Logit with multiple alternatives

Michel Bierlaire

michel.bierlaire@epfl.ch

Transport and Mobility Laboratory

Logit Model

For all $i \in \mathcal{C}_n$,

$$U_{in} = V_{in} + \varepsilon_{in}$$

- What is $\mathcal{C}_n$?
- What is ε_{in}?
- What is V_{in}?

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Choice set

Universal choice set

- All potential alternatives for the population
- Restricted to relevant alternatives

Mode choice:

driving alone	sharing a ride	taxi
motorcycle	bicycle	walking
transit bus	rail rapid transit	horse

Choice set

Individual's choice set

- No driver license
- No auto available
- Awareness of transit services
- Transit services unreachable
- Walking not an option for long distance

Choice set

Individual's choice set

Choice set generation is tricky

- How to model "awareness"?
- What does "long distance" exactly mean?
- What does "unreachable" exactly mean?

We assume here deterministic rules

Derivation of the logit model

Main assumption: ε_{in} are

- extreme value EV(0,μ),
- independent and
- identically distributed.

Comments:

- Independence: across i and n.
- Identical distribution: same scale parameter μ across i and n.

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Derivation of the logit model

Reminder: binary case

- $\mathcal{C}_n = \{i, j\}$
- $U_{in} = V_{in} + \varepsilon_{in}$
- $\varepsilon_{in} \sim \text{EV}(0, \mu)$
- Probability

$$P(i|\mathcal{C}_n = \{i, j\}) = \frac{e^{\mu V_{in}}}{e^{\mu V_{in}} + e^{\mu V_{jn}}}$$

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Derivation of the logit model

- $\mathcal{C}_n = \{1, \ldots, J_n\}$
- $U_{in} = V_{in} + \varepsilon_{in}$
- $\varepsilon_{in} \sim \text{EV}(0, \mu)$
- ε_{in} i.i.d.
- Probability

$$P(i|\mathcal{C}_n) = P(V_{in} + \varepsilon_{in} \geq \max_{j=1,\ldots,J_n} V_{jn} + \varepsilon_{jn})$$

- Assume without loss of generality (wlog) that $i = 1$

$$P(1|\mathcal{C}_n) = P(V_{1n} + \varepsilon_{1n} \geq \max_{j=2,\ldots,J_n} V_{jn} + \varepsilon_{jn})$$

Derivation of the logit model

- Define a composite alternative: “anything but one”
- Associated utility:

$$U^* = \max_{j=2,\ldots,J_n} (V_{jn} + \varepsilon_{jn})$$

- From a property of the EV distribution

$$U^* \sim \text{EV}\left(\frac{1}{\mu} \ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}, \mu\right)$$

Derivation of the logit model

- From another property of the EV distribution

$$U^* = V^* + \varepsilon^*$$

where

$$V^* = \frac{1}{\mu} \ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}$$

and

$$\varepsilon^* \sim \text{EV}(0, \mu)$$

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Derivation of the logit model

- Therefore

$$
\begin{aligned}
P(1|\mathcal{C}_n) &= P(V_{1n} + \varepsilon_{1n} \geq \max_{j=2,\ldots,J_n} V_{jn} + \varepsilon_{jn}) \\
&= P(V_{1n} + \varepsilon_{1n} \geq V^* + \varepsilon^*)
\end{aligned}
$$

- This is a binary choice model

$$
P(1|\mathcal{C}_n) = \frac{e^{\mu V_{1n}}}{e^{\mu V_{1n}} + e^{\mu V^*}}
$$

where

$$
V^* = \frac{1}{\mu} \ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}
$$

Derivation of the logit model

- We have $e^{\mu V^*} = e^{\ln \sum_{j=2}^{J_n} e^{\mu V_{jn}}} = \sum_{j=2}^{J_n} e^{\mu V_{jn}}$
- and

$$
\begin{aligned}
P(1|\mathcal{C}_n) &= \frac{e^{\mu V_{1n}}}{e^{\mu V_{1n}} + e^{\mu V^*}} \\
&= \frac{e^{\mu V_{1n}}}{e^{\mu V_{1n}} + \sum_{j=2}^{J_n} e^{\mu V_{jn}}} \\
&= \frac{e^{\mu V_{1n}}}{\sum_{j=1}^{J_n} e^{\mu V_{jn}}}
\end{aligned}
$$

Derivation of the logit model

- The scale parameter μ is not identifiable: $\mu = 1$.
- Warning: not identifiable $\neq$ not existing
- $\mu \to 0$, that is variance goes to infinity

$$\lim_{\mu \to 0} P(i|C_n) = \frac{1}{J_n} \quad \forall i \in \mathcal{C}_n$$

- $\mu \to +\infty$, that is variance goes to zero

$$\begin{aligned} \lim_{\mu\to\infty} P(i|C_n) &= \lim_{\mu\to\infty} \frac{1}{1+\sum_{j\neq i} e^{\mu(V_{jn}-V_{in})}} \\ &= \begin{cases} 1 & \text{if } V_{in} > \max_{j\neq i} V_{jn} \\ 0 & \text{if } V_{in} < \max_{j\neq i} V_{jn} \end{cases} \end{aligned}$$

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Derivation of the logit model

- $\mu \to +\infty$, that is variance goes to zero (ctd.)
- What if there are ties?
- $V_{in} = \max_{j \in \mathcal{C}_n} V_{jn}$, $i = 1, \ldots, J_n^*$
- Then

$$P(i|\mathcal{C}_n) = \frac{1}{J_n^*} \quad i = 1, \ldots, J_n^*$$

and

$$P(i|\mathcal{C}_n) = 0 \quad i = J_n^* + 1, \ldots, J_n$$

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Systematic part of the utility function

$$V_{in} = V(z_{in}, S_n)$$

where

- z_{in} is a vector of attributes of alternative i for individual n
- S_n is a vector of socio-economic characteristics of n

Outline:

- Functional form: linear utility
- Explanatory variables: What is exactly contained in z_{in} and S_n?
- Functional form: capturing nonlinearities
- Interactions

Functional form: linear utility

Notation:

$$x_{in} = (z_{in}, S_n)$$

In general, linear-in-parameters utility functions are used

$$V_{in} = V(z_{in}, S_n) = V(x_{in}) = \sum_p \beta_p (x_{in})_p$$

Not as restrictive as it may seem

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Explanatory variables: alternatives attributes

- Numerical and continuous
- $(z_{in})_p \in \mathbb{R}, \forall i, n, p$
- Associated with a specific unit

Examples:

- Auto in-vehicle time (in min.)
- Transit in-vehicle time (in min.)
- Auto out-of-pocket cost (in cents)
- Transit fare (in cents)
- Walking time to the bus stop (in min.)

Straightforward modeling

Explanatory variables: alternatives attributes

- V_{in} is unitless
- Therefore, β depends on the unit of the associated attribute
- Example: consider two specifications

$$
\begin{aligned}
V_{in} &= \beta_1 \mathsf{TT}_{in} + \cdots \\
V_{in} &= \beta_1' \mathsf{TT}_{in}' + \cdots
\end{aligned}
$$

- If TT_{in} is a number of minutes, the unit of β_1 is 1/min
- If TT_{in}' is a number of hours, the unit of β_1' is 1/hour
- Both models are equivalent, but the estimated value of the coefficient will be different

$$\beta_1 \mathsf{TT}_{in} = \beta_1' \mathsf{TT}_{in}' \Longrightarrow \frac{\mathsf{TT}_{in}}{\mathsf{TT}_{in}'} = \frac{\beta_1'}{\beta_1} = 60$$

Explanatory variables: alternatives attributes

Generic and alternative specific parameters

$$\begin{aligned} V_{\text{auto}} &= \beta_1 \text{TT}_{\text{auto}} \\ V_{\text{bus}} &= \beta_1 \text{TT}_{\text{bus}} \end{aligned}$$

or

$$\begin{aligned} V_{\text{auto}} &= \beta_1 \text{TT}_{\text{auto}} \\ V_{\text{bus}} &= \beta_2 \text{TT}_{\text{bus}} \end{aligned}$$

Modeling assumption: a minute has/has not the same marginal utility whether it is incurred on the auto or bus mode

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Explanatory variables: socio-eco. characteristics

- Numerical and continuous
- $(S_n)_p \in \mathbb{R}, \forall n, p$
- Associated with a specific unit

Examples:

- Annual income (in KCHF)
- Age (in years)

Warning: S_n do not depend on i

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Explanatory variables: socio-eco. characteristics

They cannot appear in all utility functions

$$\left.\begin{array}{rcl} V_1 & = & \beta_1 x_{11} + \beta_2 \text{income} \\ V_2 & = & \beta_1 x_{21} + \beta_2 \text{income} \\ V_3 & = & \beta_1 x_{31} + \beta_2 \text{income} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} V_1' & = & \beta_1 x_{11} \\ V_2' & = & \beta_1 x_{21} \\ V_3' & = & \beta_1 x_{31} \end{array}\right.$$

In general: alternative specific characteristics

$$\begin{array}{rcll} V_1 & = & \beta_1 x_{11} + \beta_2 \text{income} & + \beta_4 \text{age} \\ V_2 & = & \beta_1 x_{21} + \beta_3 \text{income} & + \beta_5 \text{age} \\ V_3 & = & \beta_1 x_{31} & \end{array}$$

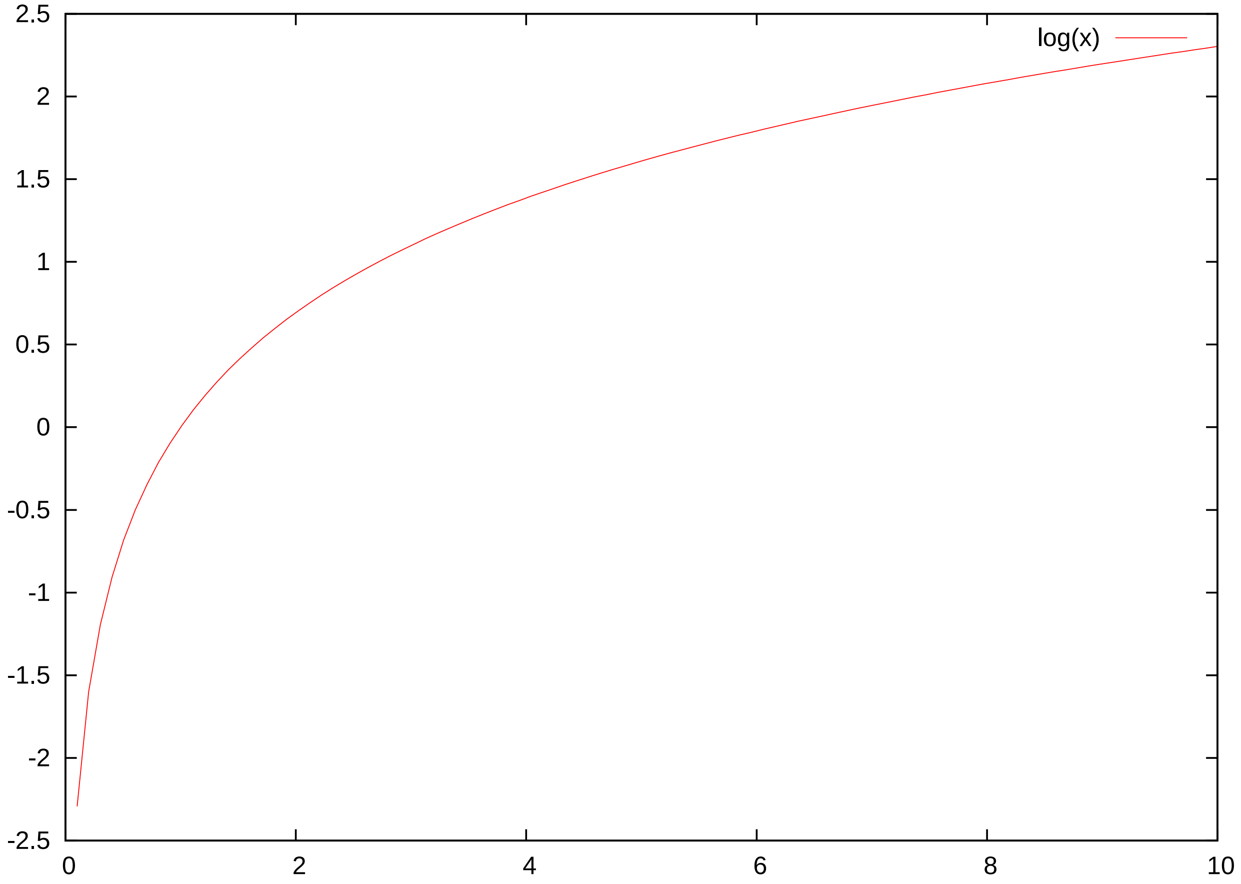
Functional form: dealing with nonlinearities

- Nonlinear transformations of the independent variables
- Discrete and qualitative variables
- Continuous variables
  - Categories
  - Splines
  - Box-Cox
  - Power series

Nonlinear transformations of the variables

- Compare a trip of 5 min with a trip of 10 min
- Compare a trip of 120 min with a trip of 125 min

Nonlinear transformations of the variables



Nonlinear transformations of the variables

Instead of

$$V_i = \beta \text{time}_i$$

one can use

$$V_i = \beta \ln(\text{time}_i)$$

It is still a linear-in-parameters form

Nonlinear transformations of the variables

Another example: disposable income

$$\max(\text{household income}(\$/\text{year}) - s \times \text{nbr of persons}, 0)$$

where s is the subsistence budget per person

$\boxed{\text{Data can be preprocessed to account for nonlinearities}}$

$$V_{in} = V(h(z_{in}, S_n)) = \sum_k \beta_k (h(z_{in}, S_n))_k$$

is linear-in-parameter, even with h nonlinear.

Discrete variables

- Mainly used to capture qualitative attributes
  - Level of comfort for the train
  - Reliability of the bus
  - Color
  - Shape
  - etc...
- or characteristics
  - Sex
  - Education
  - Professional status
  - etc.

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Discrete variables

Procedure for model specification:

- Identify all possible levels of the attribute: Very comfortable, Comfortable, Rather comfortable, Not comfortable.
- Select a base case: very comfortable
- Define numerical attributes
- Adopt a coding convention

Discrete variables

Numerical attributes

Introduce a 0/1 attribute for all levels except the base case

- z_{c} for *comfortable*
- z_{rc} for *rather comfortable*
- z_{nc} for *not comfortable*

Discrete variables

Coding convention

	z_c	z_{rc}	z_{nc}
very comfortable	0	0	0
comfortable	1	0	0
rather comfortable	0	1	0
not comfortable	0	0	1

If a qualitative attribute has n levels, we introduce $n-1$ variables (0/1) in the model

Discrete variables

Comparing two ways of coding:

	z_{vc}	z_{c}	z_{rc}	z_{nc}
very comfortable	1	0	0	0
comfortable	0	1	0	0
rather comfortable	0	0	1	0
not comfortable	0	0	0	1

$$
\begin{array}{lllllllr}
V_{in} & = & \cdots & +\beta_{\text{vc}} z_{i\text{vc}} & +\beta_{\text{c}} z_{i\text{c}} & +\beta_{\text{rc}} z_{i\text{rc}} & +\beta_{\text{nc}} z_{i\text{nc}} & \beta_{\text{vc}} = 0 \\
V'_{in} & = & \cdots & +\beta'_{\text{vc}} z_{i\text{vc}} & +\beta'_{\text{c}} z_{i\text{c}} & +\beta'_{\text{rc}} z_{i\text{rc}} & +\beta'_{\text{nc}} z_{i\text{nc}} & \beta'_{\text{c}} = 0
\end{array}
$$

Linear-in-parameter specification

Let's add a constant to all β's

Discrete variables

$$
\begin{array}{lllllllll}
V_{in} & = & \cdots & +\beta_{\text{vc}} z_{i\text{vc}} & +\beta_{\text{c}} z_{i\text{c}} & +\beta_{\text{rc}} z_{i\text{rc}} & +\beta_{\text{nc}} z_{i\text{nc}} & \beta_{\text{vc}} = 0 \\
V'_{in} & = & \cdots & +\beta'_{\text{vc}} z_{i\text{vc}} & +\beta'_{\text{c}} z_{i\text{c}} & +\beta'_{\text{rc}} z_{i\text{rc}} & +\beta'_{\text{nc}} z_{i\text{nc}} & \beta'_{\text{c}} = 0
\end{array}
$$

$$
\begin{aligned}
V_{in} &= \cdots + (\beta_{\text{vc}} + K) z_{i\text{vc}} + (\beta_{\text{c}} + K) z_{i\text{c}} + (\beta_{\text{rc}} + K) z_{i\text{rc}} + (\beta_{\text{nc}} + K) z_{i\text{nc}} \\
&= \cdots + \beta_{\text{vc}} z_{i\text{vc}} + \beta_{\text{c}} z_{i\text{c}} + \beta_{\text{rc}} z_{i\text{rc}} + \beta_{\text{nc}} z_{i\text{nc}} + K(z_{i\text{vc}} + z_{i\text{c}} + z_{i\text{rc}} + z_{i\text{nc}}) \\
&= \cdots + \beta_{\text{vc}} z_{i\text{vc}} + \beta_{\text{c}} z_{i\text{c}} + \beta_{\text{rc}} z_{i\text{rc}} + \beta_{\text{nc}} z_{i\text{nc}} + K
\end{aligned}
$$

- $K = -\beta_{\text{vc}}$: very comfortable as the base case
- $K = -\beta_{\text{c}}$: comfortable as the base case
- $K = -\beta_{\text{rc}}$: rather comfortable as the base case
- $K = -\beta_{\text{nc}}$: not comfortable as the base case

Discrete variables

Example of estimation with Biogeme:

	Model 1	Model 2
ASC	0.574	0.574
BETA_VC	0.000	0.918
BETA_C	-0.919	0.000
BETA_RC	-1.015	-0.096
BETA_NC	-2.128	-1.210

Continuous variables: categories

- Assumption: sensitivity to travel time varies with travel time
- Using βTT is not appropriate anymore
- Categories are defined: travel time in minutes

$$[0\text{–}90[,\ [90\text{–}180[,\ [180\text{–}270[,\ [270\text{–}\ [$$

- Solutions:
  - Categories with constants (inferior solution)
  - Piecewise linear specification (spline)

Continuous variables: categories

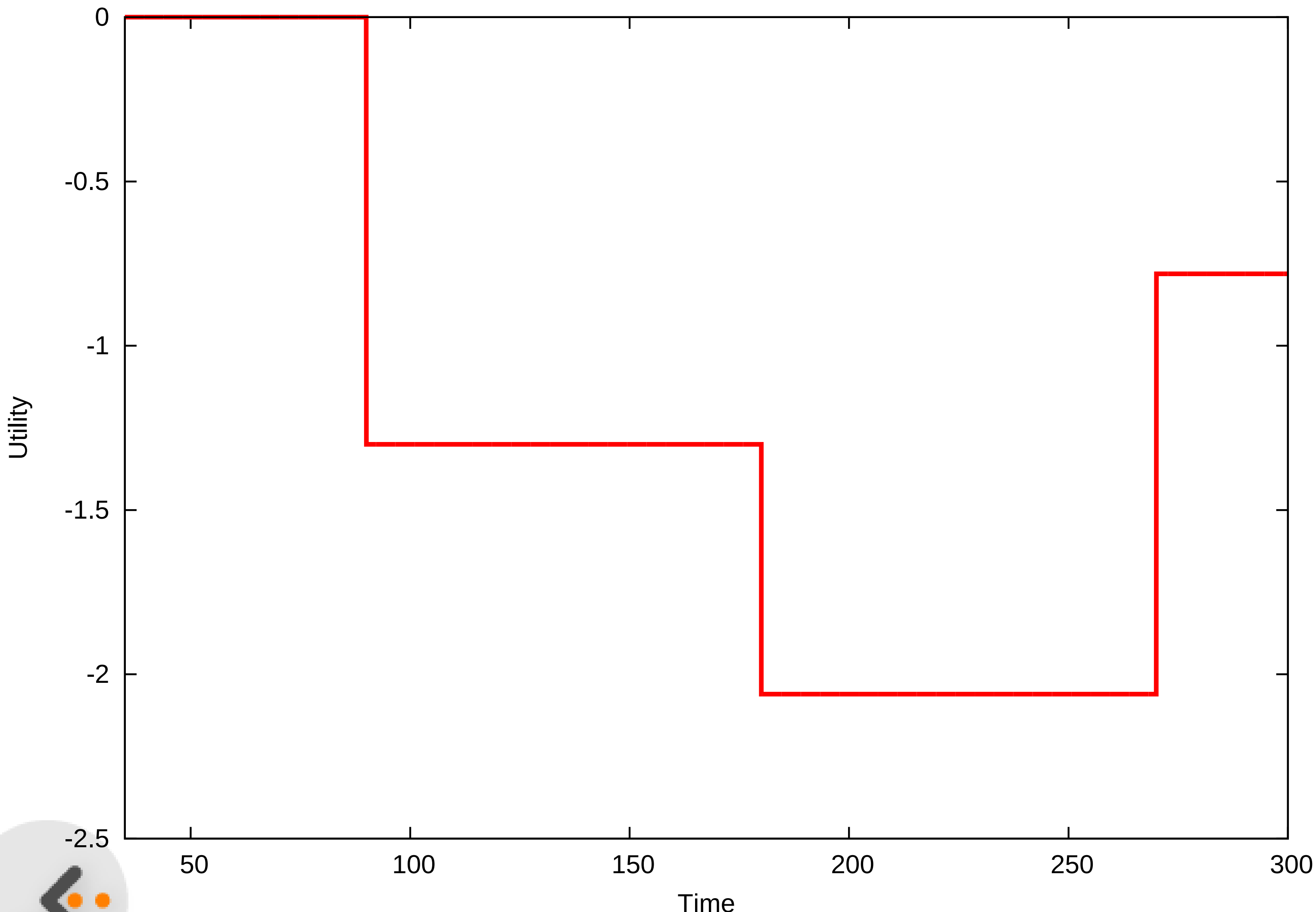
Categories with constants

- Same specification as for discrete variables

$$V_i = \beta_{T1} x_{T1} + \beta_{T2} x_{T2} + \beta_{T3} x_{T3} + \beta_{T4} x_{T4} + \ldots$$

- with
  - $x_{T1} = 1$ if $TT_i \in$ [0–90[, 0 otherwise
  - $x_{T2} = 1$ if $TT_i \in$ [90–180[, 0 otherwise
  - $x_{T3} = 1$ if $TT_i \in$ [180–270[, 0 otherwise
  - $x_{T4} = 1$ if $TT_i \in$ [270–[, 0 otherwise
- One β must be normalized to 0.

Continuous variables: categories



Continuous variables: categories

Drawbacks

- No sensitivity to travel time within the intervals
- Discontinuous utility function (jumps)
- Need for many small intervals
- Results may vary significantly with the definition of the intervals

Appropriate when

- Categories have been used in the survey (income, age)
- Definition of categories is natural (weekday)

Continuous variables: categories

Piecewise linear specification (spline)

- Capture the sensitivity within the intervals
- Enforce continuity of the utility function

Piecewise linear specification

- Specification:

$$V_i = \beta_{T1} x_{T1} + \beta_{T2} x_{T2} + \beta_{T3} x_{T3} + \beta_{T4} x_{T4} + \ldots$$

where

$$x_{T1} = \begin{cases} t & \text{if } t < 90 \\ 90 & \text{otherwise} \end{cases} \qquad x_{T2} = \begin{cases} 0 & \text{if } t < 90 \\ t - 90 & \text{if } 90 \leq t < 180 \\ 90 & \text{otherwise} \end{cases}$$

$$x_{T3} = \begin{cases} 0 & \text{if } t < 180 \\ t - 180 & \text{if } 180 \leq t < 270 \\ 90 & \text{otherwise} \end{cases} \qquad x_{T4} = \begin{cases} 0 & \text{if } t < 270 \\ t - 270 & \text{otherwise} \end{cases}$$

Piecewise linear specification

Note: coding in Biogeme for interval [a-b[

$$x_{Ti} = \begin{cases} 0 & \text{if } t < a \\ t-a & \text{if } a \leq t < a+b \\ b & \text{otherwise} \end{cases} \qquad x_{Ti} = \max(0, \min(t-a, b))$$

$$\begin{aligned} x_{T1} &= \min(t, 90) \\ x_{T2} &= \max(0, \min(t-90, 90)) \\ x_{T3} &= \max(0, \min(t-180, 90)) \\ x_{T4} &= \max(0, t-270) \end{aligned}$$

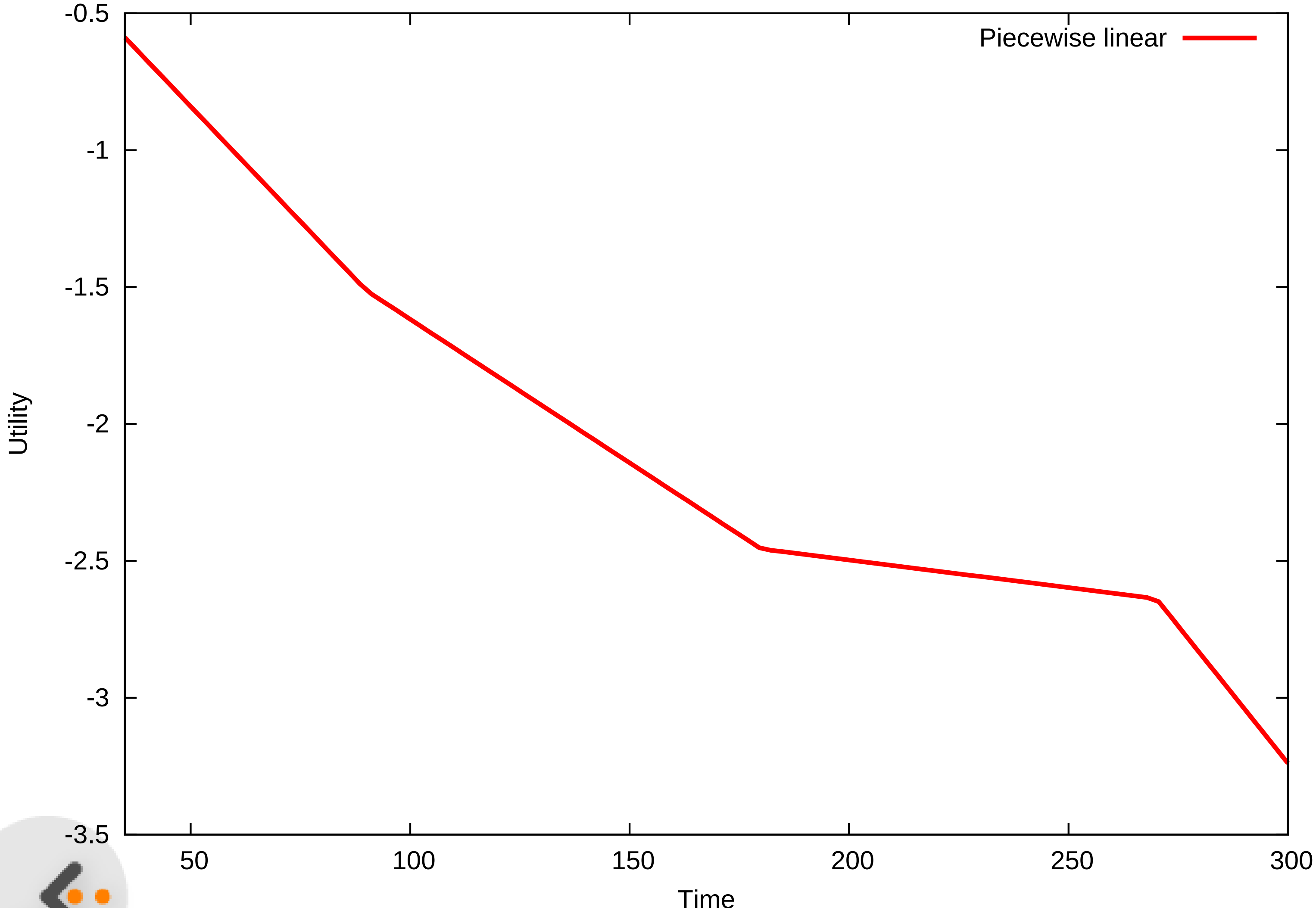
```
TRAIN_TT1  =  min( TRAIN_TT , 90)
TRAIN_TT2  =  max(0,min( TRAIN_TT - 90, 90))
TRAIN_TT3  =  max(0,min( TRAIN_TT - 180 , 90))
TRAIN_TT4  =  max(0,TRAIN_TT - 270)
```

Piecewise linear specification

Examples:

t	TT1	TT2	TT3	TT4
40	40	0	0	0
100	90	10	0	0
200	90	90	20	0
300	90	90	90	30

Piecewise linear specification



Box-Cox transforms

Box and Cox, *J. of the Royal Statistical Society* (1964)

$$V_i = \beta x_i(\lambda) + \cdots$$

where

$$x_i(\lambda) = \begin{cases} \dfrac{x_i^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln x_i & \text{if } \lambda = 0. \end{cases}$$

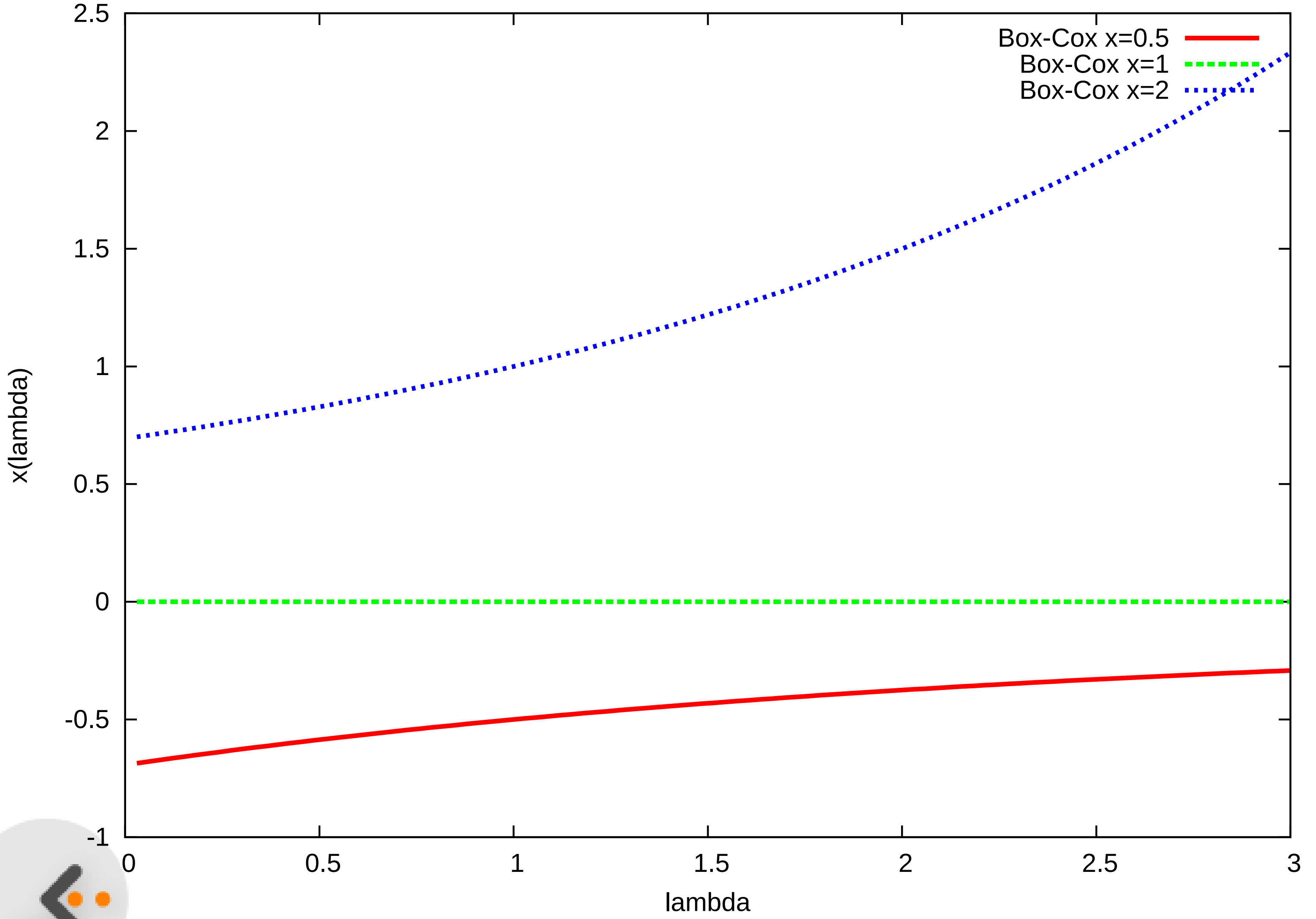
where $x_i > 0$.

Box-Cox transforms

If $x_i \leq 0$, let α such that $x_i + \alpha > 0$ and

$$x_i(\lambda, \alpha) = \begin{cases} \dfrac{(x_i + \alpha)^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \\ \ln(x_i + \alpha) & \text{if } \lambda = 0. \end{cases}$$

Box-Cox transforms



Box-Cox transforms

Other power transforms are possible:

- Manly, *Biometrics* (1971)

$$x_i(\lambda) = \begin{cases} \frac{e^{x_i \lambda} - 1}{\lambda} & \text{if } \lambda \neq 0 \\ x_i & \text{if } \lambda = 0. \end{cases}$$

- John and Draper, *Applied Statistics* (1980)

$$x_i(\lambda) = \begin{cases} \text{sign}(x_i) \frac{(|x_i|+1)^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \text{sign}(x_i) \ln(|x_i| + 1) & \text{if } \lambda = 0. \end{cases}$$

Box-Cox transforms

Other power transforms are possible:

- Yeo and Johnson, *Biometrika* (2000)

$$x_i(\lambda) = \begin{cases} \dfrac{(x_i+1)^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0, x_i \geq 0; \\[2ex] \ln(x_i+1) & \text{if } \lambda = 0, x_i \geq 0; \\[2ex] \dfrac{(1-x_i)^{2-\lambda} - 1}{\lambda - 2} & \text{if } \lambda \neq 2, x_i < 0; \\[2ex] -\ln(1-x_i) & \text{if } \lambda = 2, x_i < 0. \end{cases}$$

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Power series

$$V_i = \beta_1 T + \beta_2 T^2 + \beta_3 T^3 + \ldots$$

- In practice, these terms can be very correlated
- Difficult to interpret
- Risk of over fitting

Interactions

- All individuals in a population are not alike
- Socio-economic characteristics define segments in the population
- How to capture heterogeneity?
  - Interactions of attributes and characteristics
  - Discrete segmentation
  - Continuous segmentation

Interactions of attributes and characteristics

Combination of attributes:

- cost / income
- fare / disposable income
- out-of-vehicle time / distance

> WARNING: correlation of attributes may produce degeneracy in the model

Example: speed and travel time if distance is constant

Interactions: discrete segmentation

- The population is divided into a finite number of segments
- Each individual belongs to exactly one segment
- Example: gender (M,F) and house location (metro, suburb, perimeter areas)
- 6 segments

$$
\begin{array}{l}
\beta_{M,m} TT_{M,m} + \beta_{M,s} TT_{M,s} + \beta_{M,p} TT_{M,p} + \\
\beta_{F,m} TT_{F,m} + \beta_{F,s} TT_{F,s} + \beta_{F,p} TT_{F,p} +
\end{array}
$$

- $TT_i = TT$ if indiv. belongs to segment i, and 0 otherwise

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Interactions: continuous segmentation

- Taste parameter varies with a continuous socio-economic characteristics
- Example: the cost parameter varies with income

$$\beta_{\text{cost}} = \hat{\beta}_{\text{cost}} \left(\frac{\text{inc}}{\text{inc}_{\text{ref}}} \right)^{\lambda} \text{ with } \lambda = \frac{\partial \beta_{\text{cost}}}{\partial \text{inc}} \frac{\text{inc}}{\beta_{\text{cost}}}$$

- Warning: λ must be estimated and utility is not linear-in-parameters anymore
- Reference value is arbitrary
- Several characteristics can be combined:

$$\beta_{\text{cost}} = \hat{\beta}_{\text{cost}} \left(\frac{\text{inc}}{\text{inc}_{\text{ref}}} \right)^{\lambda_1} \left(\frac{\text{age}}{\text{age}_{\text{ref}}} \right)^{\lambda_2}$$

Heteroscedasticity

- Logit is homoscedastic
- ε_{in} i.i.d. across both i and n.
- Assume there are two different groups such that

$$\begin{aligned} U_{in_1} &= V_{in_1} + \varepsilon_{in_1} \\ U_{in_2} &= V_{in_2} + \varepsilon_{in_2} \end{aligned}$$

  and Var(ε_{in_2}) = α^2 Var(ε_{in_1})
- Then we prefer the model

$$\begin{array}{rcrcrcrcl} \alpha U_{in_1} & = & \alpha V_{in_1} & + & \alpha\varepsilon_{in_1} & = & \alpha V_{in_1} & + & \varepsilon'_{in_1} \\ U_{in_2} & = & V_{in_2} & + & \varepsilon_{in_2} & = & V_{in_2} & + & \varepsilon'_{in_2} \end{array}$$

- where ε'_{in_1} and ε'_{in_2} are i.i.d.

Heteroscedasticity

- If V_{in_1} is linear-in-parameters, that is

$$V_{in_1} = \sum_j \beta_j x_{jin_1}$$

then

$$\alpha V_{in_1} = \sum_j \alpha \beta_j x_{jin_1}$$

is nonlinear.

A case study

- Choice of residential telephone services
- Household survey conducted in Pennsylvania, USA, 1984
- Revealed preferences
- 434 observations

A case study

Telephone services and availability

	metro, suburban & some perimeter areas	other perimeter areas	non-metro areas
Budget Measured	yes	yes	yes
Standard Measured	yes	yes	yes
Local Flat	yes	yes	yes
Extended Area Flat	no	yes	no
Metro Area Flat	yes	yes	no

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A case study

Universal choice set

$$\mathcal{C} = \{\text{BM}, \text{SM}, \text{LF}, \text{EF}, \text{MF}\}$$

Specific choice sets

- Metro, suburban & some perimeter areas: {BM,SM,LF,MF}
- Other perimeter areas: $\mathcal{C}$
- Non-metro areas: {BM,SM,LF}

A case study

Specification table

	β_1	β_2	β_3	β_4	β_5
BM	0	0	0	0	$\ln$(cost(BM))
SM	1	0	0	0	$\ln$(cost(SM))
LF	0	1	0	0	$\ln$(cost(LF))
EF	0	0	1	0	$\ln$(cost(EF))
MF	0	0	0	1	$\ln$(cost(MF))

A case study

$$
\begin{array}{rcccl}
V_{\mathsf{BM}} & = & & & \beta_5 \ln(\mathsf{cost}_{\mathsf{BM}}) \\
V_{\mathsf{SM}} & = & \beta_1 & + & \beta_5 \ln(\mathsf{cost}_{\mathsf{SM}}) \\
V_{\mathsf{LF}} & = & \beta_2 & + & \beta_5 \ln(\mathsf{cost}_{\mathsf{LF}}) \\
V_{\mathsf{EF}} & = & \beta_3 & + & \beta_5 \ln(\mathsf{cost}_{\mathsf{EF}}) \\
V_{\mathsf{MF}} & = & \beta_4 & + & \beta_5 \ln(\mathsf{cost}_{\mathsf{MF}})
\end{array}
$$

A case study

Specification table II

	β_1	β_2	β_3	β_4	β_5	β_6	β_7
BM	0	0	0	0	$\ln$(cost(BM))	users	0
SM	1	0	0	0	$\ln$(cost(SM))	users	0
LF	0	1	0	0	$\ln$(cost(LF))	0	1 if metro/suburb
EF	0	0	1	0	$\ln$(cost(EF))	0	0
MF	0	0	0	1	$\ln$(cost(MF))	0	0

A case study

$$
\begin{array}{rcllllll}
V_{\text{BM}} & = & & & \beta_5 \ln(\text{cost}_{\text{BM}}) & + & \beta_6 \text{users} & & \\
V_{\text{SM}} & = & \beta_1 & + & \beta_5 \ln(\text{cost}_{\text{SM}}) & + & \beta_6 \text{users} & & \\
V_{\text{LF}} & = & \beta_2 & + & \beta_5 \ln(\text{cost}_{\text{LF}}) & & & + & \beta_7 \text{MS} \\
V_{\text{EF}} & = & \beta_3 & + & \beta_5 \ln(\text{cost}_{\text{EF}}) & & & & \\
V_{\text{MF}} & = & \beta_4 & + & \beta_5 \ln(\text{cost}_{\text{MF}}) & & & &
\end{array}
$$

Maximum likelihood estimation

Logit Model:

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}}}{\sum_{j\in\mathcal{C}_n} e^{V_{jn}}}$$

Log-likelihood of a sample:

$$\mathcal{L}(\beta_1,\ldots,\beta_K) = \sum_{n=1}^{N}\left(\sum_{j=1}^{J} y_{jn} \ln P_n(j|\mathcal{C}_n)\right)$$

where $y_{jn} = 1$ if ind. n has chosen alt. j, 0 otherwise

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Maximum likelihood estimation

$$
\begin{aligned}
\ln P_n(i|\mathcal{C}_n) &= \ln \frac{e^{V_{in}}}{\sum_{j\in\mathcal{C}_n} e^{V_{jn}}} \\
&= V_{in} - \ln(\textstyle\sum_{j\in\mathcal{C}_n} e^{V_{jn}})
\end{aligned}
$$

Log-likelihood of a sample:

$$
\mathcal{L}(\beta_1,\ldots,\beta_K) = \sum_{n=1}^{N}\sum_{i=1}^{J} y_{in}\left(V_{in} - \ln \sum_{j\in\mathcal{C}_n} e^{V_{jn}}\right)
$$

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Maximum likelihood estimation

The maximum likelihood estimation problem:

$$\max_{\beta \in \mathbb{R}^K} \mathcal{L}(\beta)$$

Maximization of a concave function with K variables
Nonlinear programming

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Maximum likelihood estimation

Numerical issue:

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}}}{\sum_{j \in \mathcal{C}_n} e^{V_{jn}}}$$

Largest value that can be stored in a computer $\approx 10^{308}$, that is

$$e^{709.783}$$

It is equivalent to compute

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}-V_{in}}}{\sum_{j \in \mathcal{C}_n} e^{V_{jn}-V_{in}}} = \frac{1}{\sum_{j \in \mathcal{C}_n} e^{V_{jn}-V_{in}}}$$

Simple models

Null model

$$U_i = \varepsilon_i \quad \forall i$$

$$P_n(i|\mathcal{C}_n) = \frac{e^{V_{in}}}{\sum_{j\in\mathcal{C}_n} e^{V_{jn}}} = \frac{e^0}{\sum_{j\in\mathcal{C}_n} e^0} = \frac{1}{\#\mathcal{C}_n}$$

$$\mathcal{L} = \sum_n \ln \frac{1}{\#\mathcal{C}_n} = -\sum_n \ln(\#\mathcal{C}_n)$$

Simple models

Constants only [Assume $\mathcal{C}_n = \mathcal{C}, \forall n$]

$$U_i = c_i + \varepsilon_i \quad \forall i$$

In the sample of size n, there are n_i persons choosing alt. i.

$$\ln P(i) = c_i - \ln(\sum_j e^{c_j})$$

If $\mathcal{C}_n$ is the same for all people choosing i, the log-likelihood for this part of the sample is

$$\mathcal{L}_i = n_i c_i - n_i \ln(\sum_j e^{c_j})$$

Simple models

Constants only

The total log-likelihood is

$$\mathcal{L} = \sum_j n_j c_j - n \ln(\sum_j e^{c_j})$$

At the maximum, the derivatives must be zero

$$\frac{\partial \mathcal{L}}{\partial c_1} = n_1 - n \frac{e^{c_1}}{\sum_j e^{c_j}} = n_1 - nP(1) = 0.$$

Simple models

Constants only

Therefore,

$$P(1) = \frac{n_1}{n}$$

$\boxed{\text{If all alternatives are always available, a model with only Alternative Specific Constants reproduces exactly the market shares in the sample}}$

Back to the case study

Alt.	n_i	n_i/n	c_i	e^{c_i}	P(i)
BM	73	0.168	0.247	1.281	0.168
SM	123	0.283	0.769	2.158	0.283
LF	178	0.410	1.139	3.123	0.410
EF	3	0.007	-2.944	0.053	0.007
MF	57	0.131	0.000	1.000	0.131
	434	1.000			

Null-model: $\mathcal{L}$ = -434 $\ln(5)$ = -698.496

Warning: these results have been obtained assuming that all alternatives are always available