Optimization and Simulation

Simulating events: the Poisson process

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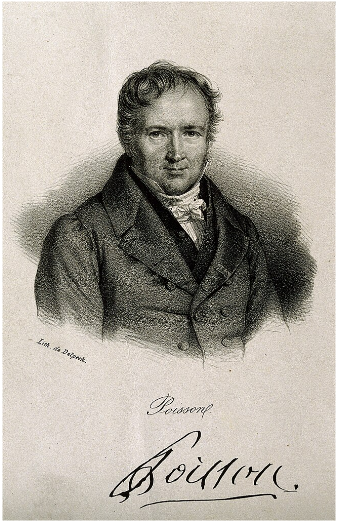
Siméon Denis Poisson

Siméon-Denis Poisson: French mathematician (1781–1840)



Outline

Poisson random variable

Poisson process

Non homogeneous Poisson process

Binomial random variable

Context

- n: number of independent trials
- p: probability of a success
- X: number of successes

Probability of k successes

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Poisson random variable

Context: continuous trials

- $n \to +\infty$: large number of trials
- $p \to 0$: low probability of a success
- $np \to \lambda$: success rate
- X: number of successes

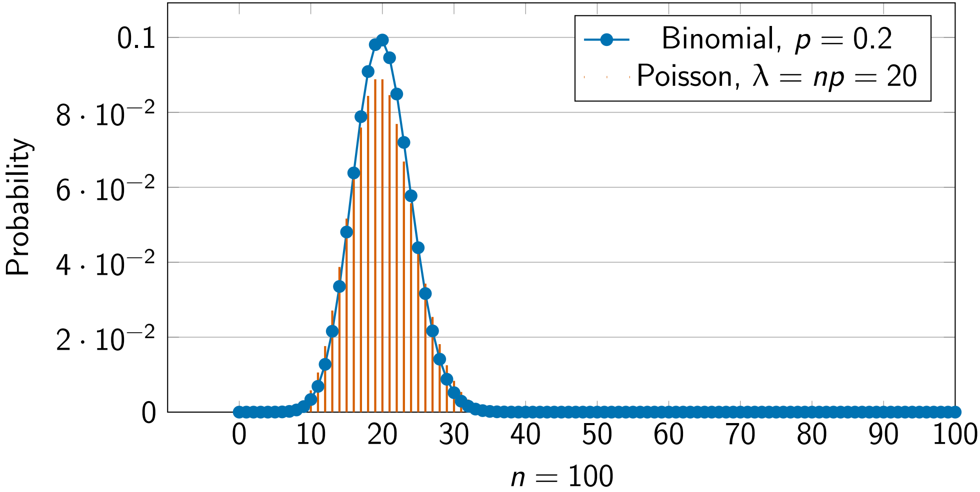
Probability of k successes

$$\Pr(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

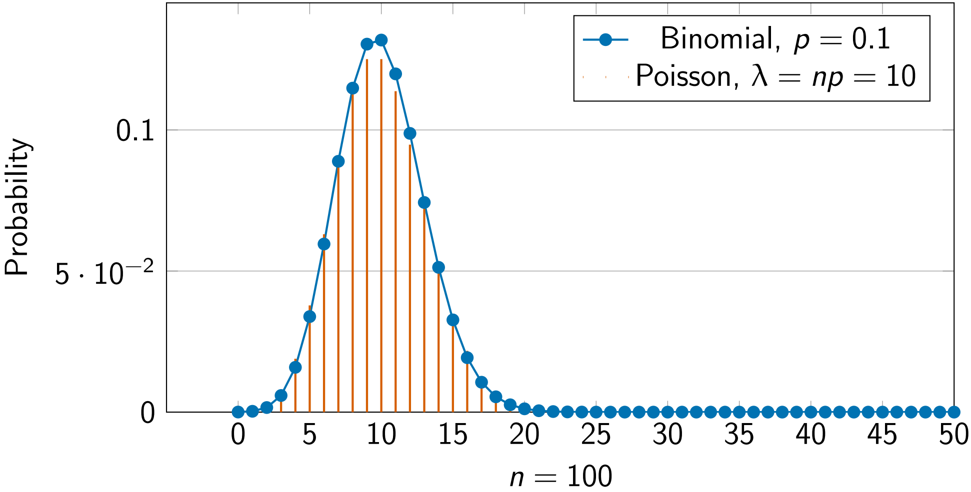
Property

$$\mathrm{E}[X] = \mathrm{Var}(X) = \lambda.$$

Poisson random variable



Poisson random variable



Outline

Poisson process

Events are occurring at random time points

$N(t)$ is the number of events during $[0, t]$

Poisson process

Poisson process with rate $\lambda > 0$ if

1. $N(0) = 0$ (no events at time zero),
2. independent increments (no interaction between disjoint intervals),
3. distribution of $N(t+s) - N(t)$ depends on s, not on t,
4. probability of one event in a small interval is approx. λh:

$$\lim_{h \to 0} \frac{\Pr(N(h) = 1)}{h} = \lambda,$$

5. probability of two events in a small interval is approx. 0:

$$\lim_{h \to 0} \frac{\Pr(N(h) \geqslant 2)}{h} = 0.$$

Poisson process

Property

$$N(t) \sim \text{Poisson}(\lambda t), \quad \Pr(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

Poisson process

Inter-arrival times

- S_k is the time when the kth event occurs,
- $X_k = S_k - S_{k-1}$ is the time elapsed between event $k-1$ and event k.
- $X_1 = S_1$
- Distribution of X_1: $\Pr(X_1 > t) = \Pr(N(t) = 0) = e^{-\lambda t}$.
- Distribution of X_k:

$$\begin{aligned}\Pr(X_k > t | S_{k-1} = s) &= \Pr(0 \text{ events in }]s, s+t] | S_{k-1} = s) \\ &= \Pr(0 \text{ events in }]s, s+t]) \\ &= e^{-\lambda t}.\end{aligned}$$

Poisson process

Inter-arrival times (ctd.)

Therefore, the CDF of X_k is, for any k,

$$F(t) = \Pr(X_k \leqslant t) = 1 - \Pr(X_k > t) = 1 - e^{-\lambda t}.$$

The pdf is

$$f(t) = \frac{dF(t)}{dt} = \lambda e^{-\lambda t}.$$

Poisson process

Conclusion

The inter-arrival times X_1, X_2,... are independent and identically distributed exponential random variables with parameter λ, and mean $1/\lambda$.

Simulation

- Simulation of event times of a Poisson process with rate λ until time T:
  1. $t = 0$, $k = 0$.
  2. Draw $r \sim U(0, 1)$.
  3. $t = t - \ln(r)/\lambda$ [inverse transform method].
  4. If $t > T$, STOP.
  5. $k = k + 1$, $S_k = t$.
  6. Go to step 2.

When is the Poisson model not appropriate?

Limitations of the Poisson assumption

- Events influence each other (aftershocks, congestion).
- Simultaneous or batched events occur.
- Arrival rate depends on past events (learning, fatigue).
- Strong periodic or seasonal patterns.

Outline

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Non homogeneous Poisson process

Non homogeneous Poisson process

Rate varies with time

$\lambda(t)$.

Non homogeneous Poisson process with rate $\lambda(t)$ if

1. $N(0) = 0$ (no events at time zero),
2. independent increments (no interaction between disjoint intervals),
3. probability of one event in a small interval is approx. $\lambda(t)h$:

$$\lim_{h \to 0} \frac{\Pr\left((N(t+h) - N(t)) = 1\right)}{h} = \lambda(t),$$

4. probability of two events in a small interval is approx. 0:

$$\lim_{h \to 0} \frac{\Pr\left((N(t+h) - N(t)) \geqslant 2\right)}{h} = 0.$$

Non homogeneous Poisson process

Mean value function

$$m(t) = \int_0^t \lambda(s)ds, \ t \geqslant 0.$$

Poisson distribution

$$N(t+s) - N(t) \sim \text{Poisson}(m(t+s) - m(t))$$

Link with homogeneous Poisson process

- Consider a Poisson process with rate λ.
- If an event occurs at time t, count it with probability $p(t)$.
- The process of counted events is a non homogeneous Poisson process with rate $\lambda(t) = \lambda p(t)$.

Non homogeneous Poisson process

Proof

1. $N(0) = 0$ [OK]
2. $\#$ of events occurring in disjoint time intervals are independent, [OK]
3. probability of one event in a small interval is approx. $\lambda(t)h$: [?]

$$\lim_{h \to 0} \frac{\Pr\left((N(t+h) - N(t)) = 1\right)}{h} = \lambda(t),$$

4. probability of two events in a small interval is approx. 0: [OK]

$$\lim_{h \to 0} \frac{\Pr\left((N(t+h) - N(t)) \geqslant 2\right)}{h} = 0.$$

Non homogeneous Poisson process

Proof (ctd.)

- $N(t)$ number of events of the non homogeneous process
- $N'(t)$ number of events of the underlying homogeneous process

$\Pr\left((N(t+h) - N(t)) = 1\right)$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \Pr\left((N'(t+h) - N'(t)) = k, 1 \text{ is counted}\right) \\
&= \Pr\left((N'(t+h) - N'(t)) = 1, 1 \text{ is counted}\right) \\
&= \Pr\left((N'(t+h) - N'(t)) = 1\right) \Pr(1 \text{ is counted}) \\
&= \Pr\left(N'(h) = 1\right) \Pr(1 \text{ is counted})
\end{aligned}
$$

$$
\begin{aligned}
\lim_{h \to 0} \frac{\Pr((N(t+h) - N(t)) = 1)}{h} &= \lim_{h \to 0} \frac{\Pr(N'(h) = 1)}{h} \Pr(1 \text{ is counted}) \\
&= \lambda p(t).
\end{aligned}
$$

Non homogeneous Poisson process

Simulation of event times of a non homogeneous Poisson process with rate $\lambda(t)$ until time T

1. Consider λ such that $\lambda(t) \leqslant \lambda$, for all $t \leqslant T$.
2. $t = 0$, $k = 0$.
3. Draw $r \sim U(0, 1)$.
4. $t = t - \ln(r)/\lambda$.
5. If $t > T$, STOP.
6. Generate $s \sim U(0, 1)$.
7. If $s \leqslant \lambda(t)/\lambda$, then $k = k + 1$, $S_k = t$.
8. Go to step 3.

Non homogeneous Poisson process

Interpretation of the algorithm

- Candidate events are generated by a homogeneous Poisson process with rate λ.
- Each event is accepted with probability $\lambda(t)/\lambda$.
- This is an accept–reject method in continuous time.

Efficiency and diagnostics

Efficiency of thinning

- Expected acceptance rate:

$$\frac{\int_0^T \lambda(t)\,dt}{\lambda T}.$$

- A loose bound λ leads to many rejected events.
- Choosing λ as tight as possible improves efficiency.

Practical diagnostics

- Monitor acceptance rate.
- Check empirical inter-arrival times against theory.
- Plot event times to verify time-varying intensity.

Bridge to discrete-event simulation

Why Poisson processes matter

- Event-driven systems evolve at random event times.
- Poisson processes provide:
  - arrival processes,
  - clocks for simulation,
  - building blocks for queues and networks.

Next step

Discrete-event simulation will generalize this idea to interacting events and state-dependent dynamics.

Summary

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- Poisson random variable
- Poisson process
- Non homogeneous Poisson process

Comments

- Main assumption: events occur continuously and independently of one another
- Typical usage: arrivals of customers in a queue
- Easy to simulate