# Optimization and Simulation

## Multi-objective optimization

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# Multi-objective optimization

## Concept

- Need for minimizing several objective functions.
- In many practical applications, the objectives are conflicting.
- Improving one objective may deteriorate several others.

## Examples

- Transportation: maximize level of service, minimize costs.
- Finance: maximize return, minimize risk.
- Survey: maximize information, minimize number of questions (burden).

# Multi-objective optimization

$$\min_{x} F(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_P(x) \end{pmatrix}$$

subject to

$$x \in \mathcal{F} \subseteq \mathbb{R}^n,$$

where

$$F : \mathbb{R}^n \to \mathbb{R}^P.$$

# Outline

# Dominance

## Dominance

Consider $x_1, x_2 \in \mathbb{R}^n$. $x_1$ is dominating $x_2$ if

1. $x_1$ is no worse in any objective

$$\forall i \in \{1, \dots, p\}, f_i(x_1) \leq f_i(x_2),$$

2. $x_1$ is strictly better in at least one objective

$$\exists i \in \{1, \dots, p\}, f_i(x_1) < f_i(x_2).$$

## Notation

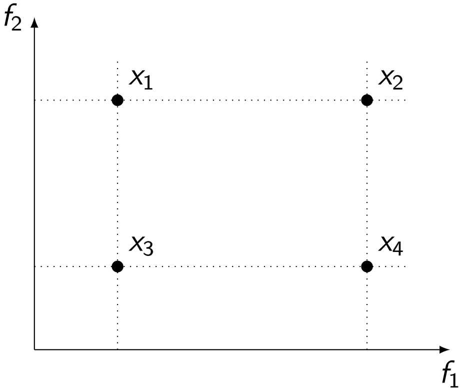
$x_1$ dominates $x_2$: $F(x_1) \prec F(x_2)$.

# Dominance

## Properties

- Not reflexive: $x \nprec x$
- Not symmetric: $x \prec y \nRightarrow y \prec x$
- Instead: $x \prec y \Rightarrow y \nprec x$
- Transitive: $x \prec y$ and $y \prec z \Rightarrow x \prec z$
- Not complete: $\exists x, y$: $x \nprec y$ and $y \nprec x$

# Dominance: example




$$
\begin{array}{lcl}
F(x_3) & \prec & F(x_2) \\
F(x_3) & \prec & F(x_1) \\
F(x_1) & \not\prec & F(x_4) \\
F(x_4) & \not\prec & F(x_1)
\end{array}
$$

# Optimality

## Pareto optimality

The vector $x^* \in \mathcal{F}$ is Pareto optimal if it is not dominated by any feasible solution:

$$\nexists x \in \mathcal{F} \text{ such that } F(x) \prec F(x^*).$$

## Intuition

$x^*$ is Pareto optimal if no objective can be improved without degrading at least one of the others.

# Optimality

## Weak Pareto optimality

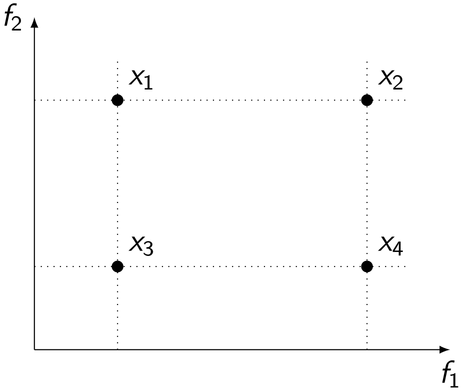
The vector $x^* \in \mathcal{F}$ is weakly Pareto optimal if there is no $x \in \mathcal{F}$ such that $\forall i = 1, \ldots, p$,

$$f_i(x) < f_i(x^*),$$

## Pareto optimality

- $P^*$: set of Pareto optimal solutions
- $WP^*$: set of weakly Pareto optimal solutions
- $P^* \subseteq WP^* \subseteq \mathcal{F}$

# Dominance: example




- $x_3$: Pareto optimal.
- $x_1$, $x_3$, $x_4$: weakly Pareto optimal.

# Pareto frontier

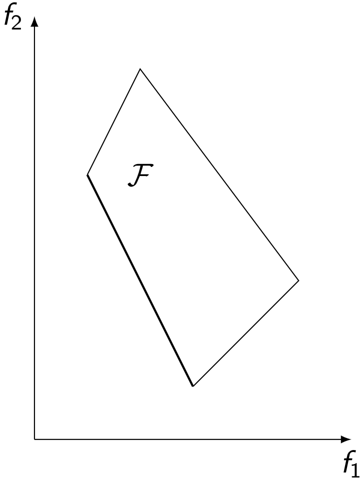
## Pareto optimal set

$$P^* = \{x^* \in \mathcal{F} | \nexists x \in \mathcal{F} : F(x) \prec F(x^*)\}$$

## Pareto frontier

$$PF^* = \{F(x^*) | x \in P^*\}$$

# Pareto frontier



# Outline

# Weighted sum

## Weights

For each $i = 1, \ldots, p$, $w_i > 0$ is the weight of objective $i$.

## Optimization

$$\min_{x \in \mathcal{F}} \sum_{i=1}^{p} w_i f_i(x). \tag{1}$$

## Comments

- Weights may be difficult to interpret in practice.
- Generates a Pareto optimal solution.
- In the convex case, if $x^*$ is Pareto optimal, there exists a set of weights such that $x^*$ is the solution of (1)

# Weighted sum: example

## Train service

- $f_1$: minimize travel time
- $f_2$: minimize number of trains
- $f_3$: maximize number of passengers

## Definition of the weights

- Transform each objective into monetary costs.
- Travel time: use value-of-time.
- Number of trains: estimate the cost of running a train.
- Number of passengers: estimate the revenues generated by the passengers.

# Goal programming

## Goals

For each $i = 1, \ldots, p$, $g_i$ is the "ideal" or "target" objective function defined by the modeler.

## Optimization

$$\min_{x \in \mathcal{F}} \|F(x) - g\|_\ell = \sqrt[\ell]{\sum_{i=1}^{p} |F_i(x) - g_i|^\ell}$$

## Issue

Not really optimizing the objectives

# Outline

# Lexicographic optimization

## Sorted objective

Assume that the objectives are sorted from the most important ($i = 1$) to the least important ($i = p$).

## First problem

$$f_1^* = \min_{x \in \mathcal{F}} f_1(x)$$

## $\ell$th problem

$$f_\ell^* = \min f_\ell(x)$$

subject to

$$\begin{aligned} x &\in \mathcal{F} \\ f_i(x) &= f_i^*, \; i = 1, \ldots, \ell - 1. \end{aligned}$$

# $\varepsilon$-lexicographic optimization

## Sorted objective and tolerances

- Assume that the objectives are sorted from the most important ($i = 1$) to the least important ($i = p$).
- For each $i = 1, \ldots, p$, $\varepsilon_i \geq 0$ is a tolerance on the objective $f_i$.

## First problem

$$f_1^* = \min_{x \in \mathcal{F}} f_1(x)$$

## $\ell$th problem

$$f_\ell^* = \min f_\ell(x)$$

subject to

$$\begin{aligned} x &\in \mathcal{F} \\ f_i(x) &\leq f_i^* + \varepsilon_i, \ i = 1, \ldots, \ell - 1. \end{aligned}$$

# Outline

# $\varepsilon$-constraints formulation

## Reference objective and upper bounds

- Select a reference objective $\ell \in \{1, \ldots, p\}$.
- Impose an upper bound $\varepsilon_i$ on each other objective.

## Constrained optimization

$$\min_{x \in \mathcal{F}} f_\ell(x)$$

subject to

$$f_i(x) \leq \varepsilon_i, \quad i \neq \ell.$$

## Property

If a solution exists, it is weakly Pareto optimal.

# Outline

# Local search

## Main difference with single objective

Maintain a set $\mathcal{P}$ of potential Pareto optimal solutions

$$\forall x, y \in \mathcal{P}, F(x) \not\prec F(y) \text{ and } F(y) \not\prec F(x).$$

## Initialization

Start with a first set $\mathcal{P}$ of candidate solutions.

## Main iteration

- Select randomly $x$ from $\mathcal{P}$ and consider $x^+$ a neighbor of $x$.
- Define
$$\mathcal{D}(x^+) = \{y \in \mathcal{P} \text{ such that } F(x^+) \prec F(y)\}.$$
- Define
$$\mathcal{S}(x^+) = \{y \in \mathcal{P} \text{ such that } F(y) \prec F(x^+)\}.$$

# Local search

## Main iteration

- If $\mathcal{S}(x^+) = \emptyset$

$$\mathcal{P}^+ = \mathcal{P} \cup \{x^+\} \setminus \mathcal{D}(x^+).$$

## Property of $\mathcal{P}^+$

$$\forall x, y \in \mathcal{P}^+, F(x) \not\leq F(y) \text{ and } F(y) \not\leq F(x).$$

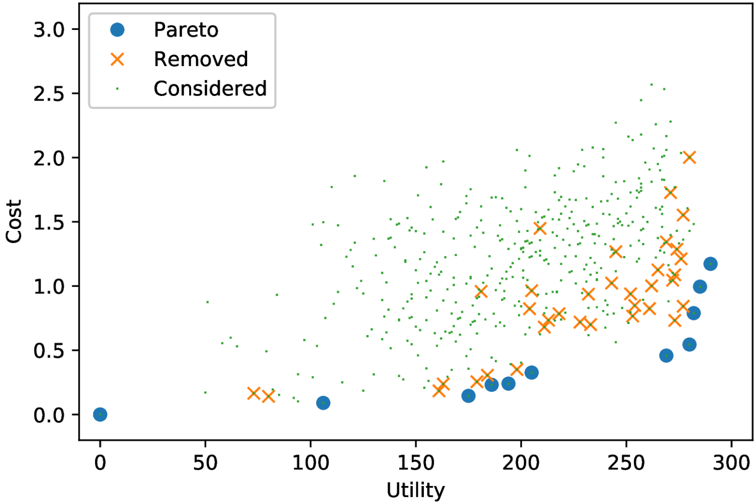
## Proof

- For $x$, $y$ different from $x^+$, already valid in $\mathcal{P}$.
- Consider $x^+$, $y \in \mathcal{P}^+$:
  - $y \in \mathcal{P}^+ \Rightarrow y \notin \mathcal{D}(x^+) \Rightarrow F(x^+) \not\leq F(y)$.
  - $x^+ \in \mathcal{P}^+ \Rightarrow \mathcal{S}(x^+) = \emptyset \Rightarrow y \notin \mathcal{S}(x^+) \Rightarrow F(y) \not\leq F(x^+)$.

# Example: priced knapsack

| Utility | Weight | Cost |
|---:|---:|---:|
| 80 | 84 | 0.50328447 |
| 31 | 27 | 0.41431774 |
| 48 | 47 | 0.07765353 |
| 17 | 22 | 0.75842330 |
| 27 | 21 | 0.14050556 |
| 84 | 96 | 0.72089439 |
| 34 | 42 | 0.11669739 |
| 39 | 46 | 0.56723896 |
| 46 | 54 | 0.02430532 |
| 58 | 53 | 0.01255171 |
| 23 | 32 | 0.03059062 |
| 67 | 78 | 0.17285314 |

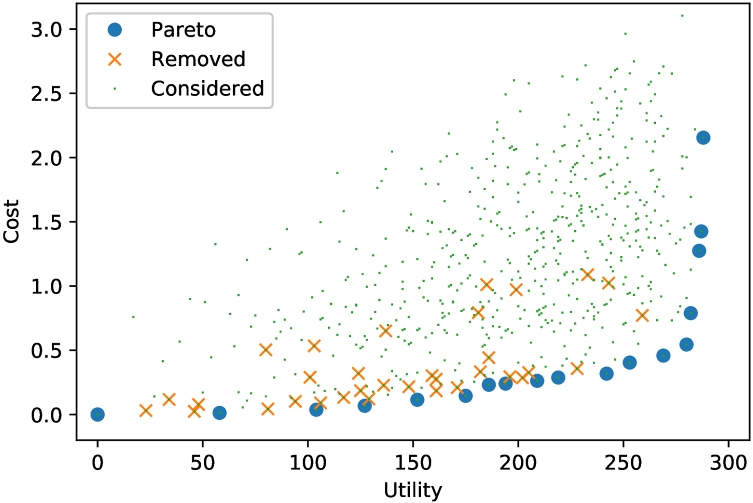
# Example: local search with neighborhood $k = 4$



# Variable Neighborhood Search

- Neighborhood of size 1 Pareto solutions: 1
- Neighborhood of size 2 Pareto solutions: 16
- Neighborhood of size 3 Pareto solutions: 16
- Neighborhood of size 4 Pareto solutions: 16
- Neighborhood of size 5 Pareto solutions: 16
- Neighborhood of size 6 Pareto solutions: 16
- Neighborhood of size 7 Pareto solutions: 18
- Neighborhood of size 8 Pareto solutions: 19
- Neighborhood of size 9 Pareto solutions: 19
- Neighborhood of size 10 Pareto solutions: 19
- Neighborhood of size 11 Pareto solutions: 19
- Neighborhood of size 12 Pareto solutions: 19
- Pareto solutions: 19

# Variable Neighborhood Search



# Conclusion

## Problem definition

- Need for trade-offs.
- Concept of Pareto frontier.

## Algorithms

- Heuristics.
- Most of time driven by problem knowledge.