ldots, +\infty, \ldots, y_{Jn}) = +\infty.$$

Multivariate Extreme Value model

- (ii) the zero property

$$F_{\varepsilon_n}(+\infty, \ldots, +\infty) = 1.$$

or

$$G(0, \ldots, 0) = 0.$$

- (iii) the strong alternating sign property:
  - Any partial derivative of F_{ε_n} defines a density function of a marginal distribution.
  - To be a valid density function, it has to be non negative.
  - For any set of $\widehat{J}_n \leq J_n$ distinct indices $i_1, \ldots, i_{\widehat{J}_n}$,

$$\frac{\partial^{\widehat{J}_n} F_{\varepsilon_n}}{\partial \varepsilon_{i_1 n} \cdots \partial \varepsilon_{i_{\widehat{J}_n} n}}(\varepsilon_{1n}, \ldots, \varepsilon_{J_n n}) \geq 0.$$

Multivariate Extreme Value model

$$F_{\varepsilon_n}(\varepsilon_{1n},\ldots,\varepsilon_{Jn}) = e^{-G(e^{-\varepsilon_{1n}},\ldots,e^{-\varepsilon_{Jn}})},$$

- (iii) the strong alternating sign property (ctd).
  - The right-hand side changes sign each time it is differentiated.
  - To obtain a non negative sign, G must also change sign each time it is differentiated.
  - For any set of $\widehat{J}_n$ distinct indices $i_1,\ldots,i_{\widehat{J}_n}$,

$$(-1)^{\widehat{J}_n-1} G_{i_1,\ldots,i_{\widehat{J}_n}} \geq 0.$$

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Multivariate Extreme Value model

We need another property: homogeneity.

- A function G is homogeneous of degree μ, or μ-homogeneous, if
$$G(\alpha y) = \alpha^{\mu} G(y), \ \forall \alpha > 0 \text{ and } y \in \mathbb{R}_{+}^{J}.$$
- It will imply two results:
  - the marginals are univariate extreme value distributions,
  - the choice model has a closed form.

Multivariate Extreme Value model

- ith marginal distribution:

$$F_{\varepsilon_n}(+\infty, \ldots, +\infty, \varepsilon_{in}, +\infty, \ldots, +\infty) = e^{-G(0,\ldots,0,e^{-\varepsilon_{in}},0,\ldots,0)}.$$

- If G is μ-homogeneous, we have

$$G(0, \ldots, 0, e^{-\varepsilon_{in}}, 0, \ldots, 0) = e^{-\mu\varepsilon_{in}} G(0, \ldots, 0, 1, 0, \ldots, 0),$$

or equivalently,

$$G(0, \ldots, 0, e^{-\varepsilon_{in}}, 0, \ldots, 0) = e^{-\mu\varepsilon_{in} + \log G(0,\ldots,0,1,0,\ldots,0)},$$

- Define $\log G(0, \ldots, 0, 1, 0, \ldots, 0) = \mu\eta$, so that

$$F_{\varepsilon_n}(+\infty, \ldots, +\infty, \varepsilon_{in}, +\infty, \ldots, +\infty) = \exp\left(-e^{-\mu(\varepsilon_{in}-\eta)}\right).$$

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Multivariate Extreme Value model

$$F_{\varepsilon_n}(\varepsilon_{1n},\ldots,\varepsilon_{Jn}) = e^{-G(e^{-\varepsilon_{1n}},\ldots,e^{-\varepsilon_{Jn}})},$$

$$F_{\varepsilon_n}(+\infty,\ldots,+\infty,\varepsilon_{in},+\infty,\ldots,+\infty) = \exp\left(-e^{-\mu(\varepsilon_{in}-\eta)}\right).$$

- Four properties (actually, three).
- Valid CDF.
- Marginals: univariate extreme value distribution.
- We have a multivariate extreme value distribution.

MEV: choice model

$$F_{\varepsilon_n}(\varepsilon_{1n}, \ldots, \varepsilon_{Jn}) = e^{-G(e^{-\varepsilon_{1n}}, \ldots, e^{-\varepsilon_{Jn}})},$$

$$P_n(i) = \int_{\varepsilon=-\infty}^{+\infty} \frac{\partial F_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{Jn}}}{\partial \varepsilon_i}(\ldots, V_{in} - V_{(i-1)n} + \varepsilon, \varepsilon, V_{in} - V_{(i+1)n} + \varepsilon, \ldots) d\varepsilon.$$

As G is μ-homogeneous, $G_i = \partial G / \partial y_i$ is $\mu - 1$-homogeneous and

$$\frac{\partial F_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{Jn}}}{\partial \varepsilon_i}(\ldots, V_{in} - V_{(i-1)n} + \varepsilon, \varepsilon, V_{in} - V_{(i+1)n} + \varepsilon, \ldots)$$

$$= e^{-\varepsilon} G_i(\ldots, e^{-V_{in} + V_{(i-1)n} - \varepsilon}, e^{-\varepsilon}, e^{-V_{in} + V_{(i+1)n} - \varepsilon}, \ldots)$$
$$\exp\left(-G(\ldots, e^{-V_{in} + V_{(i-1)n} - \varepsilon}, e^{-\varepsilon}, e^{-V_{in} + V_{(i+1)n} - \varepsilon}, \ldots)\right)$$

$$= e^{-\varepsilon} e^{-(\mu-1)\varepsilon} e^{-(\mu-1)V_{in}} G_i(\ldots, e^{V_{(i-1)n}}, e^{V_{in}}, e^{V_{(i+1)n}}, \ldots)$$
$$\exp\left(-e^{-\mu\varepsilon} e^{-\mu V_{in}} G(\ldots, e^{V_{(i-1)n}}, e^{V_{in}}, e^{V_{(i+1)n}}, \ldots)\right) .$$

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MEV: choice model

We now denote

$$e^V = \left(\ldots, e^{V_{(i-1)n}}, e^{V_{in}}, e^{V_{(i+1)n}}, \ldots\right),$$

and simplify the terms to obtain

$$\frac{\partial F_{\varepsilon_{1n},\varepsilon_{2n},\ldots,\varepsilon_{J_n}}}{\partial \varepsilon_i}(\ldots, V_{in} - V_{(i-1)n} + \varepsilon, \varepsilon, V_{in} - V_{(i+1)n} + \varepsilon, \ldots)$$

$$= e^{-\mu\varepsilon} e^{-\mu V_{in}} e^{V_{in}} G_i(e^V) \exp\left(-e^{-\mu\varepsilon} e^{-\mu V_{in}} G(e^V)\right).$$

Therefore,

$$P_n(i) = e^{-\mu V_{in}} e^{V_{in}} G_i(e^V) \int_{\varepsilon=-\infty}^{+\infty} e^{-\mu\varepsilon} \exp\left(-e^{-\mu\varepsilon} e^{-\mu V_{in}} G(e^V)\right) d\varepsilon.$$

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MEV: choice model

$$P_n(i) = e^{-\mu V_{in}} e^{V_{in}} G_i(e^V) \int_{\varepsilon=-\infty}^{+\infty} e^{-\mu\varepsilon} \exp\left(-e^{-\mu\varepsilon} e^{-\mu V_{in}} G(e^V)\right) d\varepsilon.$$

Define $t = -\exp(-\mu\varepsilon)$, so that $dt = \mu \exp(-\mu\varepsilon) d\varepsilon$:

$$P_n(i) = e^{-\mu V_{in}} e^{V_{in}} G_i(e^V) \frac{1}{\mu} \int_{\varepsilon=-\infty}^{0} \exp\left(t e^{-\mu V_{in}} G(e^V)\right) dt,$$

which simplifies to

$$P_n(i) = \frac{e^{V_{in}} G_i(e^V)}{\mu G(e^V)}.$$

From Euler's theorem:

$$P_n(i) = \frac{e^{V_{in} + \log G_i(e^V)}}{\sum_j e^{V_{jn} + \log G_j(e^V)}}.$$

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MEV: choice model

The multivariate extreme value model:

$$P_n(i) = \frac{e^{V_{in} + \log G_i\left(e^V\right)}}{\sum_j e^{V_{jn} + \log G_j(e^V)}}.$$

where $G_i = \partial G / \partial y_i$, and G verifies

- (i) the limit property: $G(y_{1n}, \ldots, +\infty, \ldots, y_{Jn}) = +\infty$.
- (ii) the strong alternating sign property: for any set of $\widehat{J}_n$ distinct indices $i_1, \ldots, i_{\widehat{J}_n}$,

$$(-1)^{\widehat{J}_n - 1} G_{i_1, \ldots, i_{\widehat{J}_n}} \geq 0.$$

- (iii) the homogeneity property:

$$G(\alpha y) = \alpha^\mu G(y), \; \forall \alpha > 0 \text{ and } y \in \mathbb{R}_+^J.$$

MEV vs GEV

- McFadden introduces the General Extreme Value model (GEV)
- In statistics, a Generalized Extreme Value distribution (Jenkinson, 1955) is a univariate distribution with CDF

$$F_X(x) = \begin{cases} e^{-(1+\xi((x-\mu)/\sigma))^{-1/\xi})} & -\infty < x \leq \mu - \sigma/\xi \quad \text{for } \xi < 0 \\ & \mu - \sigma/\xi \leq x < \infty \quad \text{for } \xi > 0 \\ e^{-e^{-(x-\mu)/\sigma}} & -\infty < x < \infty \quad \text{for } \xi = 0 \end{cases}$$

- $\xi = 0$ Type 1 EV distribution
  $\xi > 0$ Type 2 EV distribution
  $\xi < 0$ Type 3 EV distribution

MEV models

Example: $G(y) = \sum_{i=1}^{J} y_i^{\mu}$

1. $G(\alpha y) = \sum_{i=1}^{J} (\alpha y_i)^{\mu} = \alpha^{\mu} \sum_{i=1}^{J} y_i^{\mu} = \alpha^{\mu} G(y)$
2. $\lim_{y_i \to +\infty} G(y) = +\infty$, $i = 1, \ldots, J$
3. $\frac{\partial G}{\partial y_i} = \mu y_i^{\mu-1}$ and $\frac{\partial^2 G}{\partial y_i \partial y_j} = 0$

G complies with the theory

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MEV models

Example: $G(y) = \sum_{i=1}^{J} y_i^{\mu}$

$$
\begin{aligned}
F(\varepsilon_1, \ldots, \varepsilon_J) &= e^{-G(e^{-\varepsilon_1}, \ldots, e^{-\varepsilon_J})} \\
&= e^{-\sum_{i=1}^{J} e^{-\mu\varepsilon_i}} \\
&= \prod_{i=1}^{J} e^{-e^{-\mu\varepsilon_i}}
\end{aligned}
$$

Product of i.i.d EV

Logit Model

MEV models

Example: $G(e^{V_1},\ldots,e^{V_J})=\sum_{i=1}^{J} e^{\mu V_i}$

$$P(i)=\frac{e^{V_i+\ln G_i(e^{V_1},\ldots,e^{V_J})}}{\sum_{j\in C} e^{V_j+\ln G_j(e^{V_1},\ldots,e^{V_J})}}$$ with $G_i(x)=\mu x_i^{\mu-1}$

$$\begin{aligned} e^{V_i+\ln G_i(e^{V_1},\ldots,e^{V_J})} &= e^{V_i+\ln\mu+(\mu-1)\ln e^{V_i}} \\ &= e^{\ln\mu+\mu V_i} \end{aligned}$$

$$P(i)=\frac{e^{\ln\mu+\mu V_i}}{\sum_{j\in C} e^{\ln\mu+\mu V_j}}=\frac{e^{\mu V_i}}{\sum_{j\in C} e^{\mu V_j}}$$

MEV models

Example: $G(e^{V_1},\ldots,e^{V_J}) = \sum_{i=1}^{J} e^{\mu V_i}$

$$
\begin{aligned}
V_C &= \frac{1}{\mu}\left(\ln G(e^{V_1},\ldots,e^{V_J}) + \gamma\right) \\
&= \frac{1}{\mu}\ln\sum_{i=1}^{J} e^{\mu V_i} + \frac{\gamma}{\mu}
\end{aligned}
$$

Remember the NL formulation?

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MEV models

Example: Nested logit

$$G(y) = \sum_{m=1}^{M} \left(\sum_{i=1}^{J_m} y_i^{\mu_m} \right)^{\frac{\mu}{\mu_m}}$$

1. $G(\alpha y) = \sum_{m=1}^{M} \left(\sum_{i=1}^{J_m} (\alpha y_i)^{\mu_m} \right)^{\frac{\mu}{\mu_m}} = \alpha^{\mu} \sum_{m=1}^{M} \left(\sum_{i=1}^{J_m} y_i^{\mu_m} \right)^{\frac{\mu}{\mu_m}}$
2. $\lim_{y_i \to +\infty} G(y) = +\infty$, $i = 1, \ldots, J$

MEV models

Example: $G(y) = \sum_{m=1}^{M} \left(\sum_{i=1}^{J_m} y_i^{\mu_m} \right)^{\frac{\mu}{\mu_m}}$

3.

$$\frac{\partial G}{\partial y_i} = \frac{\mu}{\mu_m} \mu_m y_i^{\mu_m - 1} \left(\sum_{i=1}^{J_m} y_i^{\mu_m} \right)^{\frac{\mu}{\mu_m} - 1} \geq 0$$

$$\frac{\partial^2 G}{\partial y_i y_j} = \mu \mu_m y_i^{\mu_m - 1} y_j^{\mu_m - 1} (\frac{\mu}{\mu_m} - 1) \left(\sum_{i=1}^{J_m} y_i^{\mu_m} \right)^{\frac{\mu}{\mu_m} - 2} \leq 0$$

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MEV models

- The logit model is a MEV model
- The nested logit model is also a MEV model

$$G(y) = \sum_{m=1}^{M} \left(\sum_{i=1}^{J_m} y_i^{\mu_m} \right)^{\frac{\mu}{\mu_m}}$$

- If $\frac{\mu}{\mu_m} \leq 1$, then G complies with the theory
- Are there other such models?

Cross-Nested logit model

- MEV model with

$$G(y_1,\ldots,y_J)=\sum_{m=1}^{M}\left(\sum_{j}({\alpha_{jm}}^{1/\mu}y_j)^{\mu_m}\right)^{\frac{\mu}{\mu_m}},$$

  with $\frac{\mu}{\mu_m}\leq 1$, $\alpha_{jm}\geq 0$, and $\forall j, \exists m$ s.t. $\alpha_{jm}>0$
- Generalization of the nested-logit model

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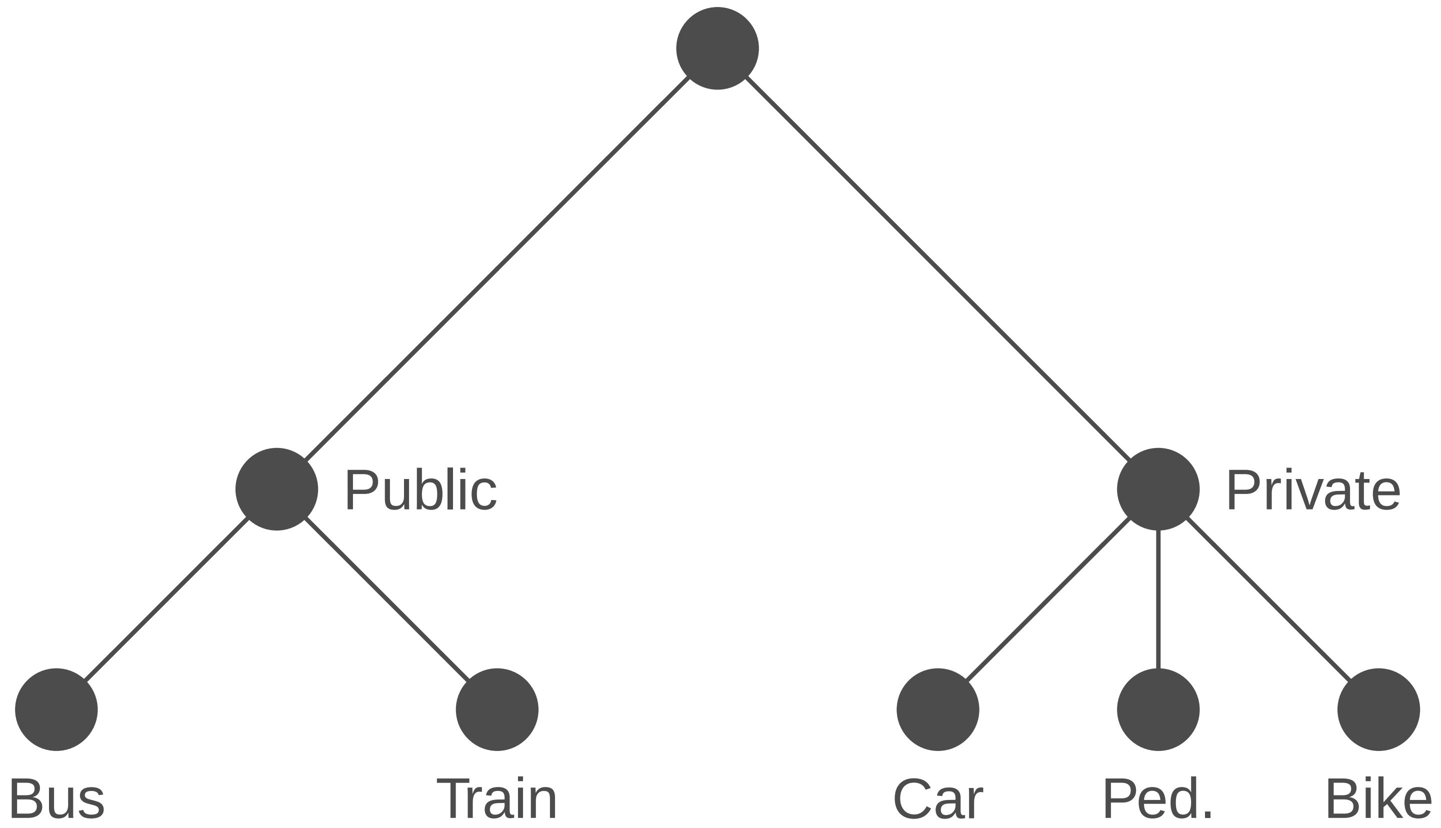
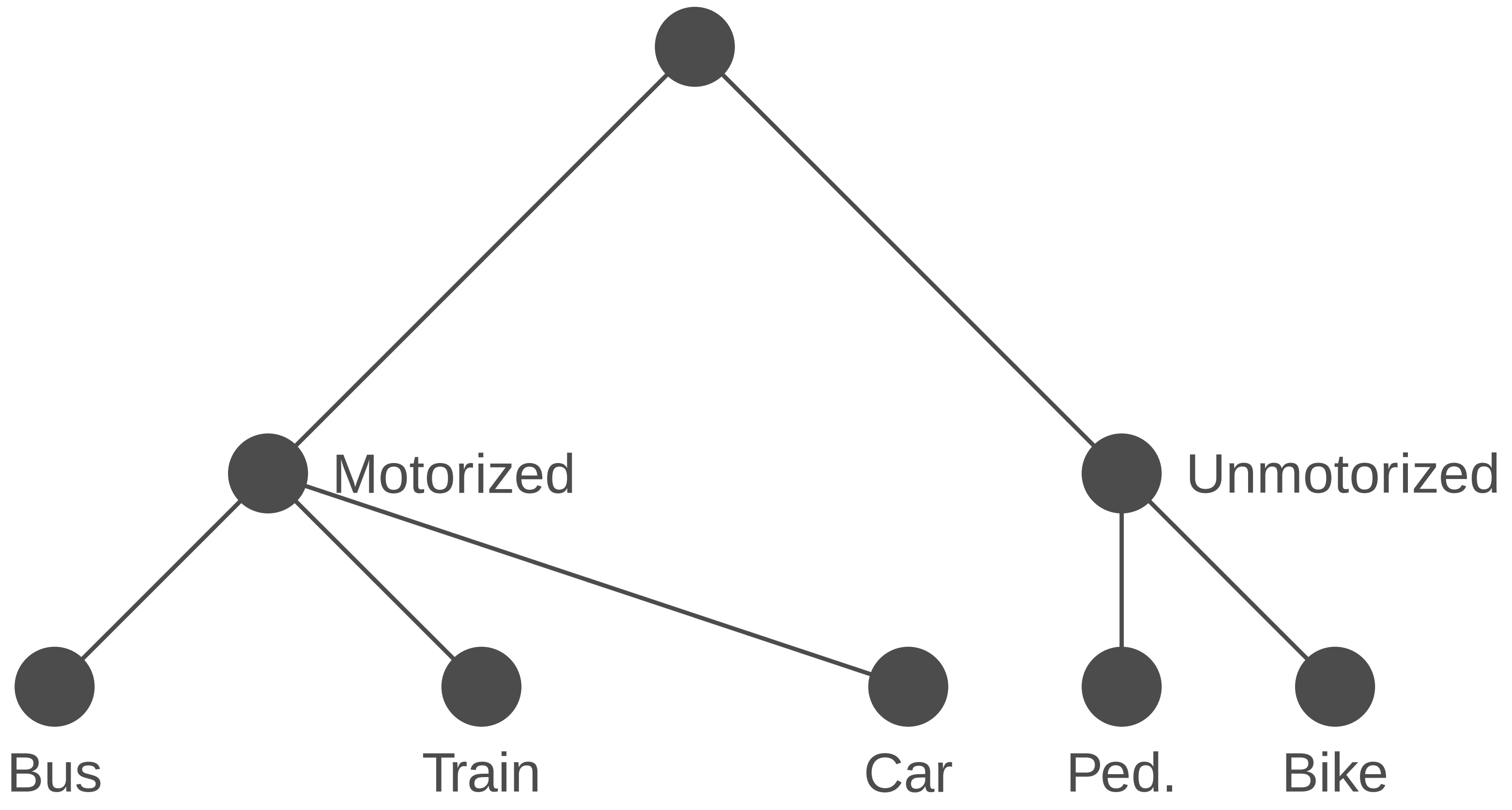
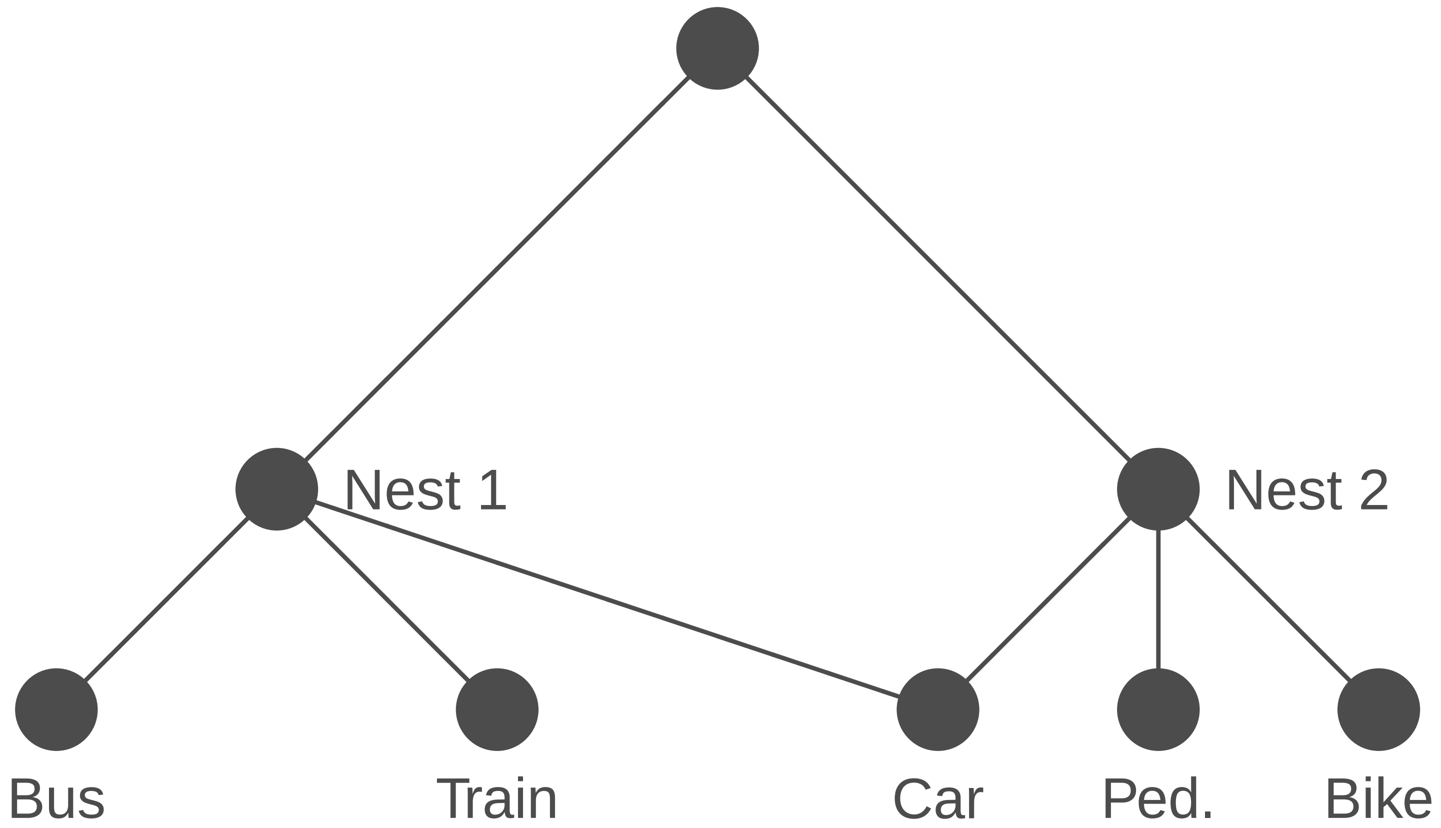
Nested Logit Model



Nested Logit Model



Cross-Nested Logit Model



Cross-Nested Logit Model

$$P(i|\mathcal{C}) = \sum_{m=1}^{M} \frac{\left(\sum_{j\in\mathcal{C}} \alpha_{jm}^{\mu_m/\mu} e^{\mu_m V_j}\right)^{\frac{\mu}{\mu_m}}}{\sum_{n=1}^{M} \left(\sum_{j\in\mathcal{C}} \alpha_{jn}^{\mu_n/\mu} e^{\mu_n V_j}\right)^{\frac{\mu}{\mu_n}}} \frac{\alpha_{im}^{\mu_m/\mu} e^{\mu_m V_i}}{\sum_{j\in\mathcal{C}} \alpha_{jm}^{\mu_m/\mu} e^{\mu_m V_j}}.$$

which can nicely be interpreted as

$$P(i|\mathcal{C}) = \sum_{m} P(m|\mathcal{C}) P(i|m).$$

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MEV models

- Provide a great deal of flexibility
- Require significant imagination
- Require heavy proofs

Network MEV

Daly & Bierlaire (2006)
Motivations:

- Extension of the tree representation for Nested Logit
- Investigate new MEV models
- Provide the proof once for all

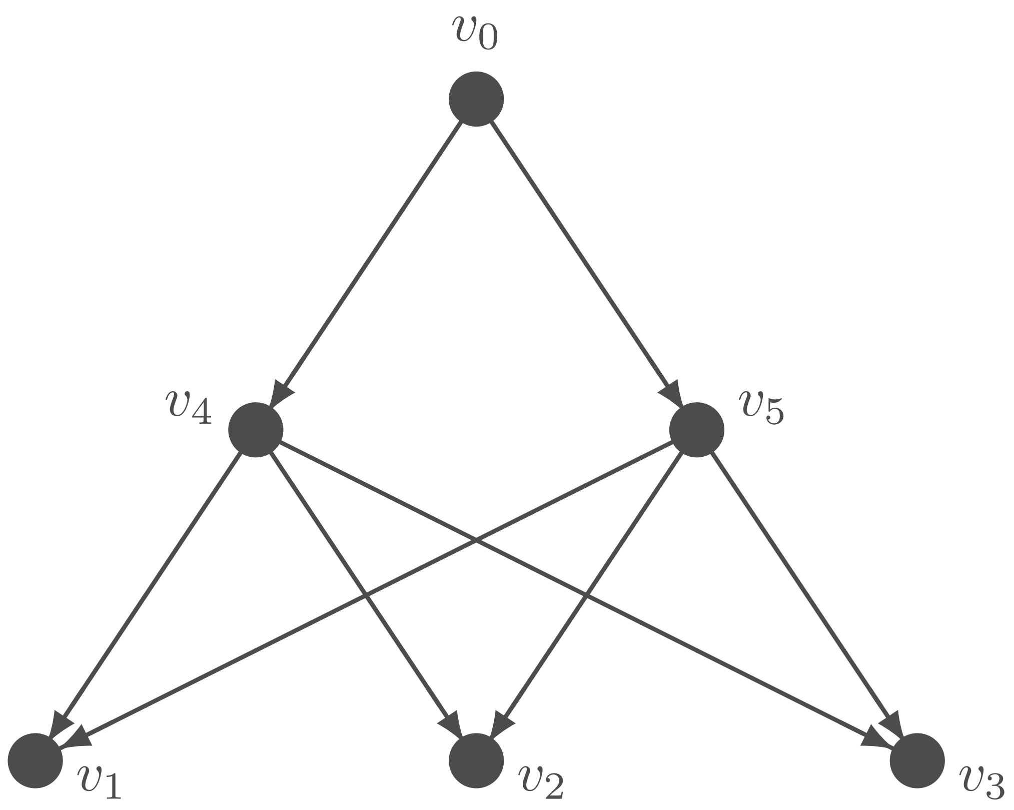
Network MEV

Let (V, E) be a network with link parameters $\alpha_{(i,j)} \geq 0$

Assumptions:

1. No circuit.
2. One node without predecessor: *root*.
3. J nodes without successor: *alternatives*.
4. For each node v_i, there exists at least one path from the root to v_i such that $\prod_{k=1}^{P} \alpha_{(i_{k-1}, i_k)} > 0$.

Network MEV



Network MEV

For each node v_i, we define

- a set of indices $I_i \subseteq \{1, \ldots, J\}$ of J_i relevant alternatives,
- a homogeneous function $G^i : \mathbb{R}^{J_i} \longrightarrow \mathbb{R}$, and
- a parameter μ_i.

Recursive definition of I_i:

- $I_i = \{i\}$ for alternatives,
- $I_i = \bigcup_{j \in \text{succ}(i)} I_j$ for all other nodes.

Network MEV

Recursive definition of G^i:

For alternatives:

$$G^i \quad : \quad \mathbb{R} \longrightarrow \mathbb{R} \quad : \quad G^i(y_i) = y_i^{\mu_i} \quad i = 1, \ldots, J$$

For all others:

$$G^i \quad : \quad \mathbb{R}^{J_i} \longrightarrow \mathbb{R} \quad : \quad G^i(y) = \sum_{j \in \text{SUCC}(i)} \alpha_{(i,j)} G^j(y)^{\frac{\mu_i}{\mu_j}}$$

Theorem

If all $G^j(y)$ are MEV generating functions, so is G^i

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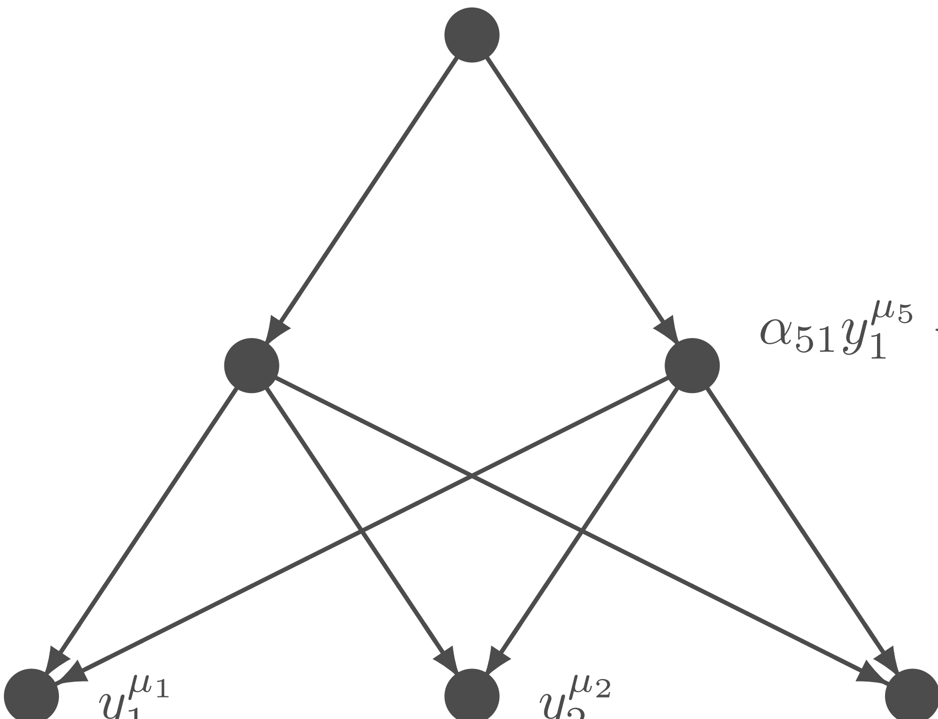
Network MEV

Example: Cross-Nested Logit

$$G = \sum_m \left(\sum_{j \in C} \alpha_{jm} y_j^{\mu_m} \right)^{\frac{\mu}{\mu_m}}$$

$\sum_{i=4,5} \alpha_{0i} \left(\alpha_{i1} y_1^{\mu_i} + \alpha_{i2} y_2^{\mu_i} + \alpha_{i3} y_3^{\mu_i} \right)^{\frac{\mu_0}{\mu_i}}$



$\alpha_{51} y_1^{\mu_5} + \alpha_{52} y_2^{\mu_5} + \alpha_{53} y_3^{\mu_5}$

$y_1^{\mu_1}$ $y_2^{\mu_2}$ $y_3^{\mu_3}$

Summary

- Need to relax the independence assumption
- Probit
- MEV family
- CNL